The Vernacular of the S-Matrix

Jacob Bourjaily
NBIA-Oxford Colloquium

The Niels Bohr International Academy
UNIVERSITY OF OXFORD

Organization and Outline

1. *Spiritus Movens*: a Parable from Collider Physics
    - A Simple, Practical Problem in Quantum Chromodynamics
2. The *Physical* Vernacular of the *S*-Matrix: On-Shell Observables
    - Physically Observable Data Describing Asymptotic States
    - Beyond (Mere) Scattering Amplitudes: On-Shell Functions
    - Basic Building Blocks: *S*-Matrices for Three Massless Particles
3. The On-Shell Analytic *S*-Matrix: All-Loop Recursion Relations
    - Building-up Diagrams with "BCFW" Bridges
    - On-Shell (Recursive) Representations of Scattering Amplitudes
    - *Exempli Gratia*: On-Shell Manifestations of Tree Amplitudes
4. The Combinatorics of Scattering (and Grassmannian Geometry)
    - Combinatorial Classification of On-Shell Functions in Planar SYM
    - Canonical Coordinates, Computation, & Auxiliary Grassmannian
5. The Ongoing Revolution: Toward a Complete Reformulation of QFT

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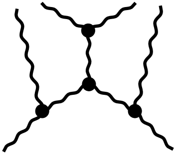

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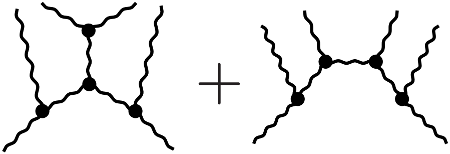

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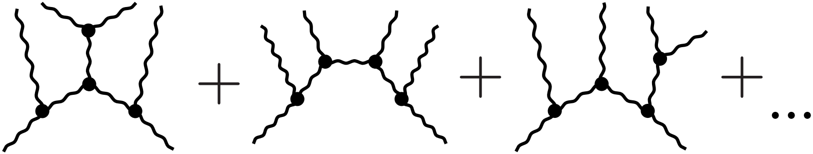

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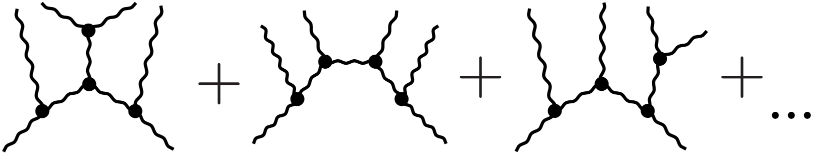

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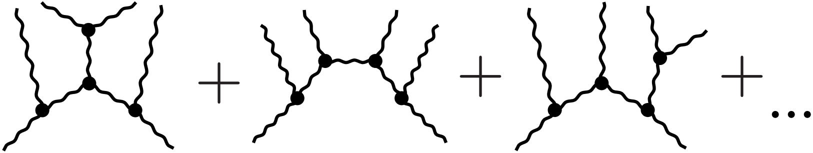

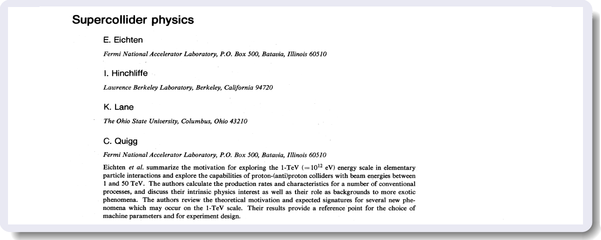

Supercollider physics

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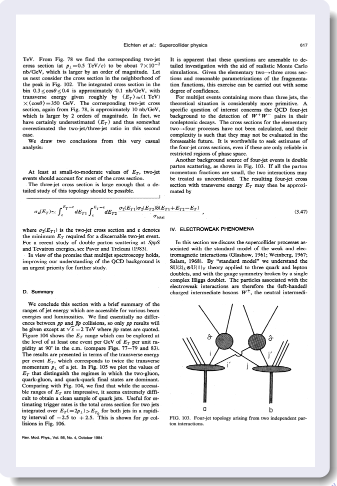

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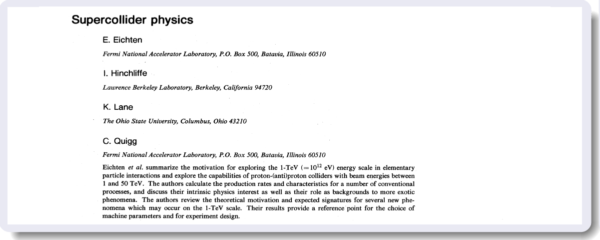

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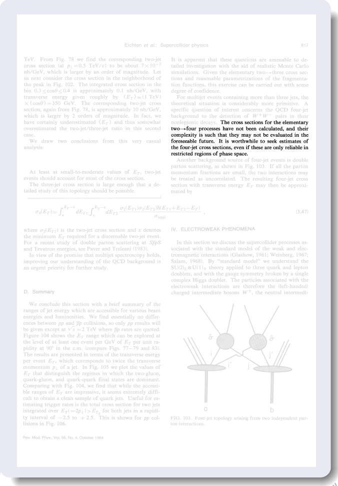

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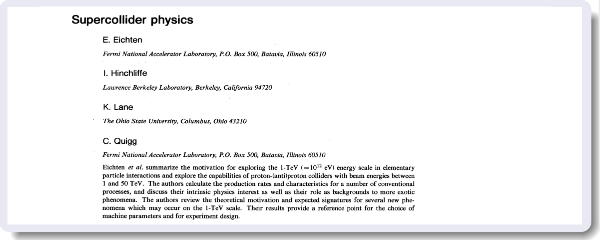

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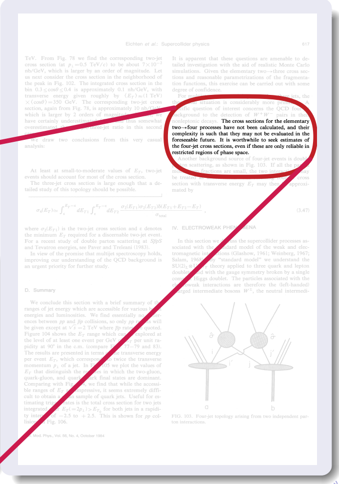

