Propagation of entanglement and Causality

Hong Liu

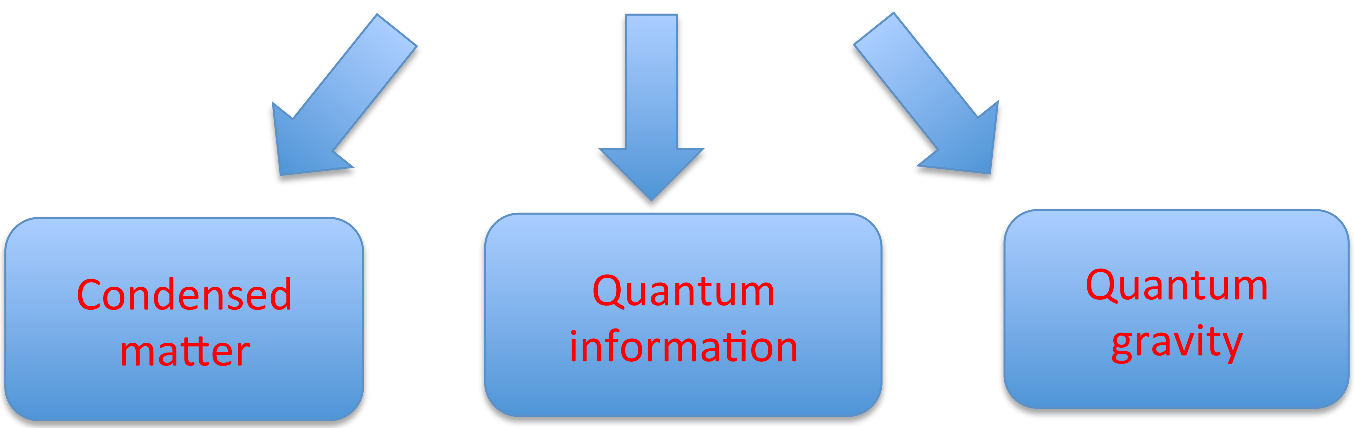
Quantum entanglement

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

"Spooky non-locality"

EPR paradox, Bell's inequality, ...

Quantum entanglement encodes subtle quantum correlations among d.o.f. of a system which often cannot be captured by more traditional observables.



Entanglement entropy

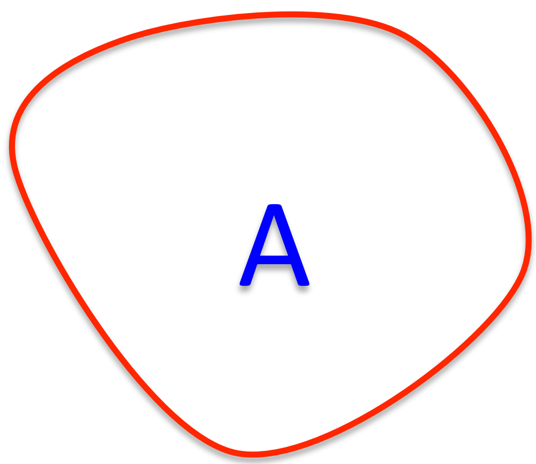
A quantum system: $A + B$ $\qquad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Wave function: $\Psi = \sum_n \psi_n(A) \otimes \chi_n(B)$

$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| \qquad S_A = -\mathrm{Tr}\rho_A \log \rho_A$

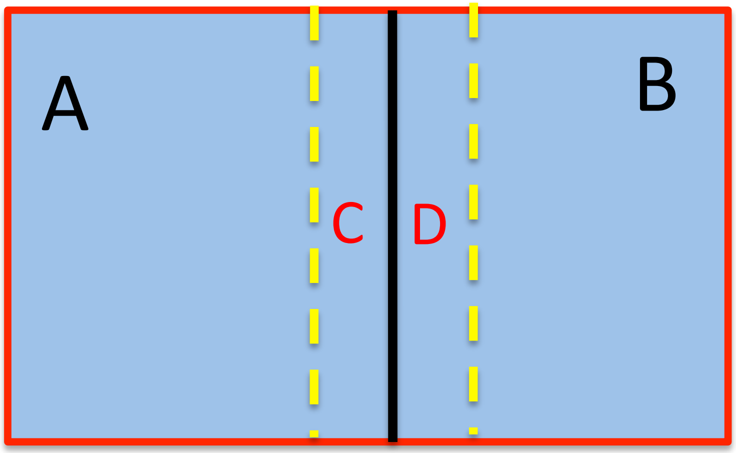
For systems with local degrees of freedom, like QFT or lattice systems,

a spatial region A



S_A

Entanglement generation



$$\psi(t=0) = \psi_A \otimes \psi_B$$

$$\psi(t) = e^{-iHt}\psi(0)$$

$$H = H_A + H_B + H_{AB}$$

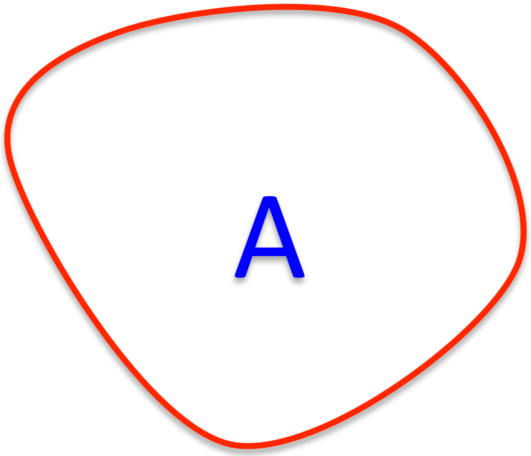
How fast can entanglement be generated?

physical systems: Local Hamiltonian $\quad H_{AB} = H_{CD}$

Effect of locality ?

Relativistic systems: constrained by causality, how?

Measure of entanglement generation



A measure: $$\frac{dS_A}{dt}$$

depends on size of A, number of d.o.f., ……

Not meaningful to compare it across different systems

Ideal to have an "intensive" quantity which can be compared among different systems and different regions.

