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(1) **Eccentric Planetesimals**.

"Particle-in-a-box" accretion models approximate the relative motion of planetesimals as a random (or relative) velocity that is a function of the particles' eccentricities and/or inclinations. Here, we derive the most commonly used relation between the relative velocity $\vec{v}_{\rm rel}$ and eccentricity e. Consider a planetesimal on a Keplerian orbit so that

$$r = \frac{a\,(1-e^2)}{1+e\cos\theta}, \tag{1}$$

where $\theta \in [0, 2\pi]$ is the angle between the instantaneous position r and periapse (the true anomaly). The periapse velocity v_p is

$$v_p = \sqrt{\frac{GM}{a}}\sqrt{\frac{1+e}{1-e}}. \tag{2}$$

Compute an orbit-averaged $\langle v_{\rm rel}^2\rangle$, where $v_{\rm rel}$ is the relative velocity of the eccentric planetesimal compared to a co-planar, locally circular orbit. Here, "local" means that you are to compare the planetesimal's instantaneous velocity to that of a circular orbit with $a = r$, where r is the planetesimal's time-varying orbital radius. Remember that angular momentum is conserved at every point on an orbit.

We parametrize the position of the planetesimal $\vec{r}$ by the radius r and true anomaly θ. The velocity of the particle in cylindrical coordinates is $\vec{v} = (\dot{r}, r\dot{\theta})$, where the first entry is the velocity in the radial direction and the second in the θ direction. We have for Keplerian motion

$$r = \frac{a\,(1-e^2)}{1+e\cos\theta}, \tag{3}$$

where $\theta = \theta(t)$. Differentiating with respect to time, we get

$$\dot{r} = \frac{ae \sin\theta \left(1 - e^2\right)}{\left(1 + e\cos\theta\right)^2} \frac{d\theta}{dt} \tag{4}$$

$$= r\frac{d\theta}{dt} \frac{e\sin\theta}{1 + e\cos\theta}. \tag{5}$$

Substituting for $\vec{v}$, we get

$$\vec{v} = r\frac{d\theta}{dt} \left(\frac{e\sin\theta}{1 + e\cos\theta}, 1\right). \tag{6}$$

Now, at periapse ($\theta = 0$), $\vec{r} = \left(a\left(1 - e\right), 0\right)$, so that $|\vec{v}(\theta = 0)| = v_p$ yields

$$a\left(1 - e\right) \left.\frac{d\theta}{dt}\right|_{\theta=0} = \sqrt{\frac{GM}{a}} \sqrt{\frac{1 + e}{1 - e}} \tag{7}$$

$$= na\sqrt{\frac{1 + e}{1 - e}}, \tag{8}$$

where $n = \sqrt{GM/a^3}$ is the mean motion. Solving for $|d\theta/dt|_{\theta=0}$ gives

$$\left.\frac{d\theta}{dt}\right|_{\theta=0} = \frac{n}{1 - e} \sqrt{\frac{1 + e}{1 - e}}. \tag{9}$$

To determine $d\theta/dt$ at other θ, we use conservation of angular momentum. For angular momentum $h = |\vec{h}|$, in a Keplerian system, we have the following expression of Kepler's Second Law:

$$h = r^2 \frac{d\theta}{dt} = \text{const.} \tag{10}$$

On substituting with the expressions at periapse, we get

$$h = \frac{na^2\left(1 - e\right)^2}{1 - e} \sqrt{\frac{1 + e}{1 - e}} \tag{11}$$

$$= na^2\sqrt{1 - e^2}, \tag{12}$$

so that

$$\frac{d\theta}{dt} = \frac{h}{r^2}. \tag{13}$$

The velocity is then

$$\vec{v} = \frac{h}{r}\left(\frac{e\sin\theta}{1 + e\cos\theta}, 1\right)$$

$$= \frac{na}{\sqrt{1 - e^2}} \left(e\sin\theta, 1 + e\cos\theta\right). \tag{14}$$

We now calculate the velocity for an *instantaneously circular Keplerian orbit*. In this case, the velocity is purely in the θ direction, and hence $\propto (0,1)$. For an instantaneously circular orbit, the Keplerian speed is $\sqrt{GM/r(\theta)} = \sqrt{GM/a(1-e^2)} \times \sqrt{1+e\cos\theta} = na/\sqrt{1-e^2} \times \sqrt{1+e\cos\theta}$, where again, n is the mean motion. The velocity for a circular orbit is then

$$\vec{v}_{\rm circ} = \frac{na}{\sqrt{1-e^2}} \left(0, \sqrt{1+e\cos\theta}\right). \tag{15}$$