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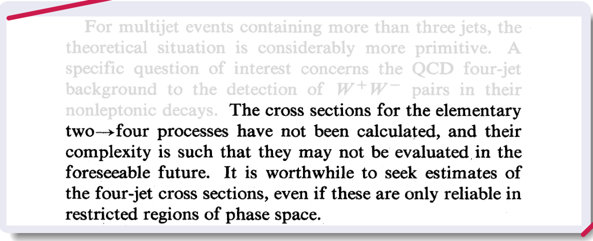

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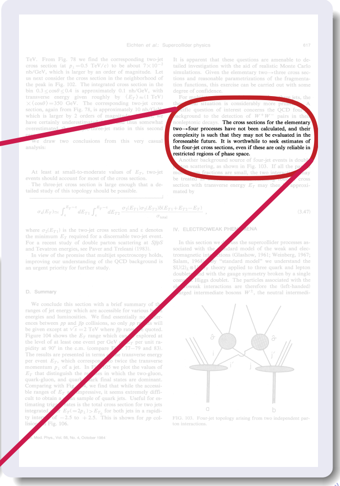

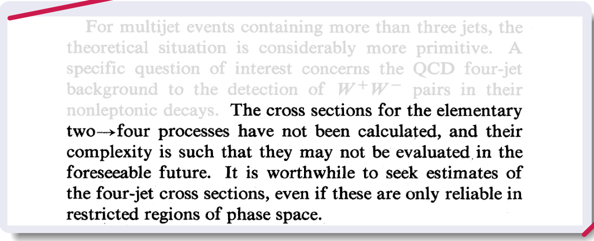

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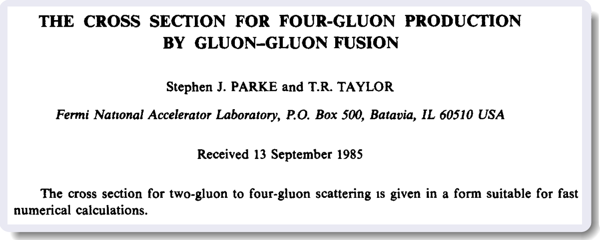

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Stephen J. PARKE and T.R. TAYLOR

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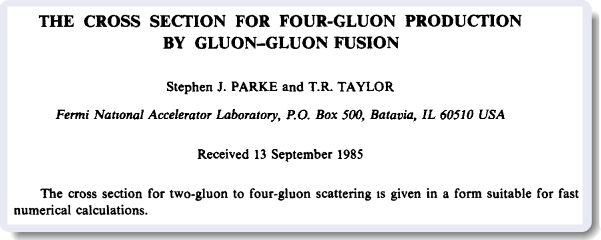

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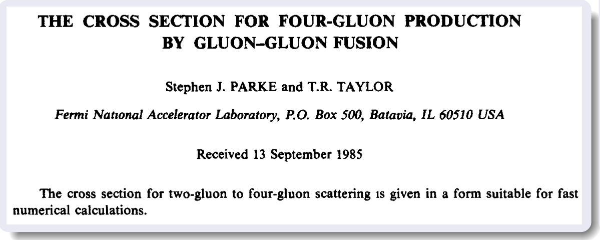

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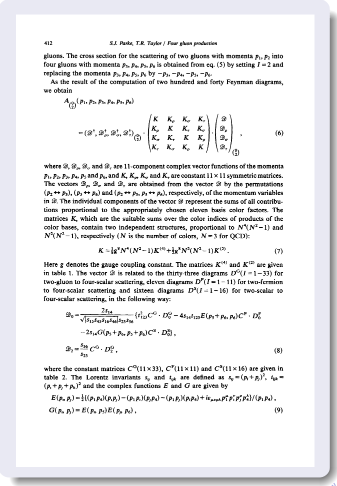

gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I = 2$ and replacing the momenta p_3, p_4, p_5, p_6 by $-p_3, -p_4, -p_5, -p_6$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathcal{D}^\dagger, \mathcal{D}^\dagger_\rho, \mathcal{D}^\dagger_\sigma, \mathcal{D}^\dagger_\tau)_{\binom{0}{2}} \cdot \begin{pmatrix} K & K_\rho & K_\sigma & K_\tau \\ K_\rho & K & K_\tau & K_\sigma \\ K_\sigma & K_\tau & K & K_\rho \\ K_\tau & K_\sigma & K_\rho & K \end{pmatrix} \cdot \begin{pmatrix} \mathcal{D} \\ \mathcal{D}_\rho \\ \mathcal{D}_\sigma \\ \mathcal{D}_\tau \end{pmatrix}_{\binom{0}{2}}, \tag{6}$$

where $\mathcal{D}$, $\mathcal{D}_\rho$, $\mathcal{D}_\sigma$ and $\mathcal{D}_\tau$ are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6, and K, K_ρ, K_σ and K_τ are constant 11×11 symmetric matrices. The vectors $\mathcal{D}_\rho$, $\mathcal{D}_\sigma$ and $\mathcal{D}_\tau$ are obtained from the vector $\mathcal{D}$ by the permutations $(p_2 \leftrightarrow p_3)$, $(p_5 \leftrightarrow p_6)$ and $(p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in $\mathcal{D}$. The individual components of the vector $\mathcal{D}$ represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, $N = 3$ for QCD):

$$K = \tfrac{1}{8} g^8 N^4 (N^2-1) K^{(4)} + \tfrac{1}{2} g^8 N^2 (N^2-1) K^{(2)} . \tag{7}$$

Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(2)}$ are given in table 1. The vector $\mathcal{D}$ is related to the thirty-three diagrams $D^{\mathrm{G}}(I = 1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^{\mathrm{F}}(I = 1-11)$ for two-fermion to four-scalar scattering and sixteen diagrams $D^{\mathrm{S}}(I = 1-16)$ for two-scalar to four-scalar scattering, in the following way:

$$\mathcal{D}_0 = \frac{2s_{14}}{\sqrt{|s_{15}s_{45}s_{16}s_{46}|}\, s_{23}s_{56}} \{ t^2_{123} C^{\mathrm{G}} \cdot D^{\mathrm{G}}_0 - 4 s_{14} t_{123} E(p_5 + p_6, p_6) C^{\mathrm{F}} \cdot D^{\mathrm{F}}_0 - 2 s_{14} G(p_5 + p_6, p_5 + p_6) C^{\mathrm{S}} \cdot D^{\mathrm{S}}_0 \} ,$$

$$\mathcal{D}_2 = \frac{s_{56}}{s_{23}} C^{\mathrm{G}} \cdot D^{\mathrm{G}}_2 , \tag{8}$$

where the constant matrices $C^{\mathrm{G}}(11 \times 33)$, $C^{\mathrm{F}}(11 \times 11)$ and $C^{\mathrm{S}}(11 \times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij} = (p_i + p_j)^2$, $t_{ijk} = (p_i + p_j + p_k)^2$ and the complex functions E and G are given by

$$E(p_i, p_j) = \tfrac{1}{2} \{ (p_1 p_4)(p_i p_j) - (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4) + i \varepsilon_{\mu\nu\rho\lambda} p_1^\mu p_i^\nu p_j^\rho p_4^\lambda \} / (p_1 p_4) ,$$

$$G(p_i, p_j) = E(p_i, p_5) E(p_j, p_6) , \tag{9}$$

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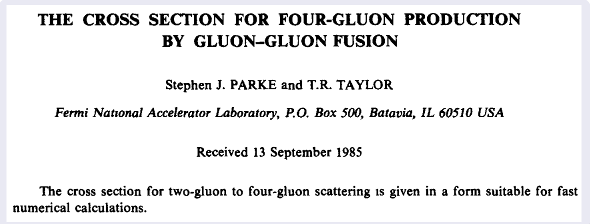

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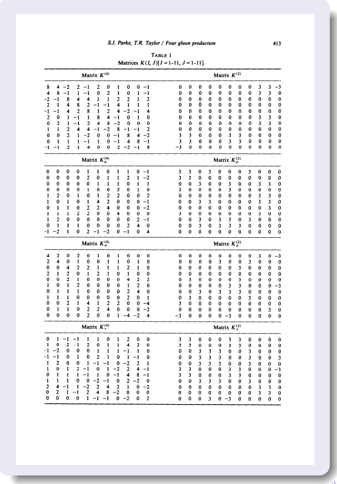

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TABLE 1

Matrices $K(I, J)[I = 1\text{-}11, J = 1\text{-}11]$.

Matrix $K^{(4)}$											Matrix $K^{(2)}$										
8	4	−2	2	−1	2	0	1	0	0	−1	0	0	0	0	0	0	0	0	3	3	−3
4	8	−1	1	−1	0	2	1	0	1	−1	0	0	0	0	0	0	0	0	3	3	0
−2	−1	8	4	4	1	1	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0
2	1	4	8	2	−1	−1	4	1	1	1	0	0	0	0	0	0	0	0	0	0	0
−1	−1	4	2	8	1	2	4	−2	−1	4	0	0	0	0	0	0	0	0	0	0	0
2	0	1	−1	1	8	4	−1	0	1	0	0	0	0	0	0	0	0	0	3	3	0
0	2	1	−1	2	4	8	−2	0	0	0	0	0	0	0	0	0	0	0	3	3	0
1	1	2	4	4	−1	−2	8	−1	−1	2	0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	−2	0	0	−1	8	4	−2	3	3	0	0	0	3	3	0	0	0	0
0	1	1	1	−1	1	0	−1	4	8	−1	3	3	0	0	0	3	3	0	0	0	0
−1	−1	2	1	4	0	0	2	−2	−1	8	−3	0	0	0	0	0	0	0	0	0	0
Matrix $K_\rho^{(4)}$											Matrix $K_\rho^{(2)}$										
0	0	0	0	1	1	0	1	1	0	−1	3	3	0	3	0	0	0	3	0	0	0
0	0	0	0	2	0	1	1	2	1	−2	3	3	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	1	0	1	1	0	0	3	0	0	3	0	0	3	3	0
0	0	0	0	1	0	0	2	0	1	0	3	0	0	0	0	3	0	0	0	0	0
1	2	0	1	0	1	2	2	0	0	2	0	0	0	0	0	0	0	0	3	3	0
1	0	1	0	1	4	2	0	0	0	−1	0	0	3	3	0	0	0	0	3	3	0
0	1	1	0	2	2	4	0	0	0	−2	0	0	0	0	0	0	0	0	0	3	0
1	1	1	2	2	0	0	4	0	0	0	3	0	0	0	0	0	0	0	3	0	0
1	2	0	0	0	0	0	0	0	2	−1	0	0	3	0	3	3	0	3	0	0	0
0	1	1	1	0	0	0	0	2	4	0	0	0	3	0	3	3	3	0	0	0	0
−1	−2	1	0	2	−1	−2	0	−1	0	4	0	0	0	0	0	0	0	0	0	0	0
Matrix $K_\sigma^{(4)}$											Matrix $K_\sigma^{(2)}$										
4	2	0	2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	3	0	−3
2	4	0	1	0	0	1	1	0	1	0	0	0	0	0	3	0	0	3	0	0	0
0	0	4	2	2	1	1	1	2	1	0	0	0	0	0	0	0	3	0	0	0	0
2	1	2	0	1	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	0	0	0	0	4	2	2	0	3	0	0	0	0	3	0	0	0	0
1	0	1	2	0	0	0	0	1	2	0	0	0	0	0	0	3	3	0	0	0	−3
0	1	1	1	0	0	0	0	2	4	0	0	0	3	0	3	3	3	0	0	0	0
1	1	1	0	0	0	0	0	2	0	1	0	3	0	0	0	0	0	3	0	0	0
0	0	2	1	4	1	2	2	0	0	−4	3	0	0	0	0	0	0	0	0	0	0
0	1	1	0	2	2	4	0	0	0	−2	0	0	0	0	0	0	0	0	0	3	0
0	0	0	0	2	0	0	1	−4	−2	4	−3	0	0	0	0	−3	0	0	0	0	0
Matrix $K_\tau^{(4)}$											Matrix $K_\tau^{(2)}$										
0	1	−1	−1	1	1	0	1	2	0	0	3	3	0	0	0	3	3	0	0	0	0
1	0	−2	−1	2	0	1	1	4	2	0	3	3	0	0	0	3	3	0	0	0	0
−1	−2	0	0	0	1	1	1	−1	1	0	0	0	3	3	3	0	0	3	0	0	0
−1	−1	0	1	0	2	1	0	1	−1	0	0	0	3	3	3	0	0	3	0	0	3
1	2	0	0	1	−1	−1	0	−2	2	1	0	0	3	3	3	0	0	3	0	0	0
1	0	1	2	−1	0	1	−2	2	4	−1	3	3	0	0	0	3	3	0	0	0	−3
0	1	1	1	−1	1	0	−1	4	8	−1	3	3	0	0	0	3	3	0	0	0	0
1	1	1	0	0	−2	−1	0	2	−2	0	0	0	3	3	3	0	0	3	0	0	0
2	4	−1	1	−2	2	4	2	1	0	−2	0	0	0	0	0	0	0	0	3	3	0
0	2	1	−1	2	4	8	−2	0	0	0	0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	1	−1	−1	0	−2	0	2	0	0	0	3	0	−3	0	0	0	0	0

