Properties of the Scattering Equations

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Based on

- work of Louise Dolan and P.G.
  - 1311.5200 Proof of the Formula of CHY for Yang-Mills Tree Amplitudes
  - 1402.7374 The Polynomial Form of the Scattering Equations
- following work of Freddy Cachazo, Song He and Ellis Yuan
  - 1306.6575 Scattering Equations and KLT Orthogonality
  - 1307.2199 Scattering of Massless Particles in Arbitrary Dimensions
  - 1309.0885 Scattering of Massless Particles: Scalars, Gluons and Gravitons
- and early work of David Fairlie and David Roberts
  - Durham preprint 72-2440 Dual Models without Tachyons - a New Approach

    see also: David Fairlie
  - 0805:2263 A Coding of Real Null Four-Momenta into World-Sheet Co-ordinates
- see also work of David Gross and Paul Mende
  - Nucl. Phys. B303 (1988) 407 String Theory Beyond The Planck Scale

General Overview

N massless particles, momenta $k_a, \quad a \in A = \{1, 2, \ldots, N\}$.

Scattering Equations:

$$\sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \qquad z_a \in \mathbb{C}, \quad a \in A.$$

N equations in N variables, but Moebius invariant, so there must be 3 relations between the equations and so, after fixing 3 of the z_a, there are $N-3$ equations in $N-3$ variables.

Typically there are $(N-3)!$ solutions.

equivalent to the polynomial form:

$$\sum_{\substack{S\subset A\\ |S|=m}} k_S^2\, z_S = 0, \qquad 2 \le m \le N-2,$$

where for $S \subset A$, $\qquad k_S = \sum_{b\in S} k_b, \qquad z_S = \prod_{a\in S} z_a,$

$N-3$ equations in N variables, but Moebius invariant, so after fixing 3 of the z_a, there are $N-3$ equations in $N-3$ variables.

Then have polynomials of degree 1 to $N-3$ implying $(N-3)!$ solns.

Note the coefficients in the polynomials are exactly all the Mandelstam variables k_S^2.

- Linear in each z_a separately, facilitating solution.

General Objective:

Cachazo, He and Yuan and others have found expressions for tree amplitudes in massless theories as integrals of rational functions of the z_a and the momenta, around poles at solutions of the scattering equations. These integrals are hence just sums over the solutions and thus, necessarily, rational functions of the coefficients in the scattering equations.

The prescription is fundamentally one of attaching an algebraic expression to the scattering equations, or the zero-dimensional variety that they describe. The integral is somewhat illusory.

Our objective is to understand the CHY expressions, etc., in this sense, in terms of natural algebraic objects attached to this "scattering variety".

Plan of Talk

- The Scattering Equations
  - Moebius Invariance
- Tree Amplitudes
  - Four-point Tree Amplitude
  - History
- Polynomial Form for the Scattering Equations
- Solutions to the Scattering Equations
- Moebius Transformations and the Polynomial Form
- One-Loop Scattering Equations
- Four-Dimensional Space-Time

The Scattering Equations

massless momenta k_a labelled by $a \in A$. $\qquad k_a^2 = 0$

$$|A| = N \qquad A = \{1, 2, \ldots, N\} \qquad \sum_{a \in A} k_a = 0$$

Scattering Equations:

$$\sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \qquad a \in A. \qquad \text{[SE]}$$

N equations in N variables $z_a \in \mathbb{C}$,

but [SE] are Moebius invariant, so solutions determined up to

$$z_a \mapsto \frac{\alpha z_a + \beta}{\gamma z_a + \delta}$$

and only $N-3$ of the [SE] are independent.

Moebius Invariance

After fixing Moebius invariance ($e.g.\ z_1 = \infty, z_2 = 1, z_N = 0$),

typically the [SE] have $(N-3)!$ solutions. Tree amplitudes are expressed effectively as a sum over the solutions to the [SE].

