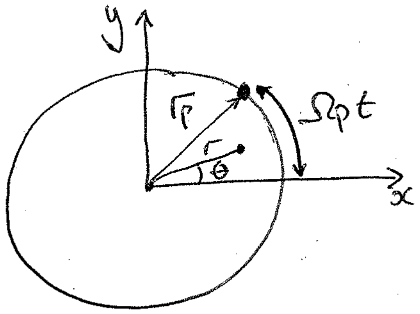
Lindblad resonances

1 Circular orbit



Planet moving on circular orbit moving with angular velocity Ω_p.

Gravitational potential due to planet

$$\Phi_p(r,\theta,t) = -\frac{GM_p}{|\underline{r}-\underline{r}_p|} = \frac{-GM_p}{\sqrt{r_p^2+r^2-2rr_p\cos(\theta-\Omega_p t)}}$$

Planet potential is periodic function in both space (θ) and time ($\Omega_p t$) and so may be expressed as a Fourier series

$$\Phi_p(r,\theta,t) = \sum_{m=0}^{\infty} \phi_m(r)\exp\left(i\left[m\theta - m\Omega_p t\right]\right)$$

$$= \sum_{m=0}^{\infty} \phi_m(r)\exp\left[im(\theta-\Omega_p t)\right] \quad (1)$$

where $\phi_m(r) = \frac{1}{2\pi^2}\int_0^{2\pi} d(\Omega_p t)\int_0^{2\pi} d\theta\, \Phi_p(r,\theta,t)\cos\left[m(\theta-\Omega_p t)\right]$

A few remarks :

(i) Observed in the inertial frame, each potential Component has m-fold azimuthal Symmetry and rotates with an angular Velocity Ω_p. This is often called the pattern speed of the potential Component, and for a Circular orbit $\Omega_{pattern} = \Omega_p$.

(ii) The potential frequency observed by a fluid element in the disc is $\pm m(\Omega - \Omega_p)$ where Ω = fluid angular velocity. This is often called the "doppler-Shifted" frequency and may be obtained by differentiating the argument of the Complex exponential in the Fourier Series eqn (1): $m\dot{\theta} - m\Omega_p = m(\Omega - \Omega_p)$. The $\pm$ depends on whether the fluid is orbiting inside or outside of the planet's orbit.

More generally we can consider an eccentric orbit. For small eccentricity we can consider the planet's motion to consist of circular motion of a guiding centre with angular velocity Ω_p and epicyclic motion around the guiding centre.

The epicyclic motion introduces new frequencies into the problem (as the planet now has motion such that its angular velocity sweeps through values that are larger and smaller than Ω_p).

Hence the Fourier series becomes:

$$\Phi_p(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{mk}(r) \exp\left\{ i \left[m(\theta - \Omega_p t) + k \kappa_p t \right] \right\}$$

where κ_p = Epicyclic frequency of the planet. (the natural frequency for radial oscillation of the planet if perturbed away from a circular orbit).

For a Keplerian orbit $\kappa_p = \Omega_p$

The pattern speed of these potential components $\Omega_{pattern} = \dfrac{m\Omega_p - k\Omega_p}{m}$

It turns out that $\phi_{mk}(r) \propto e^{|k|}$ so the strength of each potential term depends on e, and for $e << 1$ we consider only the $|k| = 1$ terms in general.

A Lindblad resonance occurs when the doppler-shifted frequency of a potential component experienced by a fluid element in the disc equals the epicyclic frequency of the fluid element :

$$m^2(\Omega - \Omega_p)^2 = \kappa^2 \quad \text{(Circular orbit case!)}.$$

$$\therefore \quad m(\Omega - \Omega_p) = \pm \Omega \quad \text{(Keplerian orbit)}$$

$$\therefore \quad \Omega = \frac{m}{m \pm 1} \Omega_p$$

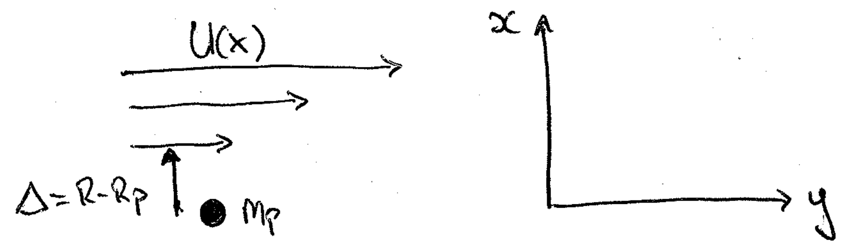
for Lindblad resonance.
+ Sign → Outer L.R.
− Sign → Inner L.R.

Simple derivation of Planet torque formula.

See Lin & Papaloizou (1979)

We work in a frame of reference with planet at radius R_p, and consider material exterior to the planet streaming past with an unperturbed speed $U = -R_p \frac{d\Omega}{dR} x = -R_p \Omega' x$

This is the velocity obtained in the shearing sheet approximation (e.g. Goldreich & Lynden-Bell 1969)



The equation of motion for the fluid (x-component)

$$\frac{dU_x}{dt} = -\frac{GM_p \, x}{(x^2+y^2)^{3/2}}$$

Set $x = \Delta$ and $y = Ut$ $\quad t$ = time.

$$\frac{dU_x}{dt} = -\frac{GM_p \Delta}{(\Delta^2 + U^2t^2)^{3/2}}$$

Change in U_x during encounter

$$\Delta U_x = -\int_{-\infty}^{\infty} \frac{GM_p \Delta}{(\Delta^2+U^2t^2)^{3/2}} \, dt$$

– assume $t=0$ corresponds to point of closest approach to planet.

