THERMOFIELD DYNAMICS AND GRAVITY

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- Thermofield dynamics gives a way of discussing mixed states (which carry entropy) in terms of a pure state description. Can they be useful in gravity? (Israel; Maldacena; Jacobson; + *others*)
- Fuzzy spaces provide approximations to a differential manifold in terms of finite-dimensional matrices.

Can we combine these two to produce some version of gravity?

- What I hope to do is
  - Describe a (slight) generalization of thermofield dynamics
  - Apply this to gravity in 2+1 dimensions

Summary

- Thermofield dynamics can be expressed by a coherent state path integral with action on two copies of $\mathbb{CP}^{N-1}$ with opposite orientation.
- It can also be expressed as a functional integral over spinor fields, with a particular limit taken at the end
- For a fuzzy space, introduce gauge fields as a way of defining the large N limit.
- Double the Hilbert space modeling a fuzzy space to $\mathcal{H}_N \otimes \tilde{\mathcal{H}}_N$, with left chirality gravitational fields ($SO(3)_L$ in 2+1) on one component and right chirality fields ($SO(3)_R$) on the tilde component
- This leads to

$$S = -\frac{1}{4\pi} \int \left[\mathrm{Tr}\left(A\, dA + \frac{2}{3} A^3 \right)_L - \mathrm{Tr}\left(A\, dA + \frac{2}{3} A^3 \right)_R \right] = \mathrm{Einstein-Hilbert\ action}$$

- For a system with Hilbert space $\mathcal{H}$, the expectation value of observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho\, \mathcal{O}) = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} \mathcal{O} \right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H} \right)$$

- We double the Hilbert space to $\mathcal{H} \otimes \tilde{\mathcal{H}}$ and introduce the pure state (called thermofield vacuum)

$$|\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{1}{2}\beta E_n} |n, \tilde{n}\rangle$$

- Then we get

$$\langle \Omega |\, \mathcal{O}\, |\Omega\rangle = \frac{1}{Z} \sum_{m,n} e^{-\frac{1}{2}\beta(E_n + E_m)} \langle m|\mathcal{O}|n\rangle \langle \tilde{m}|\tilde{n}\rangle = \mathrm{Tr}(\rho\, \mathcal{O})$$

- The Hamiltonian is taken as

$$\check{H} = H - \tilde{H} = H \otimes \mathbf{1} - \mathbf{1} \otimes H, \qquad \Longrightarrow \quad \check{H}\, |\Omega\rangle = 0$$

This formalism is very useful for considering time-dependent (nonequilibrium) effects at finite temperature.

- For a quantum system, the density matrix evolves by the Liouville equation

$$i \frac{\partial \rho}{\partial t} = H \rho - \rho H$$

- We can write an "action" for this,

$$S = \int dt \, \mathrm{Tr} \left[\rho_0 \left(U^\dagger \, i \frac{\partial U}{\partial t} - U^\dagger \, H \, U \right) \right]$$

  where U's are to be varied, and $\rho = U \rho_0 U^\dagger$.

- Our first step is to construct a similar action for thermofield dynamics.
- For this we start by using coherent states $\phi_n(z)$, $\chi_n(w)$ such that

$$\int_{\mathcal{M}} d\mu(\bar{z}, z) \;\; \phi_n^* \phi_m = \delta_{nm}, \qquad \int_{\mathcal{M}} d\mu(\bar{w}, w) \;\; \chi_n^* \chi_m = \delta_{nm}$$

  There are many choices for the space of z, $\bar{z}$ (and w, $\bar{w}$); the simplest is to use $\mathbb{CP}^{N-1}$.

- The states can be taken for this case as

$$\langle N|z\rangle = \frac{1}{\sqrt{1+\bar{z}\cdot z}}, \qquad \langle i|z\rangle = \frac{z_i}{\sqrt{1+\bar{z}\cdot z}}, \quad i = 1, 2, \cdots, (N-1)$$

- These can be made orthonormal with the integration measure corresponding to the standard Fubini-Study metric,

$$d\mu = \frac{(N-1)!}{\pi^{N-1}} \frac{\prod_i dz_i \, d\bar{z}_i}{(1+z\cdot\bar{z})^N}$$