Part I

Entanglement tsunami and hints of a measure

HL and J. Suh, Phys. Rev. Lett. 112, 011601 (2014)

HL and J. Suh, Phys. Rev. D 89, 066012 (2014)

Related work:

Hubeny, Rangamani, Takayanagi: arXiv:0705.0016

Abajo-Arrastia, Aparicio and Lopez, arXiv:1006.4090

Albash and Johnson, arXiv:1008.3027

Balasubramanian, Bernamonti, de Boer, Copland, Craps, Keski-Vakkuri, Muller, Schafer, Shigemori, Staessens arXiv:1012.4753, arXiv:1103.2683

Hartman and Maldacena arXiv:1303.1080

Kim and Huse arXiv:1306.4306

Hubeny and Maxfield arXiv: 1312.6887

Global quenches

1. Start with a QFT in the ground state.

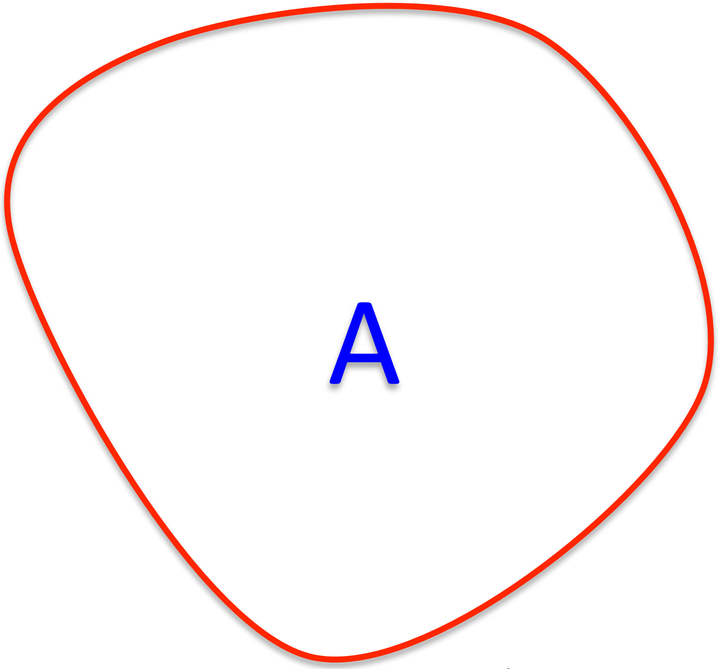
2. At t=0 in a very short time inject a uniform energy density

   - initial state homogeneous, isotropic, entanglement properties as vacuum

3. The system evolves to (thermal) equilibrium

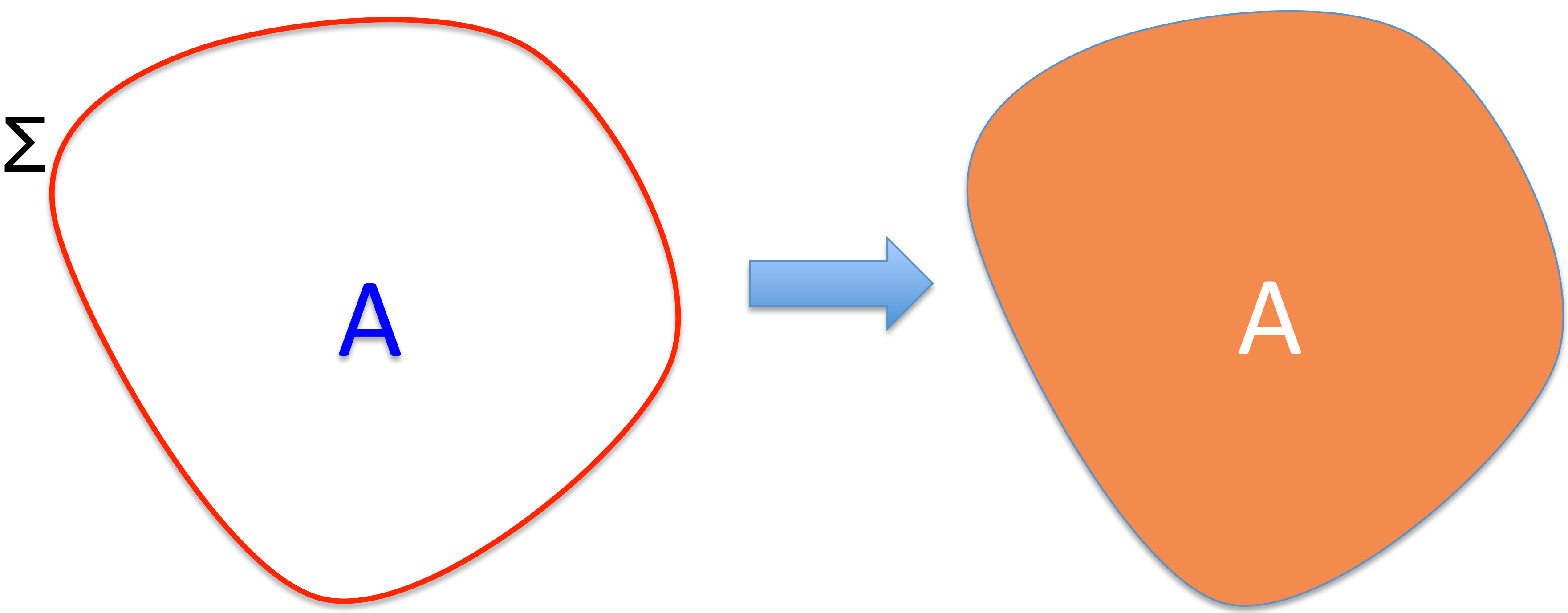
The system is in a pure state throughout.