The relative velocity $\vec{v}_{\rm rel} = \vec{v} - \vec{v}_{\rm circ}$ is

$$\vec{v}_{\rm rel} = \frac{na}{\sqrt{1-e^2}} \left(e\sin\theta, 1+e\cos\theta - \sqrt{1+e\cos\theta}\right), \tag{16}$$

so that

$$\begin{aligned} |\vec{v}_{\rm rel}|^2 &= \frac{n^2a^2}{1-e^2}\left[e^2\sin^2\theta + \left(1+e\cos\theta - \sqrt{1+e\cos\theta}\right)^2\right] \\ &= \frac{n^2a^2}{1-e^2}\left[e^2\sin^2\theta + (1+e\cos\theta)^2 - 2\,(1+e\cos\theta)^{3/2} + 1 + e\cos\theta\right]. \end{aligned} \tag{17}$$

We expand the last three terms on the right hand side out to second order in $e\cos\theta$, keeping in mind that $\cos\theta \leq 1$. In particular, the third term has the expansion $(1+e\cos\theta)^{3/2} \approx 1 + 3/2 \times e\cos\theta + 3/8 \times e^2\cos^2\theta + ...$, so that upon simplification we get

$$|\vec{v}_{\rm rel}|^2 = \frac{n^2a^2e^2}{1-e^2}\left[\sin^2\theta + \frac{\cos^2\theta}{4}\right]. \tag{18}$$

Averaged over an orbit, we have

$$\begin{aligned} \left\langle v_{\rm rel}^2 \right\rangle &= \frac{1}{2\pi}\int_0^{2\pi} |\vec{v}_{\rm rel}|^2\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{n^2a^2e^2}{1-e^2}\left[\sin^2\theta + \frac{\cos^2\theta}{4}\right] d\theta \\ &= \frac{5}{8}\frac{n^2a^2e^2}{1-e^2}. \end{aligned} \tag{19}$$

Recognizing that $na = v_K$ and that the denominator $(1-e^2)^{-1}$ is a geometric series $= 1 + e^2 + e^4 + ...$, we have to second order in e

$$\left\langle v_{\rm rel}^2 \right\rangle = \frac{5}{8} v_K^2 e^2. \tag{20}$$

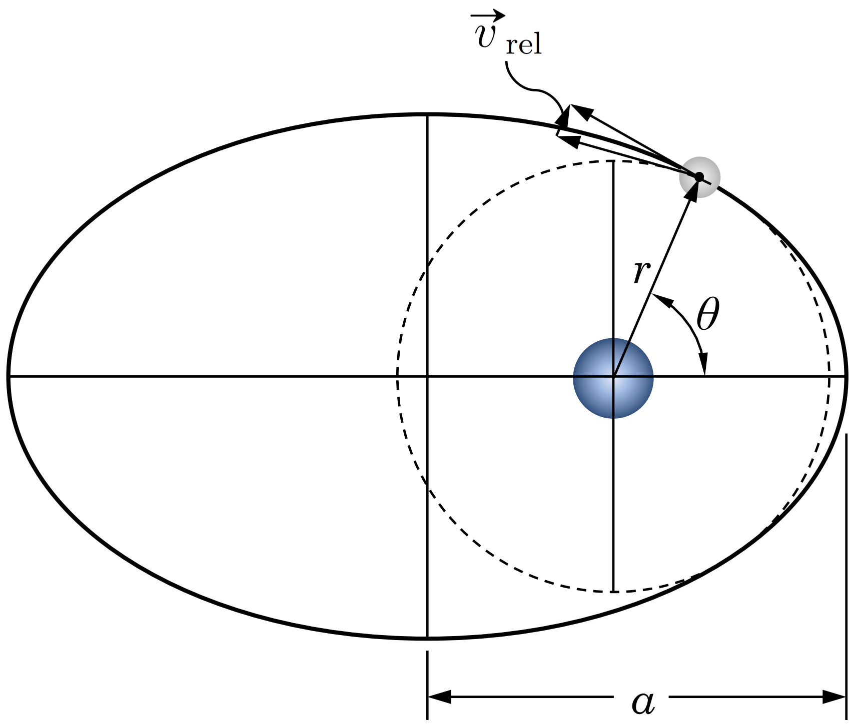

FIGURE 1. Orbit geometry for Problem 1.