- Then the thermofield state $|\Omega\rangle$ can then be represented as

$$|\Omega\rangle = \chi^*_n \, (\sqrt{\rho})_{nm} \, \phi_m, \qquad \mathcal{O}\,|\Omega\rangle = \chi^\dagger \, \sqrt{\rho} \; \mathcal{O}\phi$$

- We get, as expected,

$$\langle\Omega|\,\mathcal{O}\,|\Omega\rangle = \int \phi^*_a (\sqrt{\rho})_{ab} \, \chi_b \, \chi^*_c \, (\sqrt{\rho})_{cd} (\mathcal{O}\phi)_d = \mathrm{Tr}(\rho\,\mathcal{O})$$

- Introduce a slight change of notation,

$$\Omega(\bar{z},\bar{u}) = \sum_{n\,m} \psi_n(\bar{u})\,(\sqrt{\rho})_{nm}\,\phi_m(\bar{z}), \qquad \chi(w) \to \psi(\bar{u})$$

- Time evolution is given by a path integral

$$\phi_n(\bar{z},t) = \int [\mathcal{D}z]\, e^{i\,S(z,\bar{z},t\,|z',\bar{z}')}\,\phi_n(\bar{z}',0)$$

$$\Omega(\bar{z},\bar{u},t) = \int [\mathcal{D}z\,\mathcal{D}u]\, e^{i\,S(z,\bar{z},t\,|z',\bar{z}')}\, e^{i\,\tilde{S}(u,\bar{u},t\,|u',\bar{u}')}\,\Omega(\bar{z}',\bar{u}',0)$$

- The vacuum-to-vacuum amplitude is given by

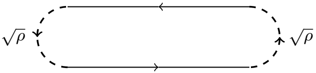
$$\begin{aligned} F &= \int [\mathcal{D}z\,\mathcal{D}u]\ \Omega^*(z,u)\ e^{i\,S(z,\bar{z},t\,|z',\bar{z}')}\ e^{i\,\tilde{S}(u,\bar{u},t\,|u',\bar{u}')}\ \Omega(\bar{z}',\bar{u}') \\ &= \sum (\sqrt{\rho})^*_{kl}\ \langle k|\,e^{-iH_z t}\,|a\rangle\ \langle l|\,e^{-iH_u t}\,|b\rangle\ (\sqrt{\rho})_{ab} \end{aligned}$$

$$e^{i\,S(z,\bar{z},t\,|z',\bar{z}')} = \langle z|\,e^{-iH_z t}\,|z'\rangle$$

- We choose $H_u = -H^T$, to be consistent with the algebra, so that

$$F = \mathrm{Tr}\left(\sqrt{\rho}^{\dagger}\ e^{-iHt}\ \sqrt{\rho}\ e^{\,iHt}\right)$$

- This may be viewed as a contour integral as



- Correlation functions (which are the observables of interest) are of the form

$$\langle A(t_1)\, B(t_2)\rangle = \mathrm{Tr}\left(\sqrt{\rho}\; U(t,t_1)\, A\, U(t_1,t_2)\, B\, U(t_2,0)\, \sqrt{\rho}\; U^\dagger(t,0)\right)$$

- This is not the Schwinger-Keldysh type contour-ordered correlator. If we define

$$\Omega_K = \sum_{n\,m} \psi_n(\bar{u})\, K_{nm}\, \phi_m(\bar{z})$$

we have

$$F = \int [\mathcal{D}z\,\mathcal{D}u]\; \Omega^*_{\sqrt{\rho}}(z,u)\; e^{i\,S(z,\bar{z},t\,|z',\bar{z}')}\; e^{i\,\tilde{S}(u,\bar{u},t\,|u',\bar{u}')}\; \Omega_{\sqrt{\rho}}(\bar{z}',\bar{u}')$$

- The Schwinger-Keldysh type contour-ordered correlator is

$$F_{1\,\rho} = \int [\mathcal{D}z\,\mathcal{D}u]\; \Omega^*_{\mathbf{1}}(z,u)\; e^{i\,S(z,\bar{z},t\,|z',\bar{z}')}\; e^{i\,\tilde{S}(u,\bar{u},t\,|u',\bar{u}')}\; \Omega_{\rho}(\bar{z}',\bar{u}')$$

- Turning to the action for the coherent states

$$S = \int dt \left[(i \bar{z}_k \dot{z}_k - \bar{z}_k H_{kl} z_l) + \Big(i \bar{u}_k \dot{u}_k + \bar{u}_k H^T_{kl} u_l \Big) \right]$$

  with the constraints

$$\bar{z}_k z_k = 1, \qquad \bar{u}_k u_k = 1$$

- The symplectic form (for $z, \bar{z}$) is $\omega = i\, d\bar{z}_k \wedge dz_k$ and lead to wave functions of the form

$$\Psi = e^{-z_k \bar{z}_k / 2} f(\bar{z})$$

  with z_k acting as $\partial / \partial \bar{z}_k$ on the the f's.

