## Question 1

The $L_1$ point lies between the two stars, so $x > x_2$ and $x < x_1$. Can use this to eliminate modulus signs:

$$\Phi = -\frac{GM_1}{x_1 - x} - \frac{GM_2}{x - x_2} - \frac{1}{2}\Omega^2 x^2.$$

$$\Rightarrow \frac{d\Phi}{dx} = -\frac{GM_1}{(x_1-x)^2} + \frac{GM_2}{(x-x_2)^2} - \Omega^2 x$$

At $L_1$ $\frac{d\Phi}{dx} = 0$, and $\Omega^2 = \frac{G(M_1+M_2)}{D^3}$.

$$-\frac{GM_1}{(x_1-x)^2} + \frac{GM_2}{(x-x_2)^2} - \frac{G(M_1+M_2)}{D^3}x = 0$$

$$\frac{-GM_1}{\left(x - \frac{M_2 D}{M_1+M_2}\right)^2} + \frac{GM_2}{\left(x + \frac{M_1}{M_1+M_2}D\right)^2} - \frac{G(M_1+M_2)x}{D^3} = 0.$$

Set $x = r + x_2 = r - \frac{M_1}{M_1+M_2}D$

$$\Rightarrow \frac{-GM_1}{\left(r - \frac{(M_1+M_2)D}{(M_1+M_2)}\right)^2} + \frac{GM_2}{\left(r - \frac{M_1 D}{M_1+M_2} + \frac{M_1}{M_1+M_2}D\right)^2} - \frac{G(M_1+M_2)}{D^3}\left(r - \frac{M_1 D}{M_1+M_2}\right) = 0$$

$$= \frac{-GM_1}{(r-D)^2} + \frac{GM_2}{r^2} - \frac{G(M_1+M_2)r}{D^3} + \frac{GM_1}{D^2} = 0$$

For small $M_2$, $M_2 << M_1 \Rightarrow M_1 + M_2 \simeq M_1$.

For small $r$ we have $r/D << 1$.

$$\therefore \frac{GM_1}{(r-D)^2} = \frac{GM_1}{(D-r)^2} = \frac{GM_1}{D^2(1-r/D)^2} = \frac{GM_1}{D^2}(1-r/D)^{-2}$$

$$\simeq \frac{GM_1}{D^2}\left(1 + \frac{2r}{D} - \frac{3r^2}{D^2} + \dots\right).$$

Substitute:

$$-\frac{GM_1}{D^2} - \frac{2GM_1 r}{D^3} + \frac{3GM_1 r^2}{D^4} + \frac{GM_2}{r^2} - \frac{G\overbrace{(M_1+M_2)}^{= M_1}r}{D^3} + \frac{GM_1}{D^2} = 0$$

$$-\frac{3GM_1 r}{D^3} + \frac{3GM_1 r^2}{D^4} + \frac{GM_2}{r^2} = 0$$

$$\Rightarrow \frac{3GM_1 r}{D^3}\underbrace{\left(1 - \frac{r}{D}\right)}_{\simeq 1} = \frac{GM_2}{r^2}$$

$$\therefore \frac{3GM_1 r}{D^3} = \frac{GM_2}{r^2}$$

$$\therefore \underline{r = D\left(\frac{M_2}{3M_1}\right)^{1/3}}. \quad \text{– The Hill radius of a planet.}$$

Question 2.


Volume swept out in time $t$ $= \sigma v t$.

Collision time $\tau_{coll} \simeq \dfrac{1}{n \sigma v}$

Velocity dispersion : $v = H_s \Omega$

We can write $n \simeq \dfrac{N}{2H_s}$

where $N$ = number per unit surface area.

$$\tau_{coll} \sim \frac{1}{\left(\frac{N}{2H_s}\right) \sigma \, (H_s \Omega)} = \frac{2}{N \sigma \Omega}$$

Cross-section $\sigma = \pi r_p^2$ where $r_p$ = planet radius.

For planets to overlap and collide the eccentricity must be quite large so we ignore gravitational focussing.

We have that $N = \dfrac{n_{planets}}{2\pi R \Delta R}$

where $n_{planets}$ is the total number of protoplanets.

$$\therefore \quad \tau_{coll} \sim \frac{2 R \Delta R}{n_{planets} \pi r_p^2} \left( \frac{2\pi}{\Omega} \right) \sim \frac{2 R \Delta R}{\pi n_{planets} r_p^2} \text{ Orbits.}$$

Working in cgs :

$R = 5 AU \qquad 1 AU = 1.5 \times 10^{13}$ cm.

$\Delta R = 1 AU.$

$r_p = R_\oplus \simeq 6 \times 10^8$ m

$$\therefore \tau_{coll} \sim 1.8 \times 10^8 \text{ Orbits} \sim 10^9 \text{ yrs at } 5AU.$$

Given gas disc lifetimes of $10^7$ yrs this model clearly does not work for the formation of gas giant planet cores.