$$S_A(t)?$$



$$R \gg \frac{1}{T}$$

t=0 | Equilibrium



$$S_0 = \frac{A_\Sigma}{\delta^{d-1}} + \cdots$$

Typical point of A essentially un-entangled with outside

$$S_{\rm eq} = s_{\rm eq} V_A + \cdots$$

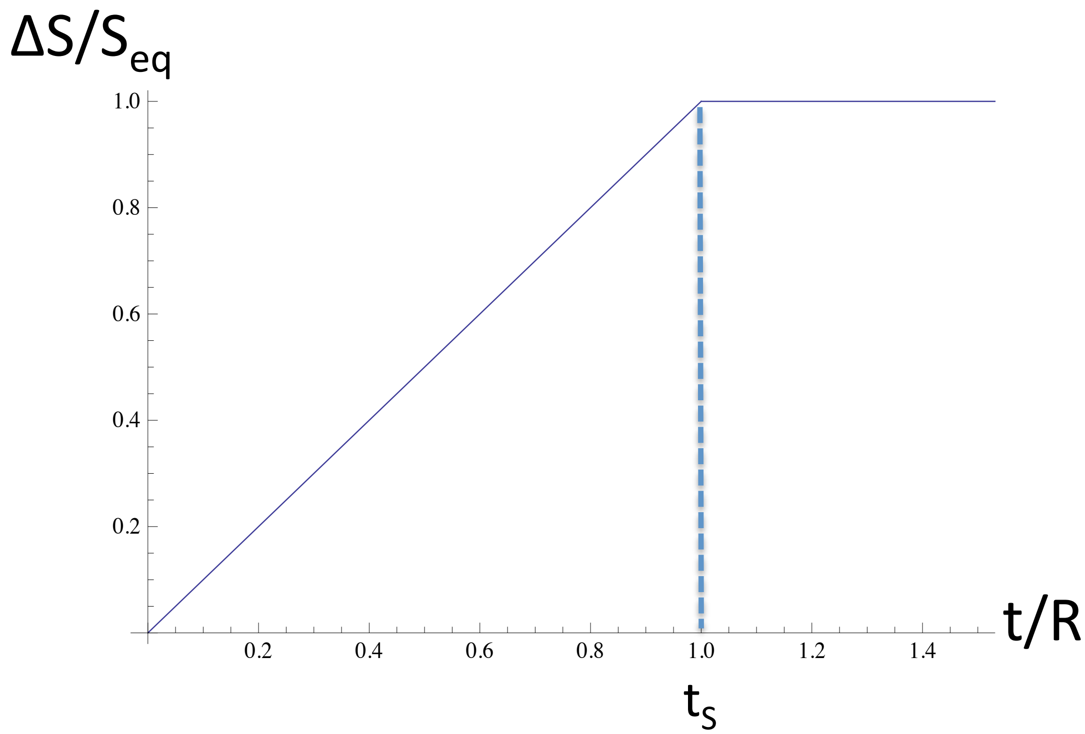
s_{eq}: equilibrium entropy density

Essentially every point of A is entangled with outside

Full time evolution: very difficult question

d=2
CFTs:

Calabrese and Cardy (2006)