- The constraint shows that the f can have one power of $\bar{z}$, which implies that $f(\bar{z}) \sim \bar{z}_k$.
- There are exactly N states, giving the rank 1 representation of $U(N)$.

- The Hamiltonian operator is

$$H = \bar{z}_k \, H_{kl} \frac{\partial}{\partial \bar{z}_l}$$

Matrix elements of this Hamiltonian $\Longrightarrow H_{kl}$.

- Story for $u, \; \bar{u}$ is similar,

$$\Psi = \exp(-u \cdot \bar{u}/2) \, f(\bar{u}), \qquad H = -\bar{u}_k \, H^T_{kl} \frac{\partial}{\partial \bar{u}_l}, \qquad \langle k | \, H \, | l \rangle = -H^T_{kl}$$

The operation $H \rightarrow -H^T$ represents conjugation in the Lie algebra of $U(N)$.

- It is useful to define

$$z_k = \xi_{k1}, \qquad \bar{u}_k = w_k = \xi_{k2}, \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which gives the action

$$S = \int dt \sum_{\alpha,\beta=1,2} P_{\alpha\beta} \left(i \bar{\xi}_{k\beta} \dot{\xi}_{k\alpha} - \bar{\xi}_{k\beta} \, H_{kl} \, \xi_{l\alpha} \right) = \int dt \, \mathrm{Tr} \left[P \, \left(i \, \xi^\dagger \, \dot{\xi} - \xi^\dagger \, H \, \xi \right) \right]$$

- The variables $z_k,\ u_k$ with the constraint define (two copies of) $\mathbb{CP}^{N-1}$. Define $\xi_{k\alpha} = U^{(\alpha)}_{k0} = \langle k|U^{(\alpha)}|0\rangle$, for two unitary matrices $U^{(\alpha)}$
- The action now takes the form

$$S = \int dt\ \Big[\Big(i\, U^{(1)\dagger}\, \dot{U}^{(1)} - U^{(1)\dagger} H\, U^{(1)} \Big)_{00} - \Big(i\, U^{(2)\dagger}\, \dot{U}^{(2)} - U^{(2)\dagger} H\, U^{(2)} \Big)_{00} \Big]$$

- The state Ω is

$$\Omega = \bar{z}_k \sqrt{\rho}_{kl}\, w_l = \bar{\xi}_{k1} \sqrt{\rho}_{kl}\ \xi_{l2} = \langle 0|U^{(1)\dagger} \sqrt{\rho}\ U^{(2)}|0\rangle$$

- We can include the factors of $\sqrt{\rho}$ as well by defining

$$\mathcal{A} = -i \left[H\, dt + \frac{i}{2\pi} \log \rho\, d\theta \right]$$

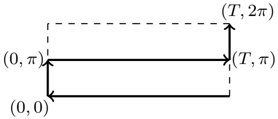
- The amplitude of interest is then

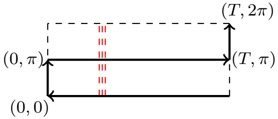
$$F_J = \int [\mathcal{D}U] \, \exp\left[\oint_C \left(-U^\dagger \, \dot{U} + U^\dagger \, \mathcal{A} \, U \right)_{00} + \oint AJ + BJ' \right]$$

- With $J = J' = 0$,

$$F_{J=0}(C) = \mathrm{Tr} \, \mathcal{P} e^{\oint_C \mathcal{A}}$$

- The contour is on $\mathbb{R} \times S^1$ of the form





- The Renyi entropy can related to multiple holonomy around the S^1 direction

$$S_R(t) = -\left(\frac{W(C, n, t) - 1}{n - 1} \right)$$

- Now we rewrite this as a field theory functional integral.

