Dust growth by sedimentation

Overview:

Dust particles in protoplanetary discs grow as they collide and stick together. The relative velocity leading to a collision can come from many sources (Brownian motion, turbulence, sedimentation, ...). Here we set up a simple model to calculate the growth of a dust particle sedimenting to the mid-plane of the disc.

You will learn about:

- The physics of the sedimentation process
- The structure of the solar nebula
- The efficiency of dust coagulation

Tasks:

Consider a spherical dust particle with mass m and radius a at height z over the mid-plane of a protoplanetary disc. The dynamical equations governing the particle vertical motion and growth are

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\Omega^2 z - \frac{1}{\tau_\mathrm{f}} v \,, \tag{1}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \pi a^2 |v| \rho_\mathrm{p} \,. \tag{2}$$

Here v is the particle velocity perpendicular to the mid-plane, Ω is the Keplerian frequency, $\rho_\mathrm{p}(z)$ is the density of dust particles and τ_f is the friction time. We assume that the dust particles are small (much smaller than 1 meter), so the friction time is given by

$$\tau_\mathrm{f} = \frac{a\rho_\bullet}{c_\mathrm{s}\rho_\mathrm{g}} \,, \tag{3}$$

with $\rho_\bullet$ the material density, c_s the sound speed of the gas and ρ_g the gas density.

Task 1: Show that the terminal velocity of the particle is

$$v = -\tau_\mathrm{f}\Omega^2 z \,. \tag{4}$$

Estimate the characteristic time-scale for reaching the terminal velocity.

Task 2: Show that the mass growth rate of the particle (equation 2) can be rewritten in terms of the radius growth as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{|v|\rho_\mathrm{p}}{4\rho_\bullet} \,. \tag{5}$$

Task 3: Find the radius of the particle as a function of the height over the mid-plane. The governing equations are

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v \,, \tag{6}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{|v|\rho_\mathrm{p}}{4\rho_\bullet} \,, \tag{7}$$

with the additional relevant connections

$$\begin{aligned}
\rho_{\rm p} &= 0.01\rho_{\rm g}\,, &(8)\\
\rho_{\rm g} &= \frac{\varSigma_{\rm g}}{\sqrt{2\pi}H}\exp[-z^2/(2H^2)]\,, &(9)\\
v &= -\tau_{\rm f}\varOmega^2 z\,, &(10)\\
\tau_{\rm f} &= \frac{a\rho_\bullet}{c_{\rm s}\rho_{\rm g}}\,. &(11)
\end{aligned}$$

Here $\rho_{\rm g}$, $\varSigma_{\rm g}$ and H are the gas density, gas column density and gas scale height, respectively. We assume a canonical dust-to-gas ratio of 0.01 and perfect mixing between gas and particles.

You can either integrate the equation system numerically to get $z(t)$ and $a(t)$ (starting the particle with radius a_0 at the height z_0 from the mid-plane), or find an analytical solution for $a(z)$. For the latter you can use the trick

$$\frac{{\rm d}a}{{\rm d}z} = \frac{{\rm d}a}{{\rm d}t}\frac{{\rm d}t}{{\rm d}z} = \frac{{\rm d}a}{{\rm d}t}\frac{1}{v} = -\frac{\rho_{\rm p}}{4\rho_\bullet}\,, \tag{12}$$

with $\rho_{\rm p}(z)$ given in equations (9)–(8). Show in that case that the analytical solution is

$$a(z) = \frac{\varSigma_{\rm p}}{8\rho_\bullet}\left\{1-{\rm erf}[z/(\sqrt{2}H)]\right\} + a_\infty\,, \tag{13}$$

where $\varSigma_{\rm p} = 0.01\varSigma_{\rm g}$ is the particle column density and a_∞ is the particle size far above the mid-plane (at $z \gg H$). Here ${\rm erf}(x)$ is the error function defined by

$${\rm erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int \exp(-x^2){\rm d}x\,. \tag{14}$$

It is possible at this point to proceed to Tasks 4, 5 and 6 and return to Task 3 later, if the integral proves difficult.

Task 4: Show that the maximum particle radius increase is

$$\Delta a = \frac{\varSigma_{\rm p}}{8\rho_\bullet} \tag{15}$$

when the particle arrives at the mid-plane. What is this size in centimeters? Consider the minimum mass solar nebula at a distance of 1 AU from the central star.

Task 5: Sketch the generic shape of the $a(z)$ curve.

Task 6: Formulate a criticism of this simple physical problem. What important ingredients are missing?