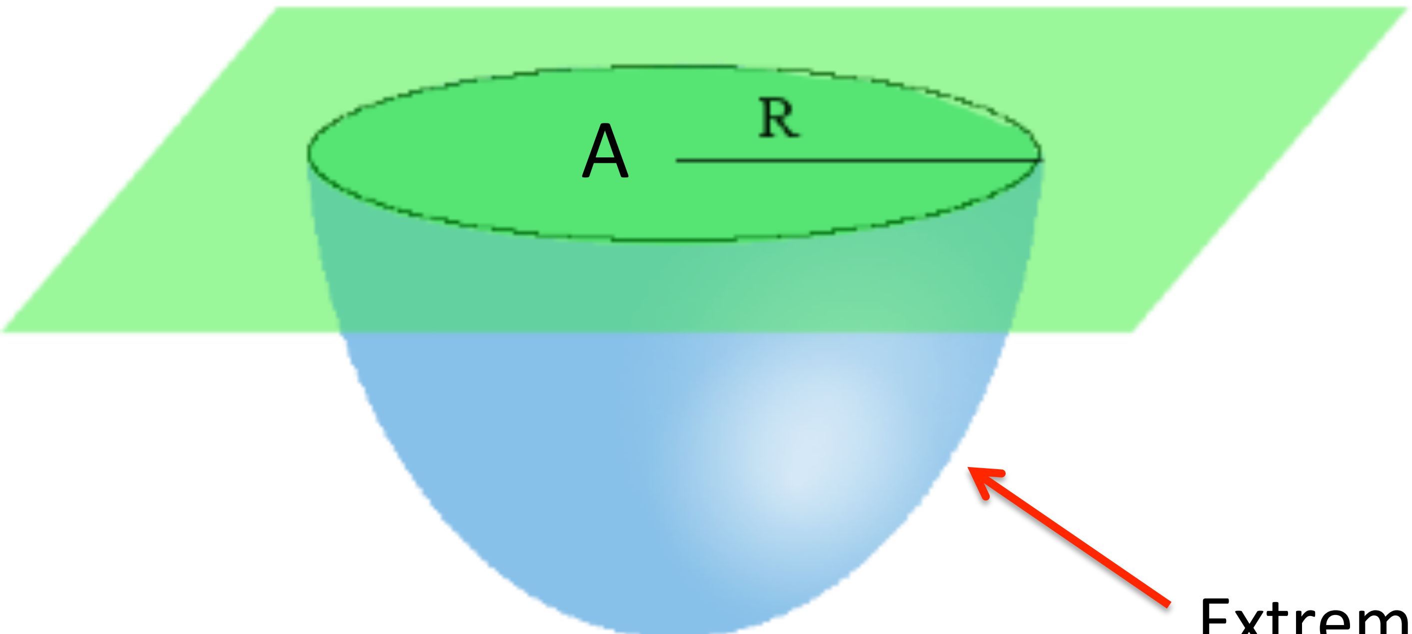
Linear growth

Slope = 1

Multiple intervals, $d > 2$: holography

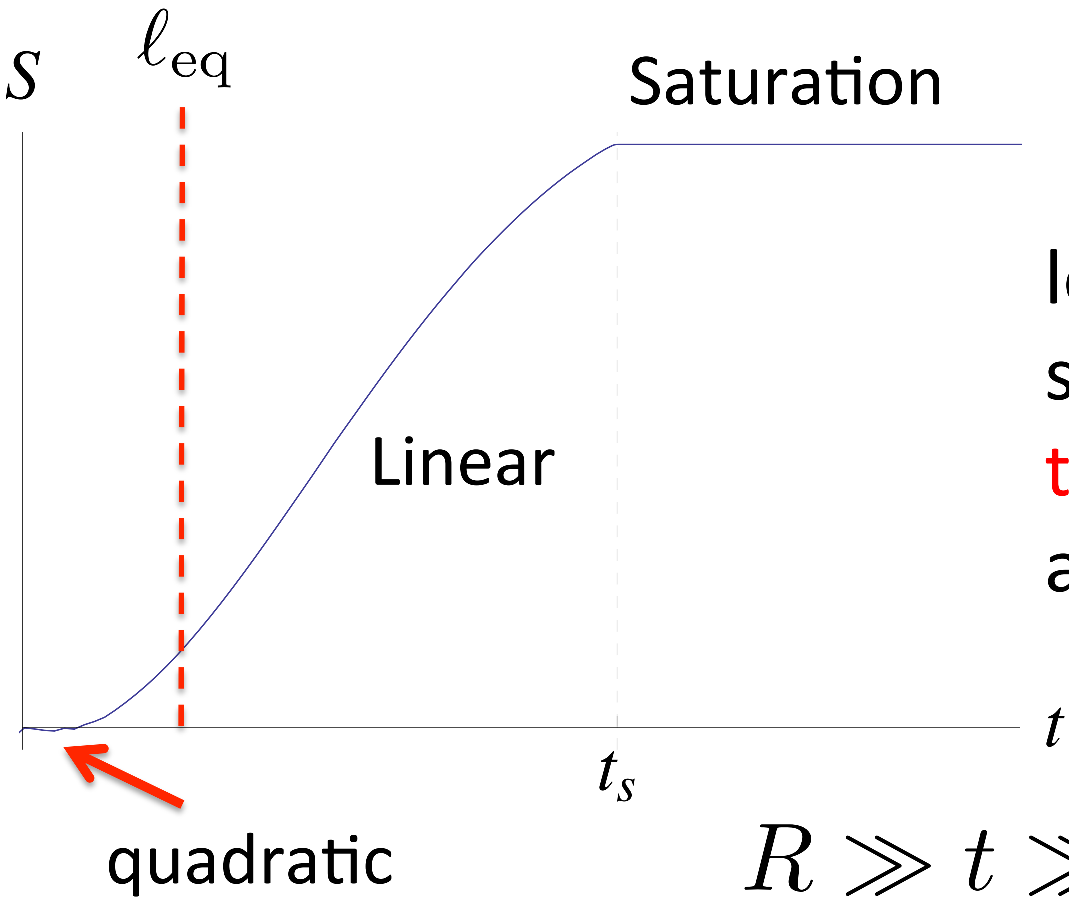
Holographic Entanglement entropy

Ryu, Takayanagi
Hubeny, Rangamani, Takayanagi



Extremal surface: M

$$S_A = \frac{\text{Area of M}}{4\mathrm{G_N}}$$



$$\ell_{\rm eq} \sim \frac{1}{T}$$

local equilibration time scale after which thermodynamics applies locally.

$$R \gg t \gg \ell_{\rm eq}$$

$$\Delta S_A(t) = v_E\, s_{\rm eq}\, A_\Sigma t + \cdots$$

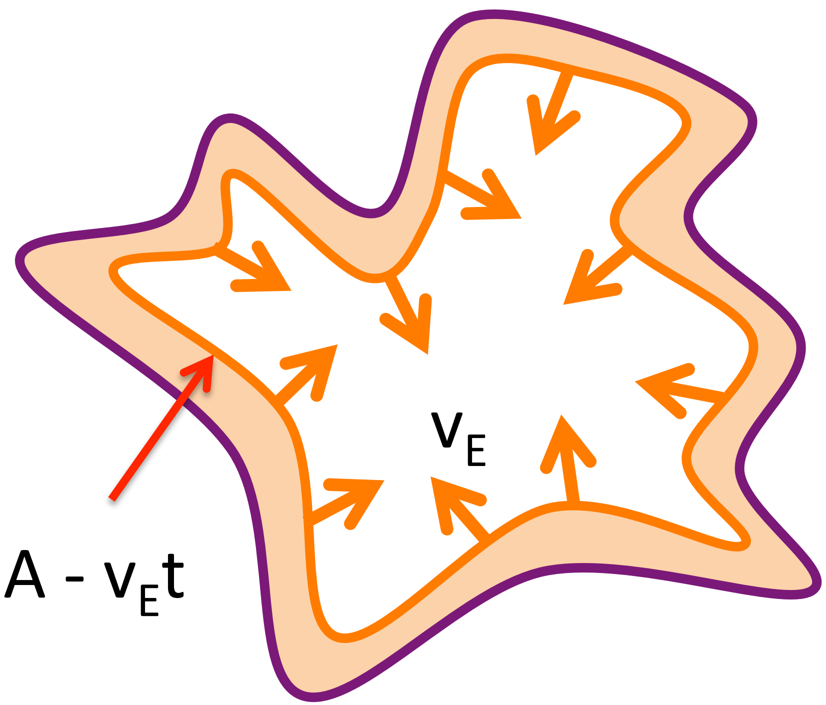
See also
Hartman, Maldacena

"universal"

v_E: dimension of velocity, characterized by final eq state.

Entanglement Tsunami

$$\Delta S_A(t) = v_E\, s_{\rm eq}\, A_\Sigma\, t = s_{\rm eq}\, (V_A - V_{A-v_E t})$$



suggests a picture of "tsunami" wave of entanglement, moving inward from boundary

d.o.f. in the region covered by the wave is now entangled with those outside A

Propose: consequence of a local Hamiltonian

Tsunami velocity

$$\Delta S_A(t) = v_E s_{\rm eq} A_\Sigma t + \cdots$$

From gravity:

$$v_E \leq v_E^{(\rm S)} = \frac{(\eta-1)^{\frac{1}{2}(\eta-1)}}{\eta^{\frac{1}{2}\eta}} = \begin{cases} 1 & d=2 \\ \frac{\sqrt{3}}{2^{\frac{4}{3}}} = 0.687 & d=3 \\ \frac{\sqrt{2}}{3^{\frac{3}{4}}} = 0.620 & d=4 \\ \frac{1}{2} & d=\infty \end{cases}$$

$$\eta \equiv \frac{2(d-1)}{d}$$

d=2: agree with previous Calabrese-Cardy's result

A measure of entanglement growth

$$\mathfrak{R}_A(t) \equiv \frac{1}{s_{\rm eq} A_\Sigma} \frac{dS_A}{dt}$$ (dimension: velocity)

can be compared among regions of different sizes, and systems of different number of d.o.f. ....

From gravity: after local equilibration

$$\mathfrak{R}_A \leq v_E^{(S)}$$

Questions

1. Generality of linear growth and tsunami picture?

2. How to relate $\mathfrak{R}_A, v_E$ directly to speed of light?

One can prove $v_E \leq 1$ H. Casini, HL, M. Mezei, to appear

Hartman, unpublished

3. Significance of $$v_E^{(S)} = \begin{cases} 1 & d=2 \\ \frac{\sqrt{3}}{2^{\frac{4}{3}}} = 0.687 & d=3 \\ \frac{\sqrt{2}}{3^{\frac{3}{4}}} = 0.620 & d=4 \\ \frac{1}{2} & d=\infty \end{cases}$$?