Consider

$$z_a \mapsto \frac{\alpha z_a + \beta}{\gamma z_a + \delta}$$

$$U(z,k) = \prod_{a<b} (z_a - z_b)^{-k_a \cdot k_b}$$

is Moebius invariant, provided that $k_a^2 = 0,\ \sum_{a \in A} k_a = 0$.

$$\frac{\partial U}{\partial z_a} = -f_a U, \qquad f_a(z,k) = \sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b}$$

implying that the system of equations $f_a = 0, a \in A$ [SE]

is Moebius invariant, and subject to the 3 relations:

$$\sum_{a \in A} f_a = 0, \quad \sum_{a \in A} z_a f_a = 0, \quad \sum_{a \in A} z_a^2 f_a = 0.$$

[6]

Tree Amplitudes

$$\mathcal{A}_N = \int \Psi_N(z,k,\epsilon) {\prod_{a\in A}}' \delta\left(f_a(z,k)\right) \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega,$$

$$= \oint_{\mathcal{O}} \Psi_N(z,k,\epsilon) {\prod_{a\in A}}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega$$

[Cachazo He Yuan]

$${\prod_{a\in A}}' \delta\left(f_a(z,k)\right) = (z_1 - z_2)(z_2 - z_N)(z_1 - z_N) \prod_{a=2}^{N-1} \delta\left(f_a(z,k)\right), \qquad d\omega = \frac{dz_1 dz_2 dz_N}{(z_1 - z_2)(z_2 - z_N)(z_1 - z_N)}$$

$\mathcal{O}$ encircles the solutions of $f_a(z,k) = 0$ and Ψ_N is Moebius invariant.

[Note that we have fixed an order on A; we have to sum over orderings.]

CHY proposed forms of Ψ_N for massless scalar ϕ^3, Yang-Mills and gravity in arbitrary space-time dimension. For massless ϕ^3, $\Psi_N = 1$.

These have been proved for ϕ^3 and Yang-Mills, using a BCFW approach, and extended to massive ϕ^3. [DG]

Tree Amplitudes

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k,\epsilon) {\prod_{a\in A}}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega$$

For massless ϕ^3,

$$\Psi_N = 1.$$

For Yang-Mills and gravity, Ψ_N is a polynomial in

$$\frac{k_a \cdot k_b}{z_a - z_b}, \qquad \frac{\epsilon_a \cdot k_b}{z_a - z_b}, \qquad \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b},$$

where ϵ_a are polarizations, and which can be described quite simply in terms of a Pfaffian, multiplied by

$$\prod_{a\in A} (z_a - z_{a+1})^n,$$

where $n = 1$ for Yang-Mills, and $n = 2$ for gravity.

[8]

Four-point Scalar Tree Amplitude

$N = 4$

$z_1 = \infty,\ z_2 = 1,\ z_3 = z,\ z_4 = 0.$

$$f_3(z,k) = \frac{k_3 \cdot k_2}{z-1} + \frac{k_3 \cdot k_4}{z}$$

$$\begin{aligned}
\mathcal{A}_4 &= \oint_{\mathcal{O}} \frac{dz}{f_3(z,k)(1-z)^2 z^2} \\
&= \oint_{z=z_*} \frac{dz}{[k_3 \cdot k_2 z + k_3 \cdot k_4 (z-1)](z-1)z}, \qquad z_* = -\frac{k_3 \cdot k_4}{k_3 \cdot k_1} \\
&= \frac{k_3 \cdot k_1}{k_3 \cdot k_4\, k_3 \cdot k_2} = -2\left(\frac{1}{s} + \frac{1}{t}\right),
\end{aligned}$$

$s = (k_3 + k_4)^2,\ t = (k_2 + k_3)^2$

History: string theory

$$X^\mu(z) = x^\mu - ip^\mu \log z + i \sum_{n \neq 0} \frac{a_n^\mu}{n} z^{-n},$$

$$P^\mu(z) = i\frac{dX^\mu(z)}{dz} = \sum_n a_n^\mu z^{-n-1}, \quad a_0^\mu = p^\mu,$$

$\langle : P(z)^2 : \rangle =$ constant [Virasoro conditions]

Fairlie and Roberts [1972]

$$x^\mu(z) = -i \sum_{a \in A} k_a^\mu \log(z - z_a),$$

$$p^\mu(z) = i\frac{dx^\mu(z)}{dz} = \sum_{a \in A} \frac{k_a^\mu}{z - z_a}.$$

$$p(z)^2 = \sum_{a,b \in A} \frac{k_a \cdot k_b}{(z - z_a)(z - z_b)} = 0.$$ [identically]

DUAL MODELS WITHOUT TACHYONS -

A NEW APPROACH

D.B.Fairlie and D.E.Roberts

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ABSTRACT

The intrinsic surface Lagrangian for the dual model is investigated. Solutions to the minimisation problem are found in which the ground state scalars have zero mass. The connection between coordinates on the surface and the momenta of the particles is explicit in this model, and manifests the homogeneous Lorentz transformations on the momenta as SL(2,C) transformations on the coordinates.