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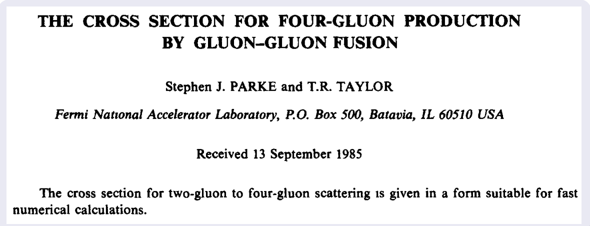

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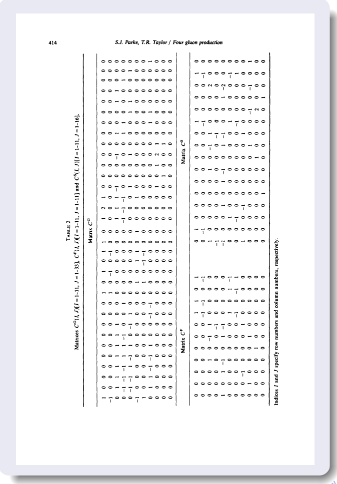

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TABLE 2

Matrices $C^G(I,J)[I=1\text{–}11, J=1\text{–}33]$, $C^F(I,J)[I=1\text{–}11, J=1\text{–}11]$ and $C^S(I,J)[I=1\text{–}11, J=1\text{–}16]$.

Matrix C^G

1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	0	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	0	0	0	0	0	0	1	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	−1	0	0	−1	0	1	0	0	1	1	0	0	0
0	−1	−1	−1	1	1	−1	0	0	1	0	0	0	0	0	0	0	−1	−1	−1	0	0	0	0	0	1	1	0	0	0	1	0	0
0	−1	−1	1	−1	1	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	1	0	0	1	0
−1	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	−1	−1	1	0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	2	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Matrix C^F

0	0	0	0	0	0	0	1	1	0	1
0	0	0	0	0	0	0	−1	−1	0	−1
0	0	0	1	0	−1	1	0	0	0	0
0	0	0	0	0	0	−1	0	0	0	0
1	0	1	−1	0	1	−1	0	0	0	0
0	0	0	0	0	0	0	1	0	1	−1
0	0	0	0	0	0	0	−1	0	−1	1
0	0	−1	0	0	0	1	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Matrix C^S

0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0
0	−1	0	0	0	0	0	0	0	0	−1	0	0	0	−1	0
1	0	0	0	0	0	1	0	−1	1	0	0	0	2	0	0
−1	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0
−1	0	0	1	0	1	−1	0	1	−1	0	0	1	−2	0	0
0	0	1	0	0	0	0	0	0	0	1	0	0	0	−1	0
0	0	−1	0	0	0	0	0	0	0	−1	0	0	0	1	0
1	0	0	−1	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	−1	0	−1	0	1
0	0	0	0	0	0	0	1	0	0	0	2	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0

Indices I and J specify row numbers and column numbers, respectively.

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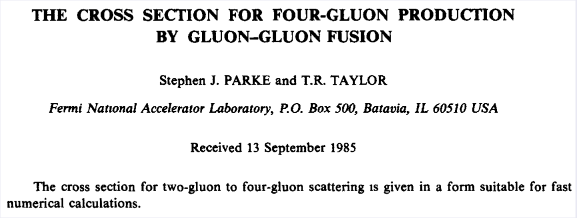

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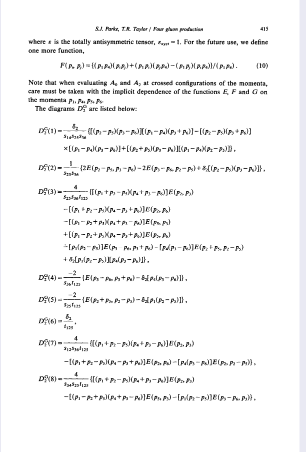

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where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt}=1$. For the future use, we define one more function,

$$F(p_i, p_j)=\{(p_1p_4)(p_ip_j)+(p_1p_i)(p_jp_4)-(p_1p_j)(p_ip_4)\}/(p_1p_4)\,. \qquad (10)$$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6.