(2) **Planet Formation.**

(a) *What is dynamical friction and (qualitatively) how does it affect protoplanet dynamical evolution?*

Dynamical friction refers to the net retarding force exerted on a larger body due to its gravitationally deflecting smaller bodies around and behind it. Gravitational focusing decreases the relative velocities of the larger body and its potential impactors, resulting in an enhancement of the gravitational focusing factor.

(b) *What is viscous stirring and (qualitatively) how does it affect protoplanet dynamical evolution?*

Viscous stirring refers to the average increase in the random velocity of a planetesimal due to gravitational interactions with bodies of similar size or with bodies larger than itself.

(c) *In terms of the radius R of a protoplanet and time t, give an equation that describes the runaway growth regime and explain briefly in words what it means.*

$$\frac{1}{R}\frac{dR}{dt} \propto R. \tag{21}$$

In runaway growth, the relative growth rate $1/R \times dR/dt$ of a protoplanet is proportional to R: larger bodies grow faster.

(d) *What is a consequence of runaway growth?*

An outcome of runaway growth is oligarchy.

(3) **Growing a Planet.**

(a) *Derive the impact velocity* $v_{\rm imp}$ *of a test particle on a planet of mass* M *and radius* R*. Express* $v_{\rm imp}$ *as a function of the relative velocity at infinity* v_∞ *and the planet's escape velocity* $v_{\rm esc}$*. Show all steps.*

By energy conservation, the impact velocity is given by

$$v_{\rm imp}^2 = v_\infty^2 + v_{\rm esc}^2, \tag{22}$$

where

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}. \tag{23}$$

(b) *Qualitatively, what is gravitational focusing?*

Gravitational focusing is when the gravity increases the effective cross section of a gravitational interaction between two bodies.

(c) *Derive the gravitational focusing factor for the situation described in (a) using conservation of angular momentum.*

We have by conservation of momentum

$$bv_\infty = Rv_{\rm imp}. \tag{24}$$

Then

$$b^2 = R^2 \frac{v_{\rm imp}^2}{v_\infty^2} \tag{25}$$

$$= R^2 \left(1 + \frac{v_{\rm esc}^2}{v_\infty^2}\right), \tag{26}$$

so that the interaction cross-section is increased over its physical cross section πR^2 by a factor of $(1 + v_{\rm esc}^2/v_\infty^2)$.

(d) *Why is there in practice a maximum value for the gravitational focusing factor that can be achieved as a test particle or planetesimal is accreted by a planet?*

Keplerian shear, so that $b_{\rm max} \sim R \times v_{\rm esc}/v_{\rm shear}$.

(4) **Isolation Masses.**

A growing planetary embryo will accumulate all of the material within an annulus of width $\Delta a = Br_H$, where B is a constant ($= 10$), r_H is the planet's Hill radius, and a is the planet's semimajor axis.

(a) *Calculate the mass of the final planet, $M_{\rm iso}$, that will form from a disk with a uniform surface density in solids Σ_s. Show all steps.*

The total mass in the annulus will be given by

$$\begin{aligned} M_{\rm iso} &= 2\pi a \Delta a \Sigma_s \\ &= 2\pi B r_H a \Sigma_s \\ &= 2\pi B \Sigma_s a^2 \left(\frac{M_{\rm iso}}{3M_\star}\right)^{1/3}, \end{aligned} \quad (27)$$

so that

$$M_{\rm iso}(a) = \left[\frac{2\pi a^2 \Sigma_s B}{(3M_\star)^{1/3}}\right]^{3/2}. \quad (28)$$

(b) *What would the radial dependence of the surface density need to be to produce an isolation mass that is independent of semimajor axis? How does this compare to the minimum mass Solar nebula?*

In order for $M_{\rm iso}(a)$ to be independent of semimajor axis, we must have $\Sigma_s \propto a^{-2}$. This is a steeper dependence than that of the MMSN, which has $\Sigma_s \propto a^{-3/2}$.

(c) *Compute $M_{\rm iso}$ for constant $\Sigma_s = 100\ \mathrm{kg/m^2}$ at 1 AU. What does this imply about the mass and approximate total number of embryos $\mathcal{N}$ expected at the end of runaway growth in the region of the terrestrial planets? Assume the mass in solids equals the total final mass in the terrestrial planets, and use your calculation at 1 AU to get an order of magnitude estimate for $\mathcal{N}$.*

Substitution gives $M_{\rm iso} = 0.1143 M_\oplus$. Thus, taking all the mass in the terrestrial planets to be roughly $3M_\oplus$, we have about $\mathcal{N} \approx 30$.