If $$p(z)^2 = \sum_{a,b\in A} \frac{k_a \cdot k_b}{(z-z_a)(z-z_b)} = 0$$ identically,

$$k_a^2 = 0 \qquad a \in A \qquad \text{[no double poles]}$$

$$f_a(z,k) = \sum_{\substack{b\in A\\ b\neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0 \qquad a \in A \qquad \text{[no single poles]}$$

These conditions are equivalent to $p(z)^2 = 0.$ [Liouville]

Gross and Mende [1987]

$f_a(z,k) = 0, a \in A,$ are stationary phase conditions for string integrand

$$U(z,k) = \prod_{a<b} (z_a - z_b)^{-k_a\cdot k_b}, \qquad f_a(z,k) = -U^{-1}\frac{\partial U}{\partial z_a}$$

determining high-energy behavior.

Polynomial Form for the Scattering Equations

For a subset $S \subset A$, define

$$k_S = \sum_{b \in S} k_b, \qquad z_S = \prod_{a \in S} z_a,$$

then the scattering equations

$$\sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \qquad a \in A. \qquad \text{[SE]}$$

are equivalent to the homogeneous polynomial equations

$$\sum_{\substack{S \subset A \\ |S| = m}} k_S^2 \, z_S = 0, \qquad 2 \leq m \leq N-2, \qquad \text{[PSE]}$$

where the sum is over all $\dfrac{N!}{m!(N-m)!}$ subsets $S \subset A$ with m elements.

Proof of the Polynomial Form for the Scattering Equations

The scattering equations are equivalent to $p(z)^2 = 0$, and so to the vanishing of

$$F(z) = 2p(z)^2 \prod_{c\in A}(z - z_c) = \sum_{a,b\in A} 2k_a\cdot k_b \prod_{\substack{c\in A\\ c\neq a,b}}(z - z_b)$$

$$= \sum_{m=0}^{N-2} z^{N-m-2} \sum_{\substack{U\subset A\\ |U|=m}} z_U \sum_{\substack{S\subset \overline{U}\\ |S|=2}} k_S^2$$

where $\overline{U} = \{b\in A : b\notin U\}$. So the [SE] are equivalent to the vanishing of

$$\tilde{h}_m = \sum_{\substack{S\subset A\\ |S|=m}} k_S^2\, z_S = 0, \qquad 2\leq m\leq N-2,$$

since $\sum_{\substack{S\subset \overline{U}\\ |S|=2}} k_S^2 = k_{\overline{U}}^2 = k_U^2.$

[13]

Polynomial Form: partially fixing Moebius invariance

Set $z_1 \to \infty,\; z_N \to 0$, using that $\tilde{h}_m$ is linear in each z_a individually,

$$h_m = \lim_{z_1 \to \infty} \frac{\tilde{h}_{m+1}}{z_1} = \sum_{\substack{S \subset A' \\ |S| = m}} k_{S_1}^2 z_S \qquad\qquad S_1 = S \cup \{1\}$$

$$= \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \in A' \\ a_i \;\; \text{uneq.}}} k_{1a_1 \ldots a_m}^2 \, z_{a_1} z_{a_2} \ldots z_{a_m},$$

$$A' = \{a \in A : a \neq 1, N\}, \qquad k_{1a_1a_2\ldots a_m} = k_1 + k_{a_1} + k_{a_2} + \ldots + k_{a_m},$$

h_m is a homogeneous polynomial of degree m in $z_2, z_3, \ldots, z_{N-1}$, linear in each of the z_a individually. The scattering equations are are equivalent to the $N-3$ equations $h_m = 0,\; 1 \leq m \leq N-3.$ Typically, they determine $(N-3)!$ solutions for the ratios of the z_a.

[Bezout's Theorem]

Amplitudes in terms of Polynomial Constraints

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \prod_{m=2}^{N-2} \frac{1}{\tilde{h}_m(z,k)} \prod_{a<b} (z_a - z_b) \prod_{a \in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega .$$

Taking $z_1 \to \infty, \; z_2$ fixed, $z_N \to 0$,

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \frac{z_2}{z_{N-1}} \prod_{m=1}^{N-3} \frac{1}{h_m(z,k)} \prod_{2 \le a < b \le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a dz_{a+1}}{(z_a - z_{a+1})^2}$$

$$= \sum_{\text{solutions}} \frac{\Psi_N(z,k)}{z_{N-1} J(z,k)} \prod_{2 \le a < b \le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a}{(z_a - z_{a+1})^2}$$

$$J(z,k) = \frac{1}{z_2} \det \left[\frac{\partial h_m}{\partial z_a} \right]_{\substack{1 \le m \le N-3 \\ 3 \le a \le N-1}} .$$