The diagrams D_2^{G} are listed below:

$$D_2^{\mathrm{G}}(1)=\frac{\delta_2}{s_{14}s_{25}s_{36}}\{[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]-[(p_2-p_5)(p_3+p_6)]$$
$$\times[(p_1-p_4)(p_3-p_6)]+[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\}\,,$$

$$D_2^{\mathrm{G}}(2)=\frac{1}{s_{25}s_{36}}\{2E(p_2-p_5,p_3-p_6)-2E(p_3-p_6,p_2-p_5)+\delta_2[(p_2-p_5)(p_3-p_6)]\}\,,$$

$$D_2^{\mathrm{G}}(3)=\frac{4}{s_{25}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2,p_3)$$
$$-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)$$
$$+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)$$
$$+[p_1(p_2-p_5)]E(p_3-p_6,p_3+p_6)-[p_4(p_3-p_6)]E(p_2+p_5,p_2-p_5)$$
$$+\delta_2[p_1(p_2-p_5)][p_4(p_3-p_6)]\}\,,$$

$$D_2^{\mathrm{G}}(4)=\frac{-2}{s_{36}t_{125}}\{E(p_3-p_6,p_3+p_6)-\delta_2[p_4(p_3-p_6)]\}\,,$$

$$D_2^{\mathrm{G}}(5)=\frac{-2}{s_{25}t_{125}}\{E(p_2+p_5,p_2-p_5)-\delta_2[p_1(p_2-p_5)]\}\,,$$

$$D_2^{\mathrm{G}}(6)=\frac{\delta_2}{t_{125}}\,,$$

$$D_2^{\mathrm{G}}(7)=\frac{4}{s_{12}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2,p_3)$$
$$-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)-[p_4(p_3-p_6)]E(p_2,p_2-p_5)\}\,,$$

$$D_2^{\mathrm{G}}(8)=\frac{4}{s_{34}s_{25}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2,p_3)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)-[p_1(p_2-p_5)]E(p_3-p_6,p_3)\}\,,$$

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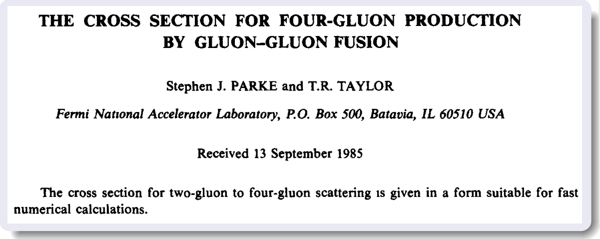

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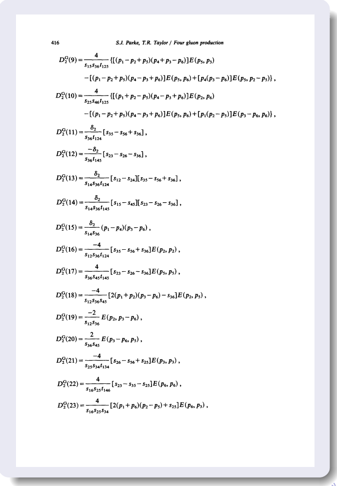

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$$D_2^G(9) = \frac{4}{s_{15}s_{36}t_{125}}\{[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)$$
$$-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_4(p_3-p_6)]E(p_5,p_2-p_5)\}\,,$$

$$D_2^G(10) = \frac{4}{s_{25}s_{46}t_{125}}\{[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)$$
$$-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_1(p_2-p_5)]E(p_3-p_6,p_6)\}\,,$$

$$D_2^G(11) = \frac{\delta_2}{s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]\,,$$

$$D_2^G(12) = \frac{-\delta_2}{s_{36}t_{145}}[s_{23}-s_{26}-s_{36}]\,,$$

$$D_2^G(13) = \frac{\delta_2}{s_{14}s_{36}t_{124}}[s_{12}-s_{24}][s_{35}-s_{56}+s_{36}]\,,$$

$$D_2^G(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}}[s_{15}-s_{45}][s_{23}-s_{26}-s_{36}]\,,$$

$$D_2^G(15) = \frac{\delta_2}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6)\,,$$

$$D_2^G(16) = \frac{-4}{s_{12}s_{36}t_{124}}[s_{35}-s_{56}+s_{36}]E(p_2,p_2)\,,$$

$$D_2^G(17) = \frac{4}{s_{36}s_{45}t_{145}}[s_{23}-s_{26}-s_{36}]E(p_5,p_5)\,,$$

$$D_2^G(18) = \frac{-4}{s_{12}s_{36}s_{45}}[2(p_1+p_2)(p_3-p_6)-s_{36}]E(p_2,p_5)\,,$$

$$D_2^G(19) = \frac{-2}{s_{12}s_{36}}E(p_2,p_3-p_6)\,,$$

$$D_2^G(20) = \frac{2}{s_{36}s_{45}}E(p_3-p_6,p_5)\,,$$

$$D_2^G(21) = \frac{-4}{s_{25}s_{34}t_{134}}[s_{26}-s_{56}+s_{25}]E(p_3,p_3)\,,$$

$$D_2^G(22) = \frac{4}{s_{16}s_{25}t_{146}}[s_{23}-s_{35}-s_{25}]E(p_6,p_6)\,,$$

$$D_2^G(23) = \frac{4}{s_{16}s_{25}s_{34}}[2(p_1+p_6)(p_2-p_5)+s_{25}]E(p_6,p_3)\,,$$

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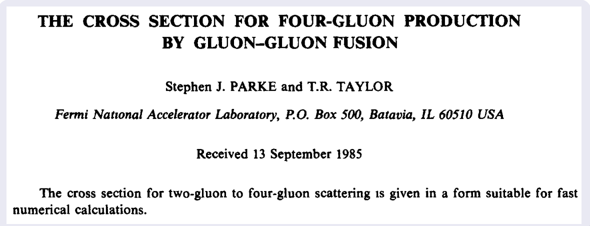

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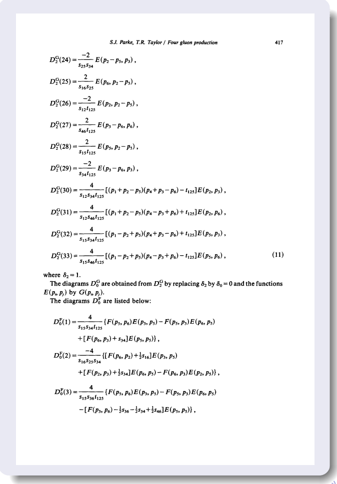