Free theory? Not available

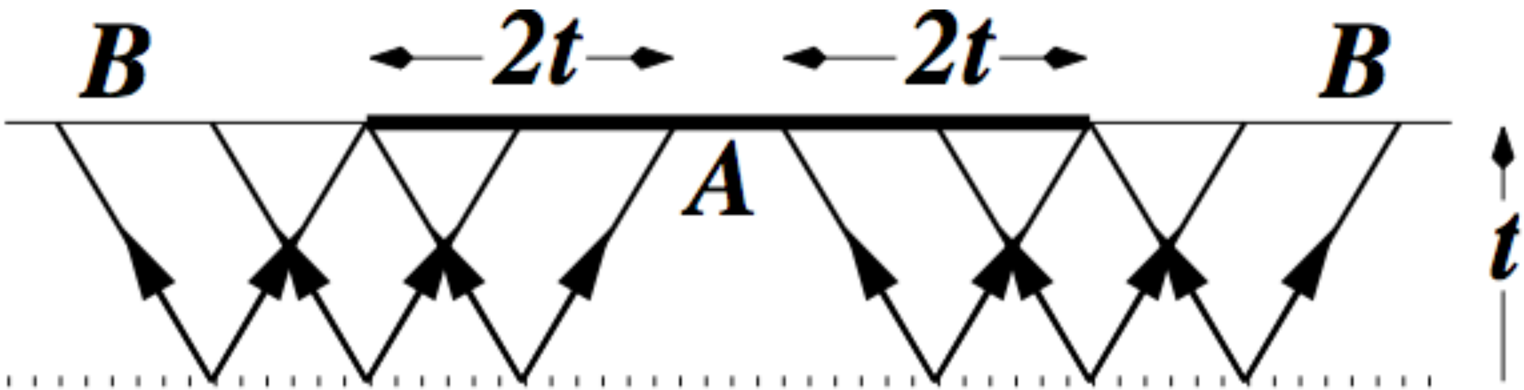
Part II

An upper bound for free propagation of entanglement

H. Casini, HL,
M. Mezei, to appear

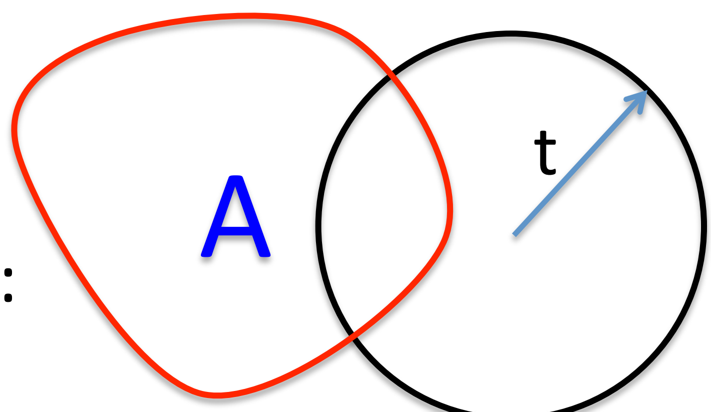
Calabrese-Cardy model

Energy injection from quench creates a finite density of EPR pairs, subsequently travel freely at the speed of light isotropically.



d=2: leads to linear growth with $v_E = 1$

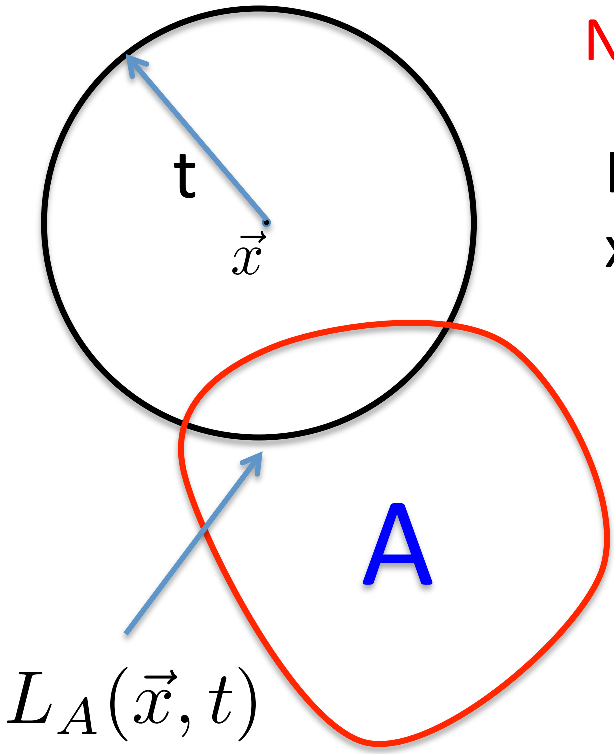
Higher Dimensions:



Entanglement spread will now depend on entanglement pattern on the light cone.

Setup

Each point is an independent source of local entanglement which subsequently spread at speed of light.



(intersection of lightcone from x with A at time t)

No interaction/interference among lightcones

For a region B on the lightcone from a point x, associate an entanglement measure $\mu[B]$:

entanglement entropy for B in the Hilbert space of the Light cone from x

Contribution from x: $\mu[L_A(\vec{x}, t)]$

$$S_A(t) = \int d^{d-1}x \, \mu[L_A(\vec{x}, t)]$$

Properties

$\mu[B]$ should have all the properties of entanglement entropy:

$\mu[B] = \mu[\bar{B}]$,   Strong subadditivity condition, etc.

It does not change with time for B with fixed angular extension.