[15]

Solutions to the Scattering Equations

- $N = 4$

$$h_1 = k_{12}^2 z_2 + k_{13}^2 z_3 = 0, \quad z_3/z_2 = -k_{12}^2/k_{13}^2 = -k_3 \cdot k_4/k_3 \cdot k_1 = z_*.$$

- $N = 5$ write $(x, y, z) = (z_2, z_3, z_4)$

$$h_1 = k_{12}^2 x + k_{13}^2 y + k_{14}^2 z = 0,$$
$$h_2 = k_{123}^2 xy + k_{124}^2 xz + k_{134}^2 yz = 0.$$

Eliminating x yields a quadratic for y/z,

with equal roots iff the five momenta lie in a 3-space

(i.e. the 3-momenta are coplanar).

$N = 5$

$$h_1 = ax + by + cz = 0,$$
$$h_2 = Ayz + Bxz + Cxy = 0.$$

eliminating x yields the quadratic:

$$Cby^2 + (Bb + Cc - Aa)yz + Bcz^2 = 0.$$

$$h_1 = a\,x + by + cz = 0,$$
$$h_2 = (Bz + Cy)x + Ayz = 0.$$

$$\begin{vmatrix} a & by + cz \\ Bz + Cy & Ayz \end{vmatrix} = \begin{vmatrix} a & ax + by + cz \\ Bz + Cy & Ayz + Bxz + Cxy \end{vmatrix} = \begin{vmatrix} h_1^x & h_1 \\ h_2^x & h_2 \end{vmatrix}$$

where $h_m^x = \dfrac{\partial h_m}{\partial x}$, etc. Note $h_m^{xx} = \dfrac{\partial^2 h_m}{\partial x^2} = 0.$

$$\frac{\partial}{\partial x}\begin{vmatrix} h_1^x & h_1 \\ h_2^x & h_2 \end{vmatrix} = \begin{vmatrix} h_1^x & h_1^x \\ h_2^x & h_2^x \end{vmatrix} + \begin{vmatrix} h_1^{xx} & h_1 \\ h_2^{xx} & h_2 \end{vmatrix} = 0.$$

[16]

- $N=6$ write $(x,y,z,u)=(z_2,z_3,z_4,z_5)$

$$h_1 = k_{12}^2 x + k_{13}^2 y + k_{14}^2 z + k_{15}^2 u = 0,$$
$$h_2 = k_{123}^2 xy + k_{124}^2 xz + k_{134}^2 yz + k_{125}^2 xu + k_{135}^2 yu + k_{145}^2 zu = 0.$$
$$h_3 = k_{1234}^2 xyz + k_{1235}^2 xyu + k_{1245}^2 xzu + k_{1345}^2 yzu = 0.$$

eliminating x,y yields a sextic for z/u. This can be written

$$\begin{vmatrix} h_1^{xy} & h_1^x & h_1^y & h_1 & 0 & 0 \\ h_2^{xy} & h_2^x & h_2^y & h_2 & 0 & 0 \\ h_3^{xy} & h_3^x & h_3^y & h_3 & 0 & 0 \\ 0 & 0 & h_1^{xy} & h_1^x & h_1^y & h_1 \\ 0 & 0 & h_2^{xy} & h_2^x & h_2^y & h_2 \\ 0 & 0 & h_3^{xy} & h_3^x & h_3^y & h_3 \end{vmatrix} = 0, \qquad h_m^{xy} = \frac{\partial^2 h_m}{\partial x \partial y}.$$

compare $\begin{vmatrix} h_1^x & h_1 \\ h_2^x & h_2 \end{vmatrix} = 0$ for $N=5$.

Write $h_m = a_m xy + b_m x + c_m y + d_m, \quad m = 1, 2, 3,$ **for** $N = 5$.

$$
\begin{aligned}
xh_1 &= a_1 x^2 y + b_1 x^2 + c_1 y + d_1 \\
xh_2 &= a_2 x^2 y + b_2 x^2 + c_2 y + d_2 \\
xh_3 &= a_3 x^2 y + b_3 x^2 + c_3 y + d_3 \\
h_1 &= a_1 xy + b_1 x + c_1 y + d_1 \\
h_2 &= a_2 xy + b_2 x + c_2 y + d_2 \\
h_3 &= a_3 xy + b_3 x + c_3 y + d_3
\end{aligned}
$$

$$
x^3 \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} h_1^{xy} & h_1^x & h_1^y & h_1 & 0 & 0 \\ h_2^{xy} & h_2^x & h_2^y & h_2 & 0 & 0 \\ h_3^{xy} & h_3^x & h_3^y & h_3 & 0 & 0 \\ 0 & 0 & h_1^{xy} & h_1^x & h_1^y & h_1 \\ 0 & 0 & h_2^{xy} & h_2^x & h_2^y & h_2 \\ 0 & 0 & h_3^{xy} & h_3^x & h_3^y & h_3 \end{vmatrix}
$$