$$\begin{aligned} \langle k | e^{-iHt} | l \rangle &= \langle 0 | a_k \, e^{-iHt} \, a_l^\dagger | 0 \rangle = \langle 0 | T \, a_k(t) \, a_l^\dagger(0) | 0 \rangle \\ &= \mathcal{N} \int [da \, da^*] \, e^{iS} \, a_k(t) \, a_l^\dagger(0) \\ S &= \int dt \, \left[a_k^* (i\partial_0) a_k - a_k^* \, H_{kl} \, a_l \right], \qquad \mathcal{N}^{-1} = \int [da \, da^*] \, e^{iS} \end{aligned}$$

- Introduce a $(z, \bar{z})$-dependent field (on $\mathcal{M}$)

$$\psi(z, \bar{z}, t) = \sum_k a_k \, z_k, \qquad \psi^\dagger(z, \bar{z}, t) = \sum_k a_k^\dagger \, \bar{z}_k$$

- The diagonal coherent state representation of operators also allows us to introduce $A_0(z, \bar{z}) = H(z, \bar{z})$ such that

$$H_{kl} = \int_{\mathcal{M}} d\mu(z, \bar{z}) \, \bar{z}_k \, H(z, \bar{z}) \, z_l = \int_{\mathcal{M}} d\mu(z, \bar{z}) \, \bar{z}_k \, A_0(z, \bar{z}) \, z_l$$

- The action now becomes

$$S = \int dt \, d\mu(z, \bar{z}) \, \left[\psi^* (i\partial_0) \psi - \psi^\dagger \, A_0(z, \bar{z}) \, \psi \right]$$

- We can go beyond fields which are "holomorphic" to general ones by considering the holomorphic ones as the lowest Landau level of a mock quantum Hall system. Use the action

$$S = \int dt\, d\mu(z,\bar{z}) \left[\psi^* \left(i\,\partial_0 - A_0(z,\bar{z}) + \frac{D^2 + E_0}{2m} \right) \psi \right]$$

- Collecting results,

$$\begin{aligned}
F &= \mathcal{N} \int [d\psi d\psi^*\, d\phi d\phi^*]\, e^{iS}\, \Omega^*(t)\, \Omega(0) \\
\Omega(\psi^*, \phi^*) &= \int_{\mathcal{M}} d\mu(z,\bar{z}) d\mu(w,\bar{w})\, \psi^*(z)\, \phi^*(w)\, (z_k \sqrt{\rho}_{kl}\, w_l) \\
S &= \int dt\, d\mu(z,\bar{z}) \left[\psi^* \left(i\,\partial_0 - A_0(z,\bar{z}) + \frac{D^2 + E_0}{2m} \right) \psi \,-\, \psi \to \phi \right] \\
&= \int dt \int_{\mathcal{M}} d\mu(z,\bar{z})\, \psi^* \left(i\,\partial_0 - A_0(z,\bar{z}) + \frac{D^2 + E_0}{2m} \right) \psi \\
&\quad + \int dt \int_{\tilde{\mathcal{M}}} d\mu(z,\bar{z})\, \phi^* \left(i\,\partial_0 - A_0(z,\bar{z}) + \frac{D^2 + E_0}{2m} \right) \phi
\end{aligned}$$

- Take the states to be of the form $|k\rangle = |\alpha\, I\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and define a set of fermion fields $\psi_I = \sum_\alpha a_{\alpha I}\, z_\alpha$. $\Longrightarrow$

$$\begin{aligned} S \;\; = & \;\; \int dt \int_{\mathcal{M}} d\mu(z,\bar{z})\, \psi_I^* \left(i\,\partial_0 \delta_{IJ} - \, (A_0(z,\bar{z})_{IJ} + \frac{D^2 + E_0}{2m}\delta_{IJ} \right) \psi_J \\ & + \int dt \int_{\tilde{\mathcal{M}}} d\mu(z,\bar{z})\, \phi_I^* \left(i\,\partial_0 \delta_{IJ} - \, (A_0(z,\bar{z}))_{IJ} + \frac{D^2 + E_0}{2m}\delta_{IJ} \right) \phi_J \end{aligned}$$

  Labels $I, J \;\sim$ some internal symmetry or degrees of freedom.