Substitute $Ut = \Delta \tan\theta \Rightarrow U\,dt = \Delta \sec^2\theta \, d\theta$

Integration limits : $\tan\theta = -\infty \Rightarrow \theta = -\pi/2$

$\tan\theta = \infty \Rightarrow \theta = \pi/2$

$$\therefore \Delta U_x = -\int_{-\pi/2}^{\pi/2} \frac{GM_p \Delta^2 \sec^2\theta \, d\theta}{U\Delta^3 (1+\tan^2\theta)^{3/2}}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\Rightarrow \Delta U_x = -\frac{GM_p}{U\Delta}\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = -\frac{2GM_p}{U\Delta}$$

$$\therefore (\Delta U_x)^2 = \frac{4(GM_p)^2}{U^2\Delta^2}$$

We want angular momentum change so need to find change in U_y.
Total change in kinetic energy during encounter $= 0$.

$$\frac{1}{2}U_y^2 = \frac{1}{2}(\Delta U_x)^2 + \frac{1}{2}(U_y + \Delta U_y)^2$$

$$= \frac{1}{2}(\Delta U_x)^2 + \frac{1}{2}U_y^2 + U_y \Delta U_y + \frac{1}{2}(\Delta U_y)^2$$

Assume that $\Delta U_y \ll U_y$:

$$\Delta U_x^2 \simeq -2U_y \Delta U_y$$

$$\Delta U_y \simeq -\frac{(\Delta U_x)^2}{2U_y} = \frac{2(GM_p)^2}{U^3\Delta^2}$$

Change in angular Momentum per unit Mass per encounter

$= - R_p \Delta U_y$

Note that reduction in $U_y \Rightarrow$ increase in angular velocity for material exterior to planet.

Time between encounters $T = \dfrac{2\pi R_p}{|R_p \Omega' \Delta|}$

Total angular Momentum exchange rate $=$ $\dfrac{\text{Exchange during encounter}}{\text{Time between encounters}}$

$$\therefore \dot{H} = 2\pi \int_{R_{in}}^{R_{out}} \frac{2(GM_p)R_p}{U^3 \Delta^2} \frac{|R_p \Omega' \Delta|}{2\pi R_p} \Sigma_p R_p \, d\Delta$$

Set $U = |\Omega' R_p U|$ and we obtain (skipping some algebra):

$$\dot{H} = \int_{\Delta_{min}}^{\infty} \frac{2(GM_p)^2 \Sigma_p \, d\Delta}{R_p |\Omega'|^2 \Delta^4}$$

For Keplerian disc $\Omega' = -\dfrac{3}{2}\dfrac{\Omega_p}{R_p}$

$$\dot{H} = \int_{\Delta_{min}}^{\infty} \frac{8}{9} \frac{(GM_p)^2 \Sigma_p R_p}{\Omega_p^2 \Delta^4} \, d\Delta$$

$$= -\left[\frac{8}{27} \frac{(GM_p)^2 \Sigma_p R_p}{\Omega_p^2 \Delta^3}\right]_{\Delta_{min}}^{\infty} = \frac{8}{27} \frac{(GM_p)^2 \Sigma_p R_p}{\Omega_p^2 \Delta_{min}^3}$$

Note that $\Omega_p^2 = \frac{GM_*}{R_p^3}$

$$\dot{H} = \frac{8}{27} \frac{(GM_p)^2 \Sigma_p R_p \Omega_p^2}{\Omega_p^4 \Delta^3} = \frac{8}{27} \left(\frac{M_p}{M_*}\right)^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{\Delta_0}\right)^3.$$

$$\underline{\dot{H} = \frac{8}{27} \left(\frac{M_p}{M_*}\right)^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{\Delta_0}\right)^3}$$

For a planet embedded in a disc without a gap, the Lindblad resonances where the torque is generated pile up at a distance $\simeq H$ from the planet. This is the distance at which the flow past the planet becomes supersonic, allowing the excitation of a wave response from the disc (e.g. Artymowicz 1993).

$$\therefore \dot{H} = \frac{8}{29} \left(\frac{M_p}{M_*}\right)^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{H}\right)^3.$$

Now let's consider the total torque on the planet coming from the inner and outer disc

$$\dot{H}_{TOT} = C_{in} \left(\frac{M_p}{M_*}\right)^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{H}\right)^3 - C_{out} \left(\frac{M_p}{M_*}\right)^2 \Sigma_p R_p^4 \Omega_p^2 \left(\frac{R_p}{H}\right)^3.$$

Torque imbalance is expected to arise because of an imbalance between Lindblad resonance strengths ~~on the~~ interior and exterior to the planet.

The fact that the disc is partially pressure supported in radius causes the disc to orbit with a sub-keplerian angular velocity profile, and this shifts the resonance positions inward slightly. Hence the outer Lindblad resonances are expected to be stronger, with the asymmetry scaling as the disc aspect ratio $\frac{H}{R}$ (Goldreich & Tremaine 1980, Ward 1997)

$$\therefore \dot{H}_{TOT} \simeq \frac{q^2}{h^2} \Sigma_p R_p^4 \Omega_p^2 \quad \text{where } h = \frac{H}{R}.$$