1	h_1	h_2	h_3	h_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x	h_1^x	h_2^x	h_3^x	h_4^x	h_1	h_2	h_3	h_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
y	h_1^y	h_2^y	h_3^y	h_4^y	0	0	0	0	h_1	h_2	h_3	h_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
z	h_1^z	h_2^z	h_3^z	h_4^z	0	0	0	0	0	0	0	0	h_1	h_2	h_3	h_4	0	0	0	0	0	0	0	0	0	0	0	0
xy	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}	h_1^y	h_2^y	h_3^y	h_4^y	h_1^x	h_2^x	h_3^x	h_4^x	0	0	0	0	h_1	h_2	h_3	h_4	0	0	0	0	0	0	0	0
xz	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}	h_1^z	h_2^z	h_3^z	h_4^z	0	0	0	0	h_1^x	h_2^x	h_3^x	h_4^x	0	0	0	0	h_1	h_2	h_3	h_4	0	0	0	0
yz	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}	0	0	0	0	h_1^z	h_2^z	h_3^z	h_4^z	h_1^y	h_2^y	h_3^y	h_4^y	0	0	0	0	0	0	0	0	h_1	h_2	h_3	h_4
xyz	h_1^*	h_2^*	h_3^*	h_4^*	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}	h_1^z	h_2^z	h_3^z	h_4^z	h_1^y	h_2^y	h_3^y	h_4^y	h_1^x	h_2^x	h_3^x	h_4^x
x^2	0	0	0	0	h_1^x	h_2^x	h_3^x	h_4^x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
y^2	0	0	0	0	0	0	0	0	h_1^y	h_2^y	h_3^y	h_4^y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
z^2	0	0	0	0	0	0	0	0	0	0	0	0	h_1^z	h_2^z	h_3^z	h_4^z	0	0	0	0	0	0	0	0	0	0	0	0
x^2y	0	0	0	0	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}	0	0	0	0	0	0	0	0	h_1^x	h_2^x	h_3^x	h_4^x	0	0	0	0	0	0	0	0
x^2z	0	0	0	0	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}	0	0	0	0	0	0	0	0	0	0	0	0	h_1^x	h_2^x	h_3^x	h_4^x	0	0	0	0
y^2x	0	0	0	0	0	0	0	0	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}	0	0	0	0	h_1^y	h_2^y	h_3^y	h_4^y	0	0	0	0	0	0	0	0
y^2z	0	0	0	0	0	0	0	0	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}	0	0	0	0	0	0	0	0	0	0	0	0	h_1^y	h_2^y	h_3^y	h_4^y
z^2x	0	0	0	0	0	0	0	0	0	0	0	0	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}	0	0	0	0	h_1^z	h_2^z	h_3^z	h_4^z	0	0	0	0
z^2y	0	0	0	0	0	0	0	0	0	0	0	0	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}	0	0	0	0	0	0	0	0	h_1^z	h_2^z	h_3^z	h_4^z
x^2yz	0	0	0	0	h_1^*	h_2^*	h_3^*	h_4^*	0	0	0	0	0	0	0	0	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}	0	0	0	0
xy^2z	0	0	0	0	0	0	0	0	h_1^*	h_2^*	h_3^*	h_4^*	0	0	0	0	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}	0	0	0	0	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}
xyz^1	0	0	0	0	0	0	0	0	0	0	0	0	h_1^*	h_2^*	h_3^*	h_4^*	0	0	0	0	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}
x^2y^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	h_1^{xy}	h_2^{xy}	h_3^{xy}	h_4^{xy}	0	0	0	0	0	0	0	0
x^2z^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	h_1^{zx}	h_2^{zx}	h_3^{zx}	h_4^{zx}	0	0	0	0
y^2z^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	h_1^{yz}	h_2^{yz}	h_3^{yz}	h_4^{yz}
x^2y^2z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	h_1^*	h_2^*	h_3^*	h_4^*	0	0	0	0	0	0	0	0
x^2yz^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	h_1^*	h_2^*	h_3^*	h_4^*	0	0	0	0
xy^2z^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	h_1^*	h_2^*	h_3^*	h_4^*