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$$D_2^{\rm G}(24) = \frac{-2}{s_{25}s_{34}}\,E(p_2-p_5,\, p_3)\,,$$

$$D_2^{\rm G}(25) = \frac{2}{s_{16}s_{25}}\,E(p_6,\, p_2-p_5)\,,$$

$$D_2^{\rm G}(26) = \frac{-2}{s_{12}t_{125}}\,E(p_2,\, p_2-p_5)\,,$$

$$D_2^{\rm G}(27) = \frac{2}{s_{46}t_{125}}\,E(p_3-p_6,\, p_6)\,,$$

$$D_2^{\rm G}(28) = \frac{2}{s_{15}t_{125}}\,E(p_5,\, p_2-p_5)\,,$$

$$D_2^{\rm G}(29) = \frac{-2}{s_{34}t_{125}}\,E(p_3-p_6,\, p_3)\,,$$

$$D_2^{\rm G}(30) = \frac{4}{s_{12}s_{34}t_{125}}[(p_1+p_2-p_5)(p_4+p_3-p_6)-t_{125}]E(p_2,\, p_3)\,,$$

$$D_2^{\rm G}(31) = \frac{4}{s_{12}s_{46}t_{125}}[(p_1+p_2-p_5)(p_4-p_3+p_6)+t_{125}]E(p_2,\, p_6)\,,$$

$$D_2^{\rm G}(32) = \frac{4}{s_{15}s_{34}t_{125}}[(p_1-p_2+p_5)(p_4+p_3-p_6)+t_{125}]E(p_5,\, p_3)\,,$$

$$D_2^{\rm G}(33) = \frac{4}{s_{15}s_{46}t_{125}}[(p_1-p_2+p_5)(p_4-p_3+p_6)-t_{125}]E(p_5,\, p_6)\,, \qquad (11)$$

where $\delta_2 = 1$.

The diagrams $D_0^{\rm G}$ are obtained from $D_2^{\rm G}$ by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p_i, p_j)$ by $G(p_i, p_j)$.

The diagrams $D_0^{\rm F}$ are listed below:

$$D_0^{\rm F}(1) = \frac{4}{s_{15}s_{34}t_{125}}\{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) + [F(p_6, p_3) + s_{34}]E(p_5, p_5)\}\,,$$

$$D_0^{\rm F}(2) = \frac{-4}{s_{16}s_{25}s_{34}}\{[F(p_6, p_2) + \tfrac{1}{2}s_{16}]E(p_3, p_5) + [F(p_2, p_3) + \tfrac{1}{2}s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5)\}\,,$$

$$D_0^{\rm F}(3) = \frac{4}{s_{15}s_{36}t_{125}}\{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) - [F(p_3, p_6) - \tfrac{1}{2}s_{36} - \tfrac{1}{2}s_{34} + \tfrac{1}{2}s_{46}]E(p_5, p_5)\}\,,$$

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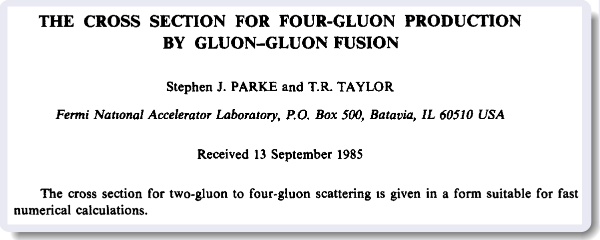

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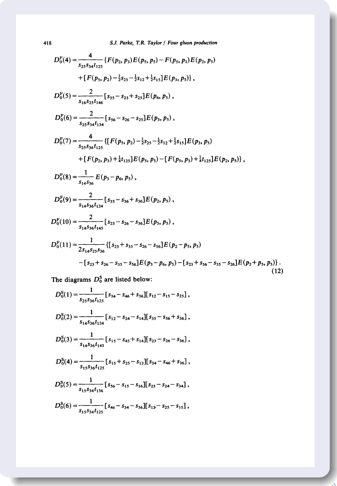

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$$D_0^{\rm F}(4) = \frac{4}{s_{25}s_{34}t_{125}}\{F(p_2,p_3)E(p_5,p_5) - F(p_5,p_3)E(p_2,p_5)$$
$$+[F(p_5,p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3,p_5)\}\,,$$

$$D_0^{\rm F}(5) = \frac{2}{s_{16}s_{25}t_{146}}[s_{35} - s_{23} + s_{25}]E(p_6,p_5)\,,$$

$$D_0^{\rm F}(6) = \frac{2}{s_{25}s_{34}t_{134}}[s_{56} - s_{26} - s_{25}]E(p_3,p_5)\,,$$

$$D_0^{\rm F}(7) = \frac{4}{s_{25}s_{36}t_{125}}\{[F(p_5,p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3,p_5)$$
$$+[F(p_2,p_3) + \tfrac{1}{4}t_{125}]E(p_5,p_5) - [F(p_5,p_3) + \tfrac{1}{4}t_{125}]E(p_2,p_5)\}\,,$$

$$D_0^{\rm F}(8) = \frac{1}{s_{14}s_{36}}E(p_3 - p_6, p_5)\,,$$

$$D_0^{\rm F}(9) = \frac{2}{s_{14}s_{36}t_{124}}[s_{35} - s_{56} + s_{36}]E(p_2,p_5)\,,$$

$$D_0^{\rm F}(10) = \frac{2}{s_{14}s_{36}t_{145}}[s_{23} - s_{26} - s_{36}]E(p_5,p_5)\,,$$

$$D_0^{\rm F}(11) = \frac{1}{2s_{14}s_{25}s_{36}}\{[s_{23} + s_{35} - s_{26} - s_{56}]E(p_2 - p_5, p_5)$$
$$-[s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5)\}\,. \tag{12}$$

