Protoplanetary Disks and Planet Formation Summer School
Problem Set for R.P. Nelson Lectures

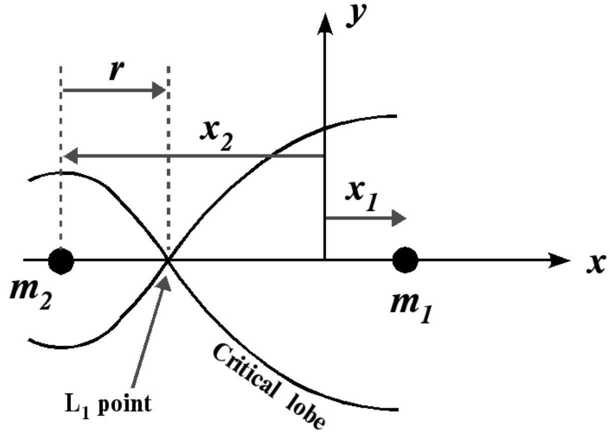
August 2015

Question 1

A binary system contains two gravitating masses m_1 and m_2, where $m_2 \ll m_1$. They rotate with angular velocity Ω in circular orbit about the centre of mass. In a frame corotating with the orbit, the line of centres lies on the x–axis and the origin is at the centre of mass. Thus

$$D \;=\; x_1 - x_2, \quad x_1 \;=\; \frac{m_2 D}{m_1 + m_2}, \quad x_2 \;=\; -\,\frac{m_1 D}{m_1 + m_2}\;,$$

where D is the distance between the stars, and x_1 and x_2 are their positions on the x–axis.



On the line of centres the gravitational and centrifugal potential is

$$\Phi \;=\; -\,\frac{Gm_1}{|x - x_1|} \;-\; \frac{Gm_2}{|x - x_2|} \;-\; \frac{1}{2}\Omega^2 x^2 \qquad \text{with} \qquad \Omega^2 \;=\; \frac{G(m_1 + m_2)}{D^3}\;.$$

At the L_1 Lagrangian point, $\dfrac{\partial \Phi}{\partial x} = 0$. Show that there (for $x > x_2$, $x < x_1$)

$$-\,\frac{Gm_1}{\left(x - \frac{m_2 D}{m_1+m_2}\right)^2} \;+\; \frac{Gm_2}{\left(x + \frac{m_1 D}{m_1+m_2}\right)^2} \;-\; \frac{G(m_1 + m_2)x}{D^3}) \;=\; 0\;.$$

Set $x = r + x_2 = r - \dfrac{m_1 D}{m_1 + m_2}$. Then show that

$$-\,\frac{Gm_1}{(r - D)^2} + \frac{Gm_2}{r^2} \;-\; \frac{G(m_1 + m_2)r}{D^3} \;+\; \frac{Gm_1}{D^2} \;=\; 0\;,$$

and hence that, for small r and m_2, approximately

$$\frac{Gm_2}{r^2} \;=\; \frac{3Gm_1 r}{D^3}$$

and thus

$$r \;=\; D\left(\frac{m_2}{3m_1}\right)^{1/3}\;.$$

Question 2

Consider a scenario in which the core of a giant planet forms *via* giant impacts between ~ 10 protoplanets, each being of one Earth mass, before gas accretion onto the giant planet core can ensue. Estimate the time scale on which the core forms by estimating the collision time scale, $\tau_{\rm coll}$, at a distance of 5 AU from a solar-type star.

The volume swept out in time t by a protoplanet of cross-sectional area σ is $\sigma v t$, where v is the velocity dispersion. The protoplanets can be considered to occupy a narrow annulus centred at 5 AU and width ~ 1 AU, where the height of the annulus, $H_{\rm s}$, is related to the velocity dispersion and the orbital angular velocity, Ω, through the expression $v = H_{\rm s}\Omega$ (these numbers give a reasonable surface density of solid material and approximation to the feeding zone width of a 10 $\rm M_\oplus$ core). Assume that v is set by mutual scattering among the protoplanets so that it approximately equals the escape velocity of the colliding bodies, hence justifying the neglect of gravitational focussing. You should assume that the radii of the protoplanets are equal to one Earth radius ($\rm R_\oplus = 6300$ km).

Given that the life times of gaseous protoplanetary discs around T Tauri stars are estimated to be $< 10^7$ yr, comment on the significance of your answer.