- If $\mathcal{M} \times \mathbb{R}$ admits spinors, we can replace the action by the Dirac type action

$$S = \int dt \int_{\mathcal{M}} d\mu(z,\bar{z})\, \bar{\Psi}_I (i\gamma^\mu D_\mu)_{IJ} \Psi_J + \int dt \int_{\tilde{\mathcal{M}}} d\mu(z,\bar{z})\, \bar{\Phi}_I (i\gamma^\mu D_\mu)_{IJ} \Phi_J$$

  Ψ and Φ are spinors, $\gamma^\mu =$ the standard Dirac matrices and $\bar{\Psi} = \Psi^\dagger \gamma^0$, $\bar{\Phi} = \Phi^\dagger \gamma^0$. The Hamiltonian for Ψ/Φ now has the form $H' + A_0$ with $H' = -i\gamma^0\gamma^i D_i$.

- Fuzzy spaces can be defined by the triple $(\mathcal{H}_N, Mat_N, \Delta_N)$
  - $\mathcal{H}_N = N$-dimensional Hilbert space
  - $Mat_N =$ matrix algebra of $N \times N$ matrices which act as linear transformations on $\mathcal{H}_N$
  - $\Delta_N =$ matrix analog of the Laplacian.
- In the large N approximation
  - $\mathcal{H}_N \longrightarrow$ Phase space $\mathcal{M}$
  - $Mat_N \longrightarrow$ Algebra of functions on $\mathcal{M}$
  - $\Delta_N \longrightarrow$ needed to define metrical and geometrical properties.
- $\mathcal{M}_F \equiv (\mathcal{H}_N, Mat_N, \Delta_N)$ defines a noncommutative and finite mode approximation to $\mathcal{M}$.
- Quantum Hall Effect on a compact space $\mathcal{M}$, lowest Landau level $\sim \mathcal{H}_N$
- Observables restricted to the lowest Landau level $\in Mat_N$
- Thermofield dynamics as a field theory functional integral is a realization of this

- Consider the $(n+1) \times (n+1)$ angular momentum matrices J^a, $n = 2j$
- Define

$$X^a = \frac{J^a}{\sqrt{j(j+1)}}$$

- These obey

$$X^a X^a = 1$$

- Functions of these matrices are functions of $\mathbf{1}, \; X^a, \; X^{(a}X^{b)} - \frac{1}{3}\delta^{ab}, \cdots$; there are $(n+1)^2$ independent functions for a basis.
- This agrees with

$$f(S^2) = \sum_0^n f_{lm} Y^l_m(\theta, \varphi), \qquad \sum_0^n (2l+1) = (n+1)^2$$

- Further, when $n \to \infty$,

$$[X^a, X^b] = i\,\epsilon^{abc} \frac{X^c}{\sqrt{j(j+1)}} \implies 0$$

- We can generalize to fuzzy versions of $\mathbb{CP}^k$, for arbitrary k. by considering QHE on $\mathbb{CP}^k$ ($U(1)$ and $SU(k)$ background fields)
- $\mathbb{CP}^k$ is given as

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)} \sim \frac{SU(k+1)}{U(1) \times SU(k)}$$

- This allows the introduction of constant background fields which are valued in $\underline{U(k)} \sim \underline{U(1)} \oplus \underline{SU(k)}$
- Useful comparison:

$$\text{Minkowski} = \text{Poincaré}/\text{Lorentz}$$

- Changing the gauge fields of $U(1) \times SU(k)$ (and more generally $SU(k+1)$) is the same gauging the isometry group. $\Longrightarrow$ suggest interpreting as gravity

- The Hilbert space $\mathcal{H} \sim \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ with states of the form $|\alpha, a, I\rangle$, where $\mathcal{H}_3$ refers to matter system of interest.
- For gravity, as a first approximation, we will not need to consider excitations of the matter system, which means that we can restrict the matter fields to the ground state. In this case, the
- States $\sim |\alpha, a, 0\rangle$ corresponding to a representation $R_1 \otimes R_2$ of G with the transformation

$$|\alpha, a, 0\rangle' = g^{(1)}_{\alpha\beta}\, g^{(2)}_{ab}\, |\beta, b, 0\rangle$$

R_1 defines $\mathcal{H}_1$, we take dim$\mathcal{H}_1 \to \infty$. R_1 is a highest weight representation $\to$can define symbols and $*$-products.