The diagrams $D_0^{\rm S}$ are listed below:

$$D_0^{\rm S}(1) = \frac{1}{s_{25}s_{36}t_{125}}[s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}]\,,$$

$$D_0^{\rm S}(2) = \frac{1}{s_{14}s_{36}t_{124}}[s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}]\,,$$

$$D_0^{\rm S}(3) = \frac{1}{s_{14}s_{36}t_{145}}[s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}]\,,$$

$$D_0^{\rm S}(4) = \frac{1}{s_{15}s_{36}t_{125}}[s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}]\,,$$

$$D_0^{\rm S}(5) = \frac{1}{s_{15}s_{34}t_{156}}[s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}]\,,$$

$$D_0^{\rm S}(6) = \frac{1}{s_{15}s_{34}t_{125}}[s_{46} - s_{34} - s_{36}][s_{12} - s_{25} - s_{15}]\,,$$

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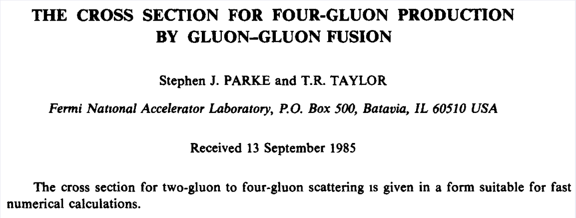

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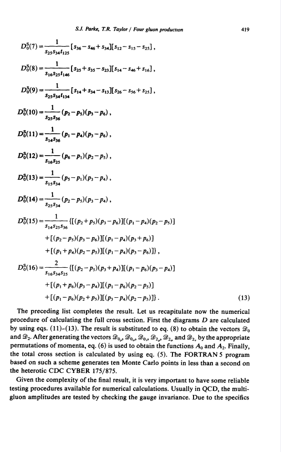

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$$D_0^S(7) = \frac{1}{s_{25}s_{34}t_{125}}[s_{36}-s_{46}+s_{34}][s_{12}-s_{15}-s_{25}]\,,$$

$$D_0^S(8) = \frac{1}{s_{16}s_{25}t_{146}}[s_{25}+s_{35}-s_{23}][s_{14}-s_{46}+s_{16}]\,,$$

$$D_0^S(9) = \frac{1}{s_{25}s_{34}t_{134}}[s_{14}+s_{34}-s_{13}][s_{26}-s_{56}+s_{25}]\,,$$

$$D_0^S(10) = \frac{1}{s_{25}s_{36}}(p_2-p_5)(p_3-p_6)\,,$$

$$D_0^S(11) = \frac{1}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6)\,,$$

$$D_0^S(12) = \frac{1}{s_{16}s_{25}}(p_6-p_1)(p_2-p_5)\,,$$

$$D_0^S(13) = \frac{1}{s_{15}s_{34}}(p_5-p_1)(p_3-p_4)\,,$$

$$D_0^S(14) = \frac{1}{s_{25}s_{34}}(p_2-p_5)(p_3-p_4)\,,$$

$$\begin{aligned} D_0^S(15) = \frac{1}{s_{14}s_{25}s_{36}}\{&[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)] \\ &+[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)] \\ &+[(p_1+p_4)(p_2-p_5)][(p_1-p_4)(p_3-p_6)]\}\,, \end{aligned}$$

$$\begin{aligned} D_0^S(16) = \frac{2}{s_{16}s_{34}s_{25}}\{&[(p_2-p_5)(p_3+p_4)][(p_1-p_6)(p_3-p_4)] \\ &+[(p_1+p_6)(p_3-p_4)][(p_1-p_6)(p_2-p_5)] \\ &+[(p_1-p_6)(p_2+p_5)][(p_3-p_4)(p_2-p_5)]\}\,. \end{aligned} \qquad (13)$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)–(13). The result is substituted to eq. (8) to obtain the vectors $\mathscr{D}_0$ and $\mathscr{D}_2$. After generating the vectors $\mathscr{D}_{0_\rho}$, $\mathscr{D}_{0_\sigma}$, $\mathscr{D}_{0_\tau}$, $\mathscr{D}_{2_\rho}$, $\mathscr{D}_{2_\sigma}$ and $\mathscr{D}_{2_\tau}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2. Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multigluon amplitudes are tested by checking the gauge invariance. Due to the specifics

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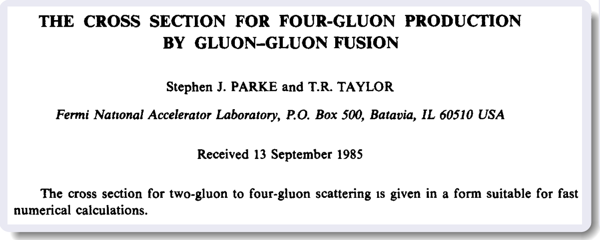

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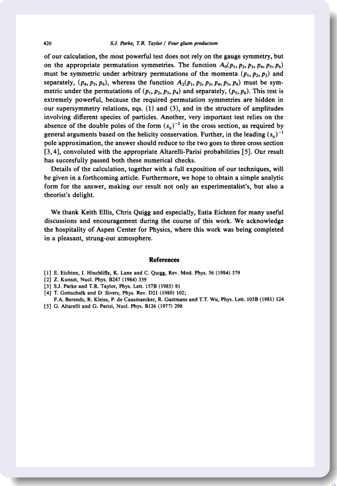

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of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A_0(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6), whereas the function $A_2(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6). This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

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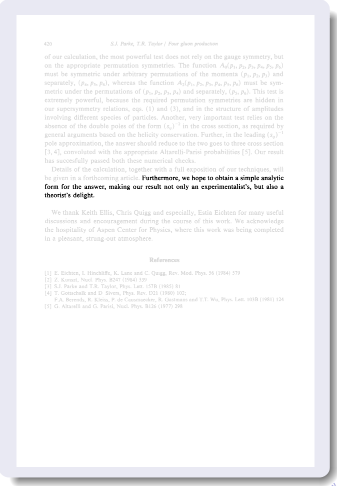

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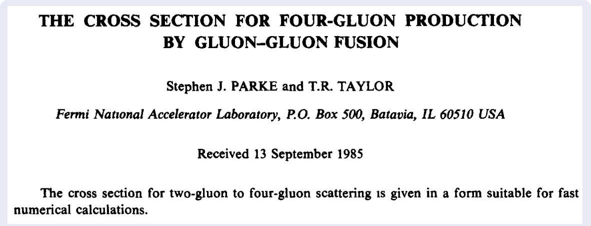