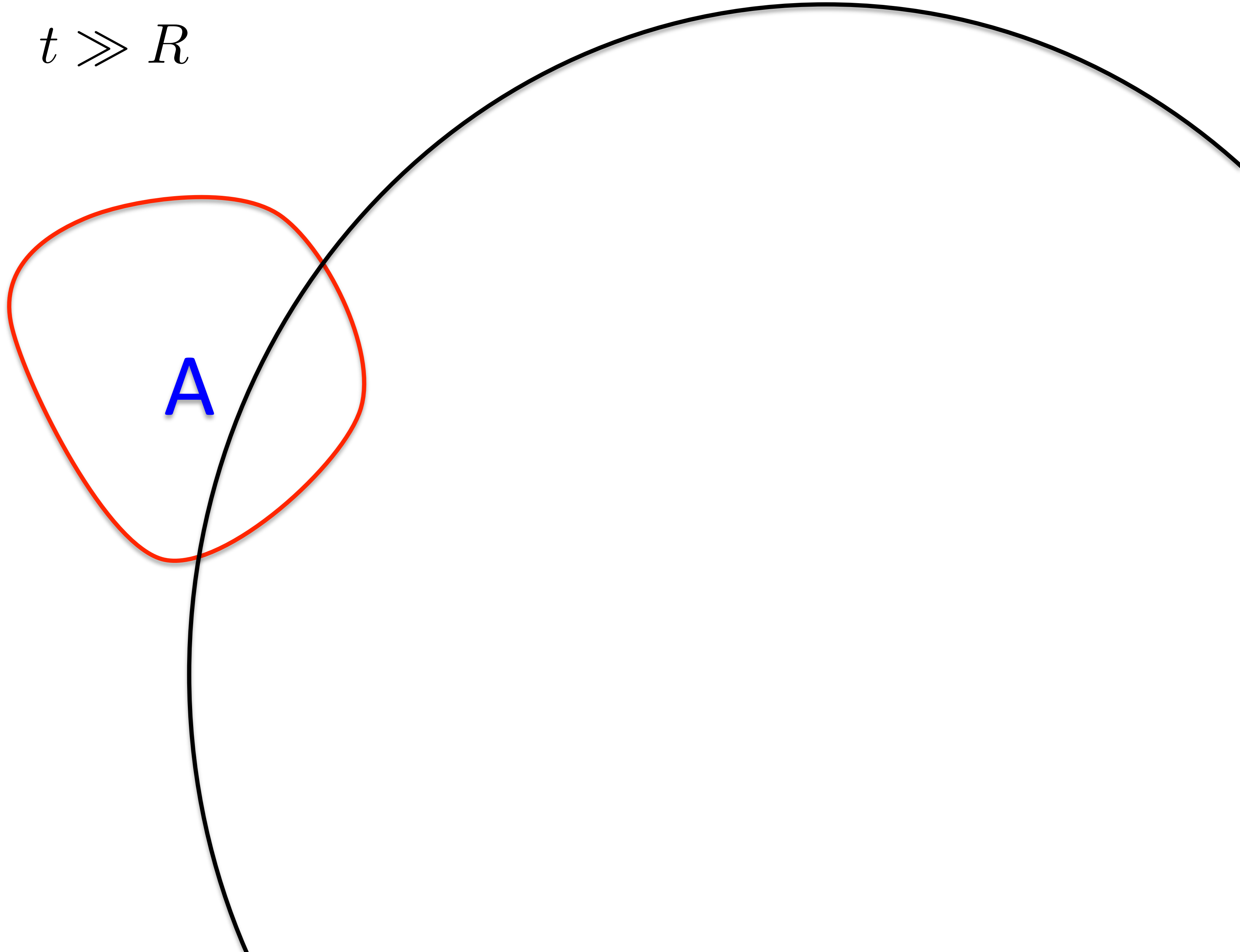
$$\lim_{B \to 0} \mu[B] = s\xi_B$$

ξ_B : normalized volume for B   e.g. Page (1992)

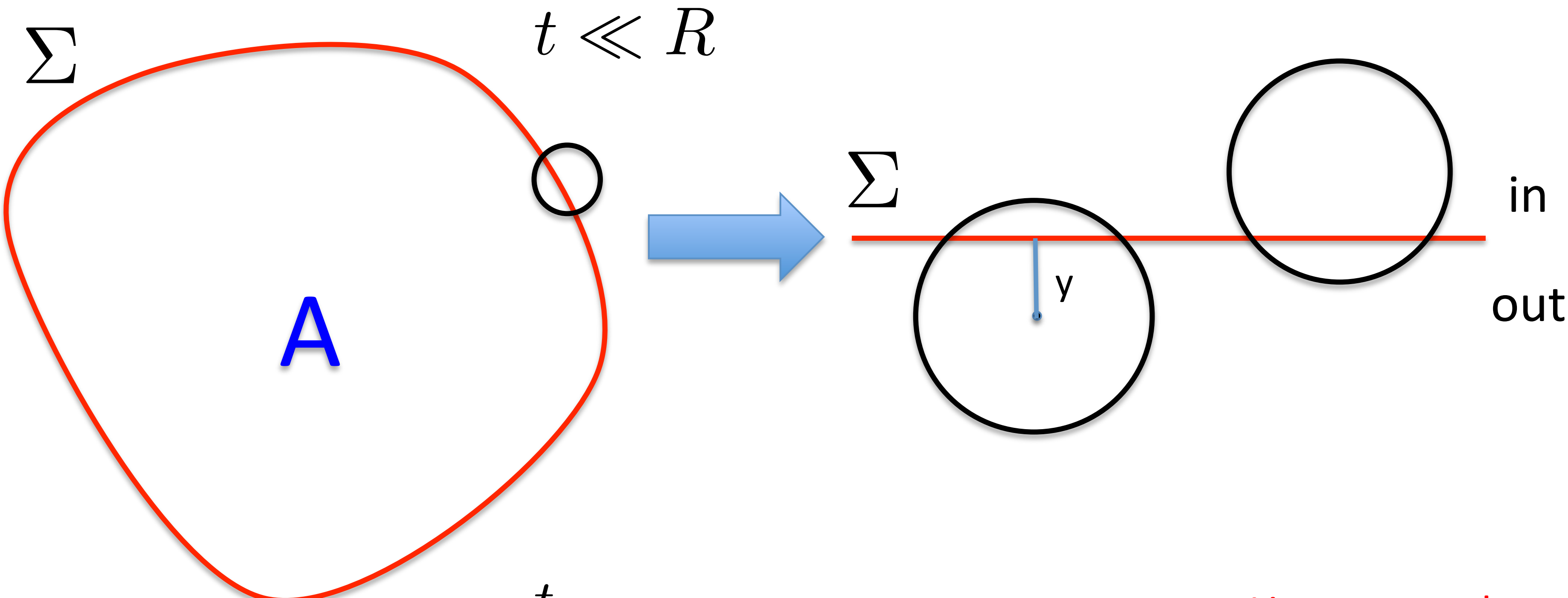
Equilibrium value:

$$S_A(t = \infty) = sV_A$$

$$\Rightarrow \quad s_{\rm eq} = s$$


$t \gg R$
A

Linear growth



$$S_A(t) = 2A_\Sigma \int_0^t dy\; \mu[\mathrm{cap}(y/t)] \propto t$$

$$v_E = \frac{2}{s}\int_0^1 dx\; \mu[\mathrm{cap}(x)]$$

Linear growth due to time independence of μ

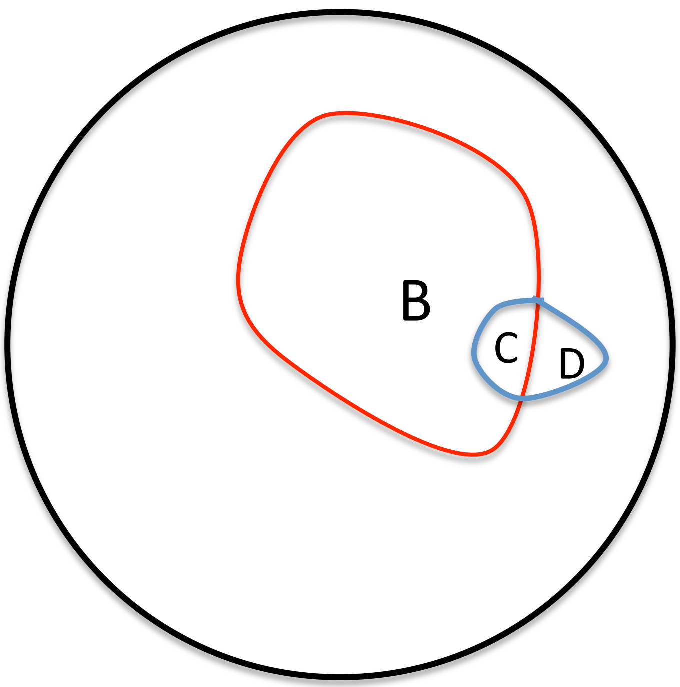
Upper bound on entanglement propagation

Random pure state measure: $\mu_R[B] \equiv s \min(\xi_B, \xi_{\bar{B}})$

Strong sub-additivity condition: $\mu[B] \leq \mu_R[B]$

$$\mu[B] + \mu[C \cup D] \geq \mu[C] + \mu[B \cup D]$$

(C,D infinitesimal)