R_2 = Fixed representation, defines $\mathcal{H}_2$

Both are unitary representations

- Since $\mathbb{CP}^1 \sim S^2 = SU(2)/U(1)$, start with choosing $g = \exp(i\sigma \cdot \theta/2) \in SU(2)$ as coordinates for the space (and a gauge direction).
- Wave functions are given by the Wigner $\mathcal{D}$-functions

$$\mathcal{D}^{(j)}_{ms}(g) = \langle j, m| \exp(iJ \cdot \theta)|j, s\rangle$$

  subject to a condition on s.
- Define right translations as $R_a\, g = g\, t_a$.
- The covariant derivatives $D_\pm = iR_\pm/r$. Since

$$[R_+, R_-] = 2R_3 \qquad \Longrightarrow \qquad [D_+, D_-] = -\frac{2R_3}{r^2}$$

  we must choose R_3 to be $-n$ for the Landau problem.
- This corresponds to a field $a = in\, \mathrm{Tr}(t_3 g^{-1} dg)$.

- The wave functions are thus

$$\Psi_m(g) \sim \mathcal{D}^{(j)}_{m,-n}(g)$$

- Choose the Hamiltonian as

$$\mathcal{H} = \frac{1}{4mr^2}\left[R_+R_- + R_-R_+\right]$$

- The left action

$$L_a\, g = t_a\, g$$

commutes with $\mathcal{H}$ and corresponds to "magnetic translations".

- The lowest Landau level (LLL) has the further condition (holomorphicity condition)

$$R_-\Psi_m(g) = 0$$

- LLL states also correspond to co-adjoint orbit quantization of $a = in\,\mathrm{Tr}(t_3 g^{-1} dg)$.

- Start with the action

$$S = \int dt \, \mathrm{Tr} \left[i\rho_0 U^\dagger \partial_t U \; - \; \rho_0 U^\dagger \mathcal{A}_0 \, U \right]$$

- The LLL has N available states, K occupied by fermions, $\quad 1 \ll K \ll N$
- Form a QH droplet, specified by the density matrix: $\rho_0 = \sum_{i=1}^{K} |i\rangle\langle i|$,

$$\rho_0 = \left[\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & 0 & \\ & & & & 0 \end{array} \right] \begin{array}{l} \updownarrow K \\ \updownarrow N-K \end{array}$$

- We will take the fully occupied case, $K = N$. ($K < N$ can be analyzed, leads to boundary terms (which are WZW actions).)

- The symbol is defined by

$$(\hat{A})_{ik} = A_{ik} = \langle -s, i|\, h^{(s)\dagger}\, \hat{A}\, h^{(s)}\, | -s, k\rangle$$

$| -s\rangle$ is the highest weight state of the spin-s representation. As a 2×2 matrix,

$$h = \frac{1}{\sqrt{1+\bar{z}z}} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

- The symbol for the product of two operators is

$$\begin{aligned}
(\hat{A}\hat{B})_{ik} &= \langle -s, i|h^\dagger\, \hat{A}\hat{B}\, h\, | -s, k\rangle \\
&= \sum_{a,j} \langle -s, i|h^\dagger\, \hat{A}\, h\, |a, j\rangle\langle a, j|\, h^\dagger\, \hat{B}\, h\, | -s, k\rangle \\
&= A_{ij}B_{jk} + \sum_{r=1}^{N-1} \langle -s, i|h^\dagger\, \hat{A}\, h\, | -s+r, j\rangle\langle -s+r, j|\, h^\dagger\, \hat{B}\, h\, | -s, k\rangle \\
&= A_{ij}B_{ik} + \sum_{r=1}^{N-1} \left[\frac{(N-1-r)!}{r!(N-1)!}\right] (R_+^r\, A)_{ij}(R_-^r\, B)_{jk} \equiv (A * B)_{ik}
\end{aligned}$$

- The action has the gauge invariance

$$U \to g\,U, \qquad \mathcal{A}_0 \to g\,\mathcal{A}_0\,g^{-1} + \dot{g}\,g^{-1}$$

For transformations g close to the identity, $g \approx 1 + \hat{\Phi}$ and

$$\hat{\mathcal{A}}_0 \to \hat{\mathcal{A}}_0 - \partial_0 \hat{\Phi} - \hat{\mathcal{A}}_0\,\hat{\Phi} + \hat{\Phi}\,\hat{\mathcal{A}}_0$$

- In terms of symbols

$$\mathcal{A}_0 \to \mathcal{A}_0 - \partial_0 \Phi - \mathcal{A}_0 * \Phi + \Phi * \mathcal{A}_0$$

This has the full content of the operator transformation. $\mathcal{A}$ and Φ are functions on $\mathbb{CP}^1 \times \mathbb{R}$ and are also 2×2 matrices.