Moebius Transformations and the Polynomial Form

It is straightforward to show that

$$\tilde{h}_m(z_a - \epsilon) = \sum_{r=0}^{m-2} \frac{(N-m+r-2)!}{r!(N-m-2)!} (-\epsilon)^r \, \tilde{h}_{m-r}(z_a),$$
$$\tilde{h}_m(\lambda z_a) = \lambda^m \tilde{h}_m(z_a),$$
$$\tilde{h}_m(1/z_a) = \tilde{h}_{N-m}(z_a)/z_A.$$

Combining translations and inversions:

$$\tilde{h}_m\left(\frac{z_a}{1+\epsilon z_a}\right) = \sum_{r=0}^{N-m-2} \frac{(m+r-2)!}{r!(m-2)!} \epsilon^r \, \tilde{h}_{m+r}(z_a) \prod_{a \in A} \frac{1}{1+\epsilon z_a}.$$

Generators of these Moebius transformations:

$$L_n = \sum_{a \in A} \left[-z_a^{n+1} \frac{\partial}{\partial z_a} + \frac{n+1}{2} z_a^n \right], \qquad n = 0, \pm 1,$$

$$[L_1, L_{-1}] = 2L_0, \qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1}$$

$$L_n = \sum_{a\in A} \left[-z_a^{n+1} \frac{\partial}{\partial z_a} + \frac{n+1}{2} z_a^n \right], \qquad n = 0, \pm 1, \tag{$*$}$$

is the N-fold tensor product of the single variable representation

$$L_n = -z^{n+1} \frac{d}{dz} + \frac{n+1}{2} z^n, \qquad n = 0, \pm 1,$$

which, acting on polynomials in z, has Moebius spin $\frac{1}{2}$, and is reducible but not decomposable; it has a two-dimensional invariant subspace consisting of functions ggg in z.

Correspondingly, the representation $(*)$, acting on polynomials in the z_a, has a 2^N-dimensional invariant subspace consisting of polynomials linear in the $z_a$taken separately, which is its largest finite-dimensional invariant subspace. This subspace decomposes into: one representation of spin $\frac{1}{2}N$; $N-1$ representations of spin $\frac{1}{2}N-1$; $\frac{1}{2}N(N-3)$ representations of spin $\frac{1}{2}N-2$, etc.

It is the $\frac{1}{2}N(N-3)$ representations of spin $\frac{1}{2}N-2$ that provide exactly the right number of equations, $N-3$, to determine a finite set of solutions, up to Moebius invariance, typically $(N-3)!$.
The representation of spin $\frac{1}{2}N$ and the $N-1$ representations of spin $\frac{1}{2}N-1$ are over-determined so that, typically, there are no solutions; the remaining representations of spin $\frac{1}{2}N-n, 3 \leq n \leq \frac{1}{2}N,$ determine varieties of dimension $2n-4$.

The representations of spin $\frac{1}{2}N-2$ all correspond to scattering equations for some light-like momenta. That is, if

$$\varphi_m = \sum_{\substack{U \subset A \\ |U|=m}} \varphi_U z_U, \qquad 2 \leq m \leq N-2,$$

is a basis for a Moebius representation of spin $\frac{1}{2}N-2$, then

$$\varphi_U = k_U^2, \quad \text{for some } k_a, a \in A, \text{with } k_a^2 = 0, \sum_{a \in A} k_a = 0$$

If

$$\varphi_m = \sum_{\substack{U\subset A\\ |U|=m}} \varphi_U z_U, \qquad 2\le m\le N-2,$$

is a basis for a Moebius representation of spin $\frac{1}{2}N-2$, then

$$\varphi_U = k_U^2, \quad \text{for some } \; k_a, a\in A, \; \text{with } \; k_a^2=0, \; \sum_{a\in A} k_a = 0,$$

where $k_a \in \mathbb{R}^{1,D-1}$ for some sufficiently large D, e.g. $D \ge N-1$.
The invariants for such k_a have exactly $\frac{1}{2}N(N-3)$ degrees of freedom.
which is the number of Moebius spin representations $\frac{1}{2}N-2$.

- It is unclear whether the representation of Moebius spin $\frac{1}{2}N-n$ and dimension $N-2n+1, n>2$, have interesting interpretations.

- We can also consider representing Moebius transformations by $L_n = \sum_{a\in A}\left[-z_a^{n+1}\frac{\partial}{\partial z_a} + M\frac{n+1}{2}z_a^n\right]$, for some $M>1$ corresponding to polynomials in the z_a of degree M. Interpetation unclear.