$$\mu[B \cup D] - \mu[B] \leq s\,\xi_D$$

$$v_E \leq v_E^{\text{free}} \equiv 2\int_0^1 dx\,\xi_{\text{cap}}(x)$$

$$\mathfrak{R}_A(t) \leq v_E^{\text{free}}$$

Free propagation

$$\frac{dS_A}{dt} \leq v_E^{\text{free}} s_{\text{eq}} A_\Sigma$$

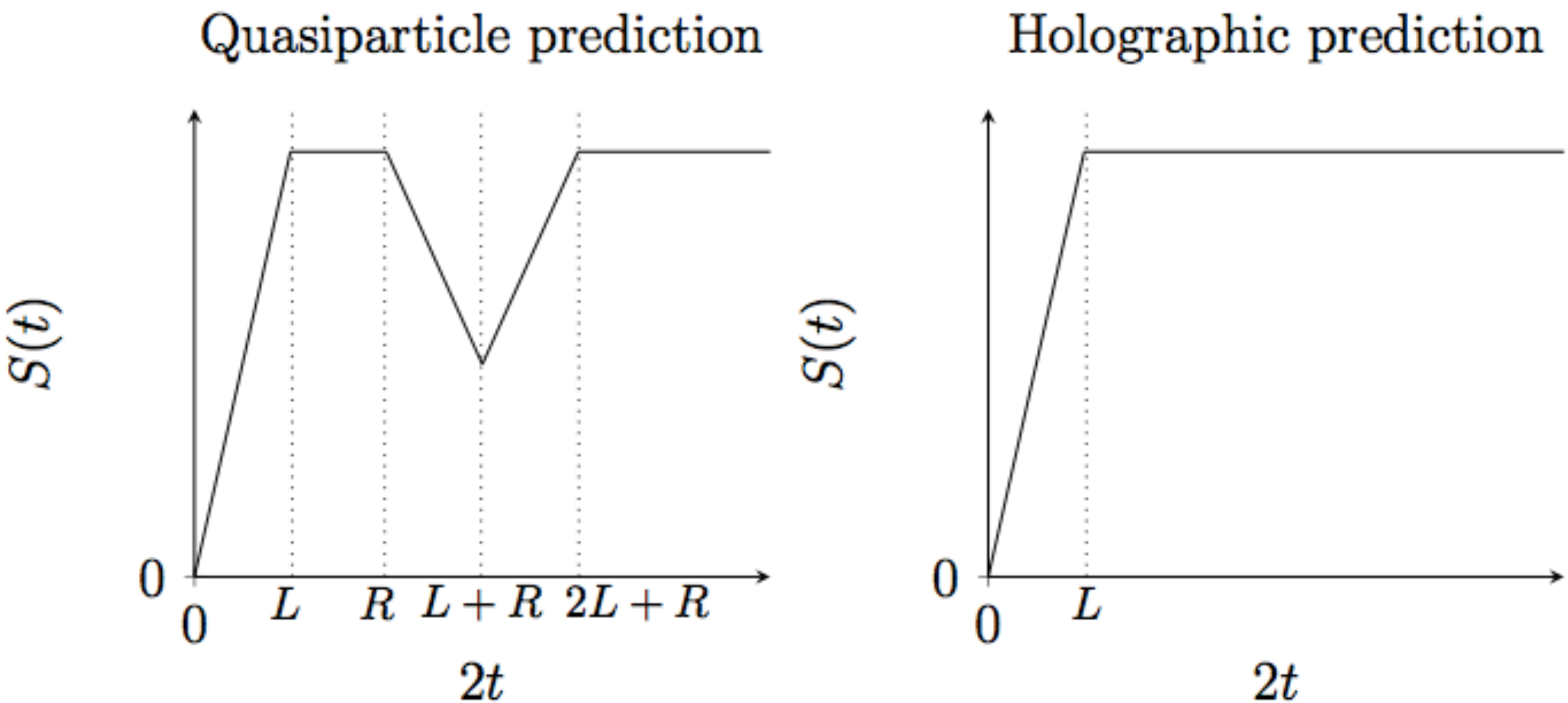
$$v_E^{\text{free}} = \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})} = \begin{cases} 1 & d=2 \\ \frac{2}{\pi} = 0.637 & d=3 \\ \frac{1}{2} & d=4 \\ \sqrt{\frac{2}{\pi d}} & d=\infty \end{cases}$$

$$v_E^{(S)} = \begin{cases} 1 & d=2 \\ \frac{\sqrt{3}}{2^{\frac{4}{3}}} = 0.687 & d=3 \\ \frac{\sqrt{2}}{3^{\frac{3}{4}}} = 0.620 & d=4 \\ \frac{1}{2} & d=\infty \end{cases}$$

In strongly coupled systems, entanglement propagates faster than that from free particles at speed of light !

Success of Cardy-Calabrese model in d=2 for a single interval is likely an accident.

Multiple intervals in d=2: fail significantly at qualitative level



Two intervals of length L separated by R > L

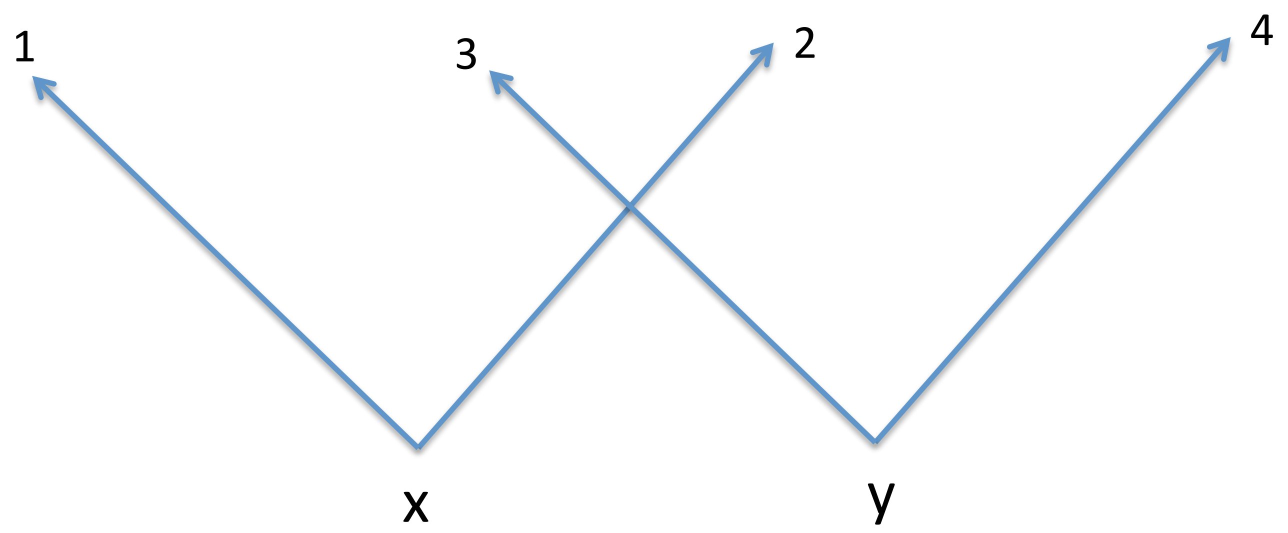
from Leichenauer and Moosa arXiv:1505.04225