- It is convenient to introduce $A_\mu dx^\mu$ and a function Λ such that

$$\left.\begin{aligned} A_0 &\to A_0 + \partial_0 \Lambda + [A_0, \Lambda] \\ A_i &\to A_i + \partial_i \Lambda + [A_i, \Lambda] \end{aligned}\right\} \Longrightarrow \mathcal{A}_0 + \partial_0 \Phi + \mathcal{A}_0 * \Phi - \Phi * \mathcal{A}_0$$

where $\mathcal{A}_0$ and Φ are functions of A_μ and Λ.

- The solution is given by (KARABALI; VPN)

$$
\begin{aligned}
\mathcal{A}_0 &= A_0 + \frac{P^{ab}}{2n}\left[\partial_a A_0\, A_b - A_a \partial_b A_0 + F_{a0} A_b - A_a F_{b0}\right] + \cdots \\
\Phi &= \Lambda + \frac{P^{ab}}{2n}(\partial_a \Lambda\, A_b - A_a \partial_b \Lambda) + \cdots \\
P^{ab} &\equiv \frac{1}{2}\left[\frac{g^{ab}}{2\pi} + i\,(\omega_K^{-1})^{ab}\right]
\end{aligned}
$$

- We get

$$
\begin{aligned}
\int dt\, \mathrm{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \hat{\mathcal{A}}_0 &= \int dt \int_{\mathcal{M}} \mathrm{Tr}_{\mathcal{H}_2} \mathcal{A}_0 = -\frac{1}{4\pi} \int \mathrm{Tr}_{\mathcal{H}_2}\left[(a+A)\, d(a+A) + \frac{2}{3}(a+A)^3\right] \\
&= -\frac{1}{4\pi} \int \mathrm{Tr}\left(A\, dA + \frac{2}{3} A^3\right), \qquad a + A \to A
\end{aligned}
$$

- Including the tilde sector

$$
S = -\frac{1}{4\pi} \int \left[\mathrm{Tr}\left(A\, dA + \frac{2}{3} A^3\right)_L - \mathrm{Tr}\left(A\, dA + \frac{2}{3} A^3\right)_R\right]
$$

- The basic proposal is that , for the gravitational part of $\mathcal{H} \otimes \tilde{\mathcal{H}}$, $SO(3)_L$ fields couple to $\mathcal{H}$ while $SO(3)_R$ fields couple to $\tilde{\mathcal{H}}_R$. i.e., $A_L \sim SO(3)_L$, $A_R \sim SO(3)_R$.
- We identify

$$A = -i\, P_a\, e^a - \frac{i}{2}\, S_{ab}\, \omega^{ab}$$

$$P_a = \frac{\gamma_3\, \gamma_a}{2il}, \qquad S_{ab} = \frac{1}{4i}\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right), \qquad a, b = 0, 1, 2.$$

$$\begin{aligned} S &= -\frac{l}{32\pi G}\int \mathrm{Tr}\left[\gamma_5\left(A\, dA + \frac{2}{3}A^3\right)\right], \qquad l/8G \to 1 \\ &= \frac{1}{16\pi G}\int d^3x\, \det e\left(R - \frac{3}{2l^2}\right) \end{aligned}$$

- A_i are auxiliary fields introduced for simplicity of representing the transformation. So it must be eliminated.
- It is also not clear what A_0 should be for gravity. Eliminating both A_0 and A_i via the equations of motion gives the gravitational field equations.

- We obtain dynamical gravity as a large N effect.
- The level number is 1 so far, we need multiplicity ($l/8G$) for a large level number.
- One can continue to Minkowski space using the field theory representation for the thermofield path integral.
- One can use the $SL(2,\mathbb{R})$ orbits of the Virasoro group to carry out a similar construction. One has to use large-central-charge limit to simplify the action.

Thank you