Question 3

$$\Phi(r,\theta,t) = \sum_{m=0}^{\infty} \phi_m(r) \exp\left[im(\theta - \Omega_p t)\right]$$

Lindblad resonance: occur where the doppler-shifted frequency of a potential component experienced by a fluid element in a disc equals the epicyclic frequency of the fluid element.

We obtain the doppler-shifted frequency by differentiating the argument of the complex exponential with respect to time.

$$m(\dot{\theta} - \Omega_p) = m(\Omega - \Omega_p).$$

Lindblad resonance: $m^2(\Omega - \Omega_p)^2 = \kappa^2$

For Keplerian disc $\kappa^2 = \Omega^2$.

$$\therefore \quad m^2(\Omega - \Omega_p)^2 = \Omega^2.$$

$$\therefore \quad \Omega = \frac{m}{m \pm 1} \Omega_p.$$

$+$ Sign $\rightarrow$ Outer L.R.
$-$ Sign $\rightarrow$ Inner L.R.

Use $\Omega = \sqrt{\frac{GM_*}{r^3}}$ and $\Omega_p = \sqrt{\frac{GM_*}{r_p^3}}$ to get resonance positions.

2:1 3:2 4:3 4:5 3:4 2:3 1:2

Star ($m=1$) | $m=2$ | $m=3$ | $m=4$ | Planet | $m=4$ | $m=3$ | $m=2$ | $m=1$

Note that frequency of the $m=1$ ILR $= \infty$ placing it at the Star's location.

$$\Phi(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{m,k}(r) \exp\left\{ i\left[ m(\theta - \Omega_p t) - k\kappa_p t \right] \right\}$$

Doppler-shifted frequency: $m(\dot{\theta} - \Omega_p) - k\kappa_p$

$$= m(\Omega - \Omega_p) - k\kappa_p$$

Lindblad Resonance: $\left[ m(\Omega - \Omega_p) - k\Omega_p \right]^2 = \underbrace{\kappa^2 = \Omega^2}_{\text{Keplerian disc.}}$

$$\therefore \quad m(\Omega - \Omega_p) - k\Omega_p = \pm \Omega.$$

$$\therefore \quad (m \pm 1)\Omega = (m+k)\Omega_p$$

$$\Omega = \frac{m+k}{m \pm 1}\Omega_p.$$

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$m=2, k=1$ : $\Omega_I = \frac{3}{1}\Omega_p$ ; $\Omega_o = \frac{3}{3}\Omega_p = \Omega_p$.

$$\Omega_{CR} = \Omega_{pattern} = \frac{m\Omega_p + k\Omega_p}{m} = \frac{3}{2}\Omega_p$$

$m=2, k=-1$ : $\Omega_I = \Omega_p$ ; $\Omega_o = \frac{\Omega_p}{3}$ ; $\Omega_{CR} = \frac{\Omega_p}{2}$.

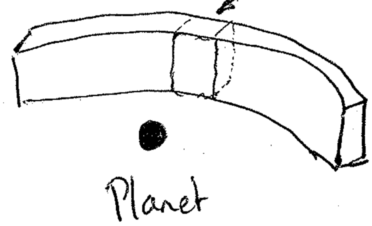
$k=1$ | 3:1 $\Omega_I$ | 3:2 $\Omega_{CR}$ | $\Omega_o / \Omega_I$ Planet | 1:2 $\Omega_{CR}$ | 1:3 $\Omega_o$

## Question 4

$$\Gamma_p = \frac{8}{27}\left(\frac{M_p}{M_*}\right)^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{\Delta}\right)^3$$

We will assume that $\Delta = R_H$ - the planet Hill radius.

We consider the viscous force per unit area exerted on a ring of gas at the gap edge




$$F_{visc} = \rho \nu \underbrace{R \frac{d\Omega}{dR}}_{\text{Keplerian shear.}}$$

We are using 2D flat disc approximation, so we need to integrate the viscous force over height and azimuth to get total force acting on ring.

$$F_{visc} = \int_0^{2\pi}\int_{-\infty}^{\infty} \rho \nu R \frac{d\Omega}{dR} \underbrace{R\,d\theta\,dz}_{\text{element of area.}}$$

Assume $\nu$ and $\Omega$ independent of $z$ and $\theta$:

$$F_{visc} = 2\pi R^2 \nu \Sigma \frac{d\Omega}{dR} \qquad \text{where } \Sigma = \int_{-\infty}^{\infty} \rho\, dz$$

$\therefore$ Viscous torque $\Gamma_{visc} = 2\pi R^3 \nu \Sigma \frac{d\Omega}{dR}$.