[22]

$n=3$ Moebius spin $\frac{1}{2}N-3$.

Multiplet is $\varphi_m = L_1^{m-3}\varphi_3/(m-3)!$, $3 \le m \le N-3$, where

$$\varphi_3 = \sum_{a<b<c} \lambda_{abc} z_a z_b z_c$$

$$\lambda_{abc} = \lambda_{acb} = \lambda_{bca}, \qquad \lambda_{aab} = 0, \qquad \sum_a \lambda_{abc} = 0.$$

$$\varphi_m = \sum_{\substack{S\subset A\\|S|=m}} \lambda_S z_S, \qquad \lambda_S = \sum_{\substack{U\subset S\\|U|=3}} \lambda_U.$$

The equations $\varphi_m = 0, \quad 3 \le m \le N-3,$ are equivalent to

$$\sum_{a,b,c} \frac{\lambda_{abc}}{(z-z_a)(z-z_b)(z-z_c)} = 0,$$

and so also equivalent to

$$\sum_{b,c} \frac{\lambda_{abc}}{(z_a-z_b)(z_a-z_c)} = 0, \qquad a \in A$$

One-Loop Scattering Equations

$$P^\mu(\nu,\tau) = p^\mu + \sum_{a\in A} k_a^\mu \zeta(\nu-\nu_a)$$

$$\zeta(\nu+1) = \zeta(\nu) + 2\eta(\tau); \qquad \zeta(\nu+\tau) = \zeta(\nu) - 2\pi i + 2\eta(\tau)\tau.$$

$P^\mu(\nu,\tau)$ is defined on the torus provided that $\sum_{a\in A} k_a = 0$

$$P^\mu(\nu+1,\tau) = P^\mu(\nu,\tau), \qquad P^\mu(\nu+\tau,\tau) = P^\mu(\nu,\tau).$$

In terms of the Weierstrass function $\wp$

$$P^\mu(\nu,\tau) = p^\mu + \sum_{a\in A} k_a^\mu \zeta(\nu-\nu_a) = k^\mu + \frac{1}{2}\sum_{a\in A} k_a^\mu \frac{\wp'(\nu)+\wp'(\nu_a)}{\wp(\nu)-\wp(\nu_a)}$$

$P(\nu,\tau)^2$ has no double poles provided that $k_a^2 = 0$, and no single poles provided that the genus scattering equations hold:

$$2k\cdot k_a + \sum_{b\neq a} k_a\cdot k_b \frac{\wp'(\nu_a)+\wp'(\nu_b)}{\wp(\nu_a)-\wp(\nu_b)} = 0, \qquad a\in A.$$

In that case, $P(\nu,\tau)^2 = k^2$ and so vanishes if, additionally, $k^2 = 0$.

These equations also have a polynomial form in terms of $\wp(\nu_a)$, $\wp'(\nu_a)$,

$$\sum_{|S|=1} k\cdot k_S \wp_S = 0, \quad \sum_{|S|=m} k\cdot k_S \wp_S = \frac{1}{2} \sum_{|S|=m-1} \sum_{a\in\overline{S}} k_a \cdot k_S \wp'_a \wp_S, \quad 2\le m\le N-1,$$

where $\overline{S}$ is the complement of S in A .

These equations, together with momentum conservation, are equivalent to a second polynomial form:

$$\sum_{|U|=m} \sum_{a\in\overline{U}} k\cdot k_a \wp'_a \wp_U = -\sum_{|U|=m+2} k_U^2 \wp_U + \frac{g_2}{4} \sum_{|U|=m} k_U^2 \wp_U - \frac{g_3}{4} \sum_{|U|=m-1} k_U^2 \wp_U$$
$$+\frac{1}{4} \sum_{|U|=m-1} \sum_{a,b\in\overline{U}} k_a \cdot k_b \wp'_a \wp'_b \wp_U, \qquad 0\le m\le N-1.$$

Four-Dimensional Space-Time

[after CHY unpublished]

$$p^\mu(z) = \sum_{a\in A} \frac{k_a^\mu}{z - z_a}$$

$$\hat{p}^\mu(z) = p^\mu(z) \prod_{a\in A}(z - z_a) = \sum_{a\in A} k_a^\mu \prod_{\substack{b\in A\\ b\neq a}}(z - z_b)$$

$$\hat{p}(z) = \hat{p}^\mu(z)\sigma_\mu = \begin{bmatrix} \hat{p}^{0-3}(z) & -\hat{p}^{1-i2}(z) \\ -\hat{p}^{1+i2}(z) & \hat{p}^{0+3}(z) \end{bmatrix}$$

$$\hat{p}^{0\pm3}(z) = \hat{p}^0(z) \pm \hat{p}^3(z), \quad \hat{p}^{1\pm i2}(z) = \hat{p}^1(z) \pm i\hat{p}^2(z)$$