Question 3

A planet exerts a gravitational torque on a protoplanetary disc through the launching of spiral density waves at Lindblad resonances. In linear perturbation theory it is customary to expand the gravitational potential of a planet on a circular orbit into a Fourier series:

$$\Phi(r, \theta, t) = \sum_{m=0}^{\infty} \phi_m(r) \exp\left[im(\theta - \Omega_{\rm p} t)\right],$$

where $\Omega_{\rm p}$ is the orbital angular velocity of the planet and m is the azimuthal mode number. Lindblad resonances occur at disc locations where the angular frequency of the potential component experienced by local fluid elements equals the epicyclic frequency. Obtain an expression that relates the orbital angular velocity of material at a Lindblad resonance, Ω, to the orbital angular frequency of the planet and the azimuthal mode number. Considering the $m = 1$, 2, 3 and 4 components of the potential, make a sketch showing the positions of the inner and outer Lindblad resonances and the orbital period commensurabilities associated with them. Write down an expression that relates the orbital radius of the Lindblad resonances, r_L, and the orbital radius of the planet, $r_{\rm p}$.

The gravitational potential of a planet on an eccentric orbit may be written as a double Fourier series:

$$\Phi(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{m,k}(r) \exp\left\{i[m(\theta - \Omega_{\rm p} t) - k\kappa t]\right\}.$$

Obtain an expression for the angular velocity at Lindblad resonances in terms of the angular velocity of the planet and the integers m and k. By considering the $m = 2$ and $k = 1$ mode, draw a sketch showing the locations of the Lindblad resonances relative the planet's radial location. A corotation resonance also exists at the location where the local angular velocity equals the pattern speed of the potential component. Show the location of the corotation resonance in your diagram. Do the same for the $m = 2$ and $k = -1$ mode.

Question 4
The torque exerted by a planet on the outer parts of the protoplanetary disc in which it is embedded can be written

$$\Gamma_{\mathrm{p}} = \frac{8}{27}\left(\frac{m_{\mathrm{p}}}{M_*}\right)^2 \Sigma_{\mathrm{p}} R_{\mathrm{p}}^4 \Omega_{\mathrm{p}}^2 \left(\frac{R_{\mathrm{p}}}{\Delta}\right)^3,$$

where m_{p} is the mass of the planet, M_* is the mass of the central star, Σ_{p} is the surface density at the planet location, Ω_{p} is the angular velocity at the planet location, and Δ is the distance between the planet and the disc material that is experiencing the torque. The torque exerted by the planet opens a gap in the disc, and this is opposed by the viscous torques in the disc. The viscous force per unit area acting within a ring of gas near the edge of the gap is given by the expression

$$F_{\mathrm{visc}} = \rho \nu R \frac{d\Omega}{dR},$$

where ρ is the volume density of the gas, ν is the kinematic viscosity and Ω is the angular velocity. Show that the following condition must be satisfied in order for the gap to be maintained:

$$\frac{m_{\mathrm{p}}}{M_*} \gtrsim \frac{10\nu}{R^2 \Omega_{\mathrm{p}}}.$$

Note: To obtain this expression, you must first integrate the viscous force per unit area over the disc height and azimuth to obtain the viscous torque for a Keplerian disc: $\Gamma_{\mathrm{visc}} = -3\pi\nu\Sigma R^2 \Omega$. You should also assume that Δ is equal to the Hill radius of the planet.

Assuming disc parameters $H/R \sim 0.04$ and $\alpha \sim 10^{-2}$ estimate the planet mass required to open and maintain a gap, noting that the 'alpha' model for the kinematic viscosity gives $\nu = \alpha H^2 \Omega$.

Question 5
We can estimate the shape of the gap that is formed when viscous torques balance planet torques. In order to do this we need to obtain a differential equation for the surface density Σ that takes account of the torque balance. Considering the viscous torque, $\Gamma_{\mathrm{visc}} = -3\pi\nu\Sigma R^2\Omega$, it is clear that on the local scales of interest Σ is the most rapidly varying quantity. Similarly, when considering the torque due to the planet,

$$\Gamma_{\mathrm{p}} = \frac{8}{27}\left(\frac{m_{\mathrm{p}}}{M_*}\right)^2 \Sigma_{\mathrm{p}} R_{\mathrm{p}}^4 \Omega_{\mathrm{p}}^2 \left(\frac{R_{\mathrm{p}}}{\Delta}\right)^3,$$

we see that the last factor in this expression varies locally with radius (noting that $\Delta = (R - R_{\mathrm{p}})$). By obtaining approximate expressions for the torque per unit radius, show that the surface density profile in the gap can be approximated by $\Sigma(R) = \Sigma_\infty exp(-X)$ where Σ_∞ is the unperturbed surface density at large radius from the planet and

$$X = \frac{8}{27}\frac{q^2 R_{\mathrm{p}}^2 \Omega_{\mathrm{p}}}{3\pi\nu}\left(\frac{R_{\mathrm{p}}}{\Delta}\right)^3.$$

Comment on how the surface density behaves at large and small distances from the planet, drawing a sketch to illustrate the points that you make.