Part III

An interacting model

H. Casini, HL,
M. Mezei, to appear

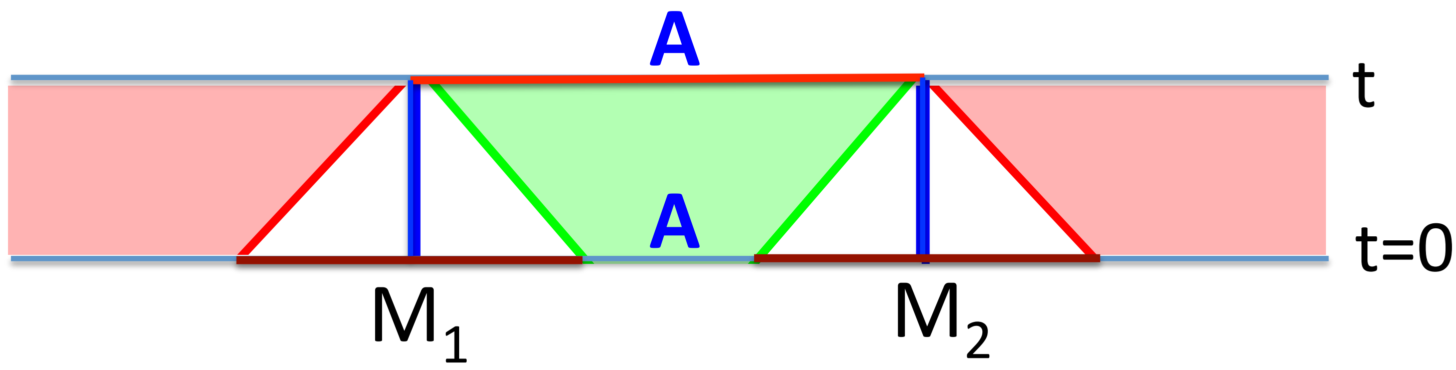
Scattering



Quantum state of the system can no longer be described as a direct product of those resulting from each point at t=0.

We then face the standard difficulties of how to characterize the quantum state of an interacting many-body system.

Domain of dependence



Red-shaded region: $\mathcal{D}_-(\bar{A})$ (past domain dependence of $\bar{A}$)

Scatterings in this region amounts to unitary transformations in $\mathcal{H}_{\bar{A}}$

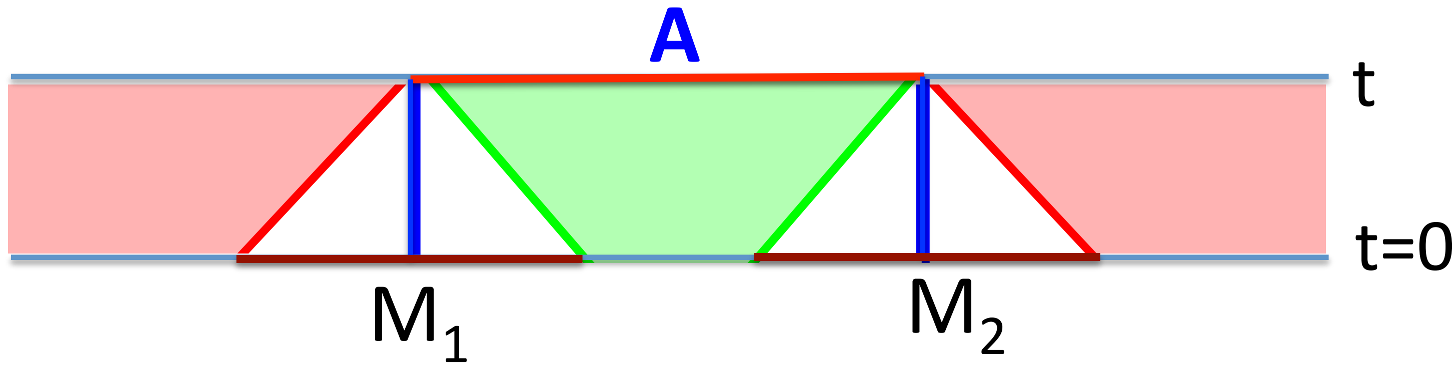
Will not affect S_A (t)

Green-shaded region: $\mathcal{D}_-(A)$

Scatterings in this region amounts to unitary transformations in $\mathcal{H}_A$

Will not affect S_A (t) either

Only particles from M_1, M_2 and scatterings in white regions relevant!



Only particles from M_1, M_2 and scatterings in white regions relevant!

A particle from M_1 and a particle from M_2 do not have effective scatterings. So M_1 and M_2 can be treated independently.

In a strongly coupled theory particles within M_1 scatter with one another many times before reaching A.

Appears natural to apply random pure state measure to the full Hilbert space of all particles in M_1 (similarly with M_2):

$$S_A = s \min(N_A(t), N_{\bar{A}}(t))$$

N_A: number of particles from M_1 falling in A

General formulation

$$M(t) \equiv \mathcal{M} - (\mathcal{D}_-(A) \cap \mathcal{M}) - (\mathcal{D}_-(\bar{A}) \cap \mathcal{M})$$

$\mathcal{M}$: spatial manifold at t=0

$$M(t) = \sum_i M_i$$

$$S_A(t) = \nu_{\mathrm{eq}} \sum_i \min\left(\int_{M_i(t)} n_A(x,t), \int_{M_i(t)} n_{\bar{A}}(x,t) \right)$$

Always larger than free propagation results derived earlier.
Likely an upper limit for interacting theories.

Results

d=2:

One interval: $v_E = 1$

Two intervals: precisely recover holographic results

Free propagation: not

Asplund and Bernamonti
Leichenauer and Moosa

Three intervals: generally same, but can be larger than holographic results for certain time intervals

Appear to be the same as a recent proposal of Leichenauer and Moosa in 1505.04225.

d >2: $v_E = 1$

However, this might be an unachievable upper bound.

Part IV

Constraining tsunami velocity by speed of light

A Proof that: $v_E \leq 1$

H. Casini, HL, M. Mezei, to appear

Hartman has also found a different proof (private communication)

Thank You