$p(z)^2 = 0$ implies $\hat{p}^{0-3}(z)\hat{p}^{0+3}(z) = \hat{p}^{1-i2}(z)\hat{p}^{1+i2}(z).$

$$\frac{\hat{p}^{0-3}(z)}{\hat{p}^{1+i2}(z)} = \frac{\hat{p}^{1-i2}(z)}{\hat{p}^{0+3}(z)} = -\frac{\pi^1(z)}{\pi^2(z)},$$ say,

where $\pi^1(z), \pi^2(z)$ are polynomials with no common factor. Then

$$\hat{p}^{0-3}(z) = \pi^1(z)\bar{\pi}^1(z), \qquad -\hat{p}^{1-i2}(z) = \pi^1(z)\bar{\pi}^2(z),$$

$$\hat{p}^{0+3}(z) = \pi^2(z)\bar{\pi}^2(z) \qquad -\hat{p}^{1+i2}(z) = \pi^2(z)\bar{\pi}^1(z),$$

$$\hat{p}^\mu(z) = p^\mu(z)\prod_{a\in A}(z-z_a) = \sum_{a\in A} k_a^\mu \prod_{\substack{b\in A\\ b\neq a}}(z-z_b)$$

$$\hat{p}^\mu(z)\sigma_\mu = \begin{bmatrix} \hat{p}^{0-3}(z) & -\hat{p}^{1-i2}(z) \\ -\hat{p}^{1+i2}(z) & \hat{p}^{0+3}(z) \end{bmatrix} = \begin{bmatrix} \pi^1(z) \\ \pi^2(z) \end{bmatrix} \begin{bmatrix} \bar{\pi}^1(z) & \bar{\pi}^2(z) \end{bmatrix} = \pi(z)\bar{\pi}(z)^T$$

$\hat{p}^\mu(z)$ is a polynomial of degree $N-2$ [momentum conservation]

$\pi^\mu(z)$ is a polynomial of degree $n-1$, say, $n-1 \leq N-2$;

$\bar{\pi}^\mu(z)$ is a polynomial of degree $N-n-1$.

n is a topological attribute of the solution, and is the number of negative helicities in the gauge theory context.

If $\mathcal{N} \subset A$ is any subset with $|\mathcal{N}| = n$, and $k_a = \pi_a\bar{\pi}_a^T$, then for some λ_a

$$\pi_r = \sum_{s\in\mathcal{N}} \frac{\lambda_r}{\lambda_s(z_r-z_s)}\pi_s, \quad \bar{\pi}_s = -\sum_{r\in\mathcal{P}} \bar{\pi}_r \frac{\lambda_r}{\lambda_s(z_r-z_s)}, \qquad r\in\mathcal{P}=\overline{\mathcal{N}}, s\in\mathcal{N},$$

equations of twistor string theory for n $-$ and $N-n$ $+$ heliticities. [27]

If $D \geq N$, the $(N-3)!$ solutions to the scattering equations

$$\sum_{\substack{b\in A\\ b\neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \qquad a \in A, \qquad \text{[SE]}$$

or, equivalently,

$$\sum_{\substack{S\subset A\\ |S|=m}} k_S^2\, z_S = 0, \qquad 2 \leq m \leq N-2, \qquad \text{[PSE]}$$

can be continued into one another by deformation of the momenta, k_a

But for $D = 4$, they split into $N-3$ classes, corresponding to the solutions of the twistor equations

$$\pi_r = \sum_{s\in\mathcal{N}} \frac{\lambda_r}{\lambda_s(z_r - z_s)}\pi_s, \quad \bar{\pi}_s = -\sum_{r\in\mathcal{P}} \bar{\pi}_r \frac{\lambda_r}{\lambda_s(z_r - z_s)}, \qquad r \in \mathcal{P} = \overline{\mathcal{N}}, s \in \mathcal{N},$$

where $n = |\mathcal{N}|$ and $2 \leq n \leq N-2$. The number of solutions is given by the Eulerian number $\left\langle {N-3 \atop n-2} \right\rangle$, $\quad \sum_{n=2}^{N-2} \left\langle {N-3 \atop n-2} \right\rangle = (N-3)!$

[28]