For Keplerian disc $\frac{d\Omega}{dR} = -\frac{3}{2}\frac{\Omega}{R}$.

$\therefore \Gamma_{visc} = -3\pi R^2 \nu \Sigma \Omega$.

For gap formation $\Gamma_p + \Gamma_{visc} \gtrsim 0$

$$\therefore \frac{8}{27}\left(\frac{M_p}{M_*}\right)^2 \cancel{\Sigma_p} R_p^4 \Omega_p^2 \left(\frac{R_p}{R_H}\right)^3 \gtrsim 3\pi R_p^2 \nu \cancel{\Sigma_p} \Omega_p$$

$$\frac{8}{27}\left(\frac{M_p}{M_*}\right)^2 R_p^4 \Omega_p^2 \frac{R_p^3}{R_p^3}\left(\frac{M_p}{3M_*}\right)^{-1} \gtrsim 3\pi R_p^2 \nu \Omega_p.$$

$$\therefore \frac{8}{9}\frac{M_p}{M_*} > \frac{3\pi \nu}{R_p^2 \Omega_p}$$

$$\therefore \frac{M_p}{M_*} \gtrsim \frac{10\nu}{R_p^2 \Omega_p} \qquad \nu = \alpha H^2 \Omega$$

$$\therefore \frac{M_p}{M_*} \gtrsim 10 \frac{\alpha H^2 \Omega_p}{R_p^2 \Omega_p} \sim 10\, \alpha \left(\frac{H}{R}\right)^2$$

$$\therefore \frac{M_p}{M_*} \gtrsim 1.6 \times 10^{-4}$$

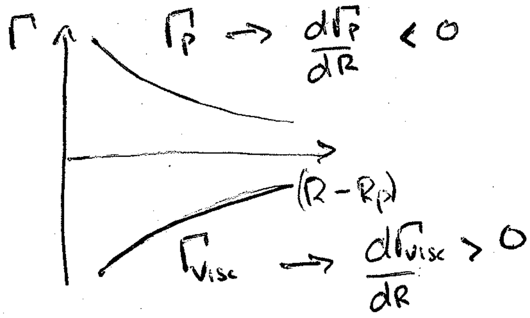
About $\frac{1}{2}$ of Saturn's mass.

Question 5

Torque balance between planet torques and viscous torques at the outer gap edge gives

$$\frac{8}{27} q^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{R-R_p}\right)^3 - 3\pi \nu \Sigma R_p^2 \Omega_p = 0$$

The torques may be sketched as follows




$$\therefore \frac{d\Gamma_{visc}}{dR} + \frac{d\Gamma_p}{dR} = 0$$

Assume ~~that~~ near gap edge we can write

$$\frac{d\Gamma_{visc}}{dR} \sim 3\pi \nu R_p^2 \Omega_p \frac{d\Sigma}{dR}$$

$$\frac{d\Gamma_p}{dR} \sim -\frac{24}{27} q^2 \Sigma_p R_p^4 \Omega_p^2 \frac{R_p^3}{(R-R_p)^4}$$

$$\therefore \frac{1}{\Sigma} \frac{d\Sigma}{dR} \sim +\frac{24}{27} \frac{q^2 R_p^4 \Omega_p^2}{3\pi \nu R_p^2 \Omega_p} \frac{R_p^3}{(R-R_p)^4}.$$

$$\int_{\Sigma(\infty)}^{\Sigma(R)} \frac{d\Sigma}{\Sigma} \sim \frac{24}{27} \frac{q^2 R_p^4 \Omega_p^2}{3\pi \nu R_p^2 \Omega_p} \int_{R=\infty}^{R=R} \frac{R_p^3}{(R-R_p)^4} dR$$

Then

$$\int_{\Sigma'(R=\infty)}^{\Sigma'(R)} \frac{d\Sigma}{\Sigma} = \frac{24}{27} \frac{q^2 R_p^4 \Omega_p^2}{3\pi\nu R_p^2 \Omega_p} \int_{R=\infty}^{R=R} \frac{R_p^3}{(R-R_p)^4} \, dR.$$

$$\ln\left(\frac{\Sigma'(R)}{\Sigma_\infty}\right) = -\frac{8}{27} \frac{q^2 R_p^2 \Omega_p}{3\pi\nu} \frac{R_p^3}{(R-R_p)^3}$$

$$\therefore \quad \Sigma'(R) = \Sigma_\infty \exp\left[-\frac{8}{27} \cdot \frac{q^2 R_p^2 \Omega_p}{3\pi\nu} \left(\frac{R_p}{R-R_p}\right)^3\right].$$

We see that for $R > R_p$ the solution tends towards $\Sigma_\infty$, which we can take to be the background surface density profile.

As $R - R_p \to 0$ we see that the $\exp(-x)$ term starts to dominate, causing the surface density to undergo an exponential decrease near the planet.