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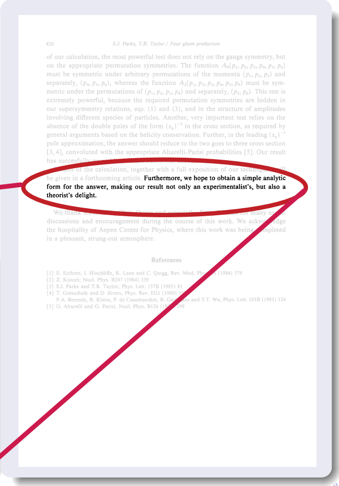

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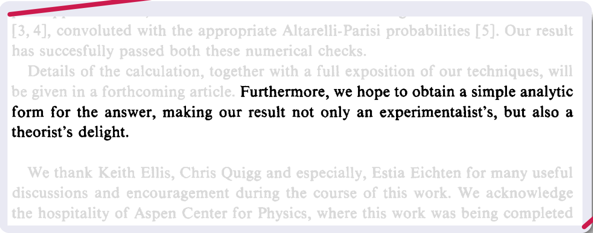

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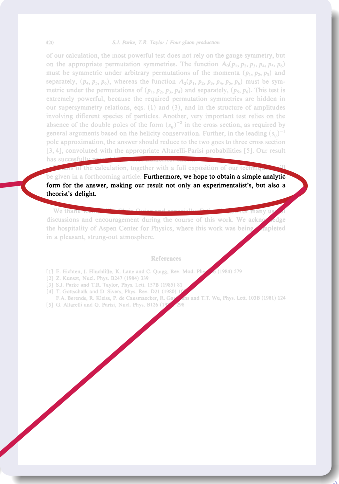

420 *S.J. Parke, T.R. Taylor / Four gluon production*

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References

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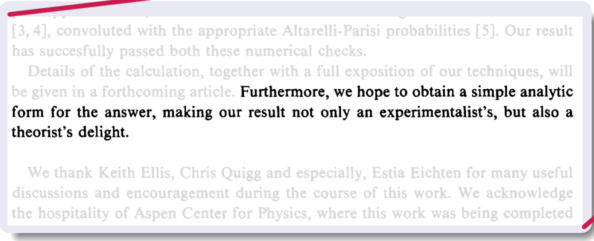

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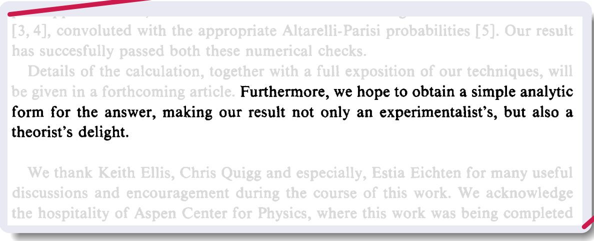

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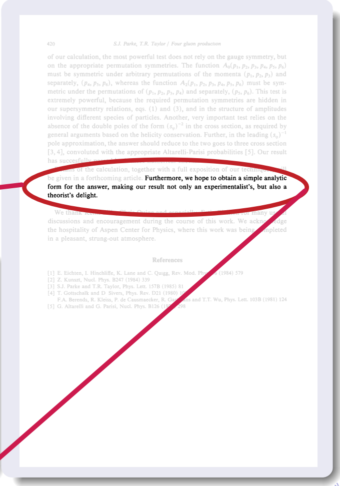

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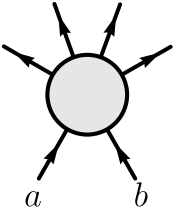

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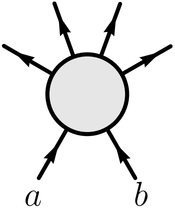

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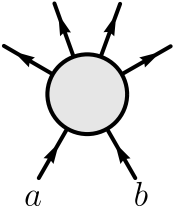

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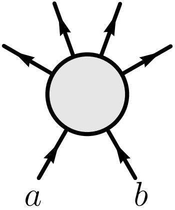

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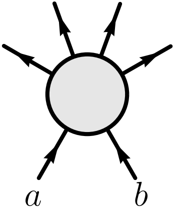

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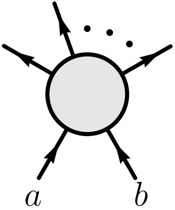

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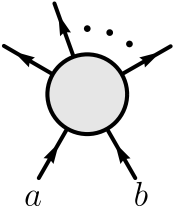

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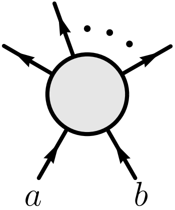

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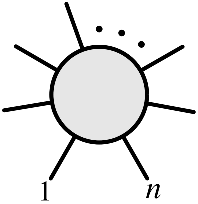

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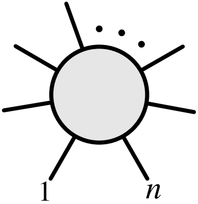

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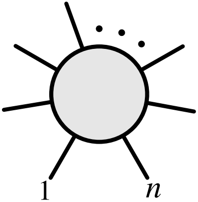

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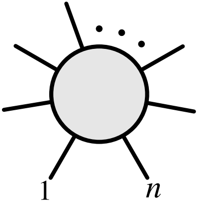

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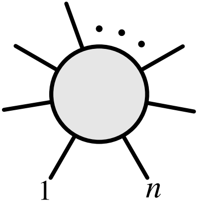

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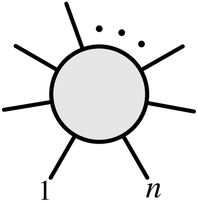
$\mathcal{A}_n \equiv$ 

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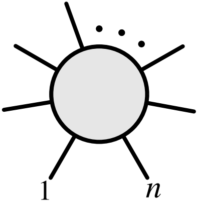
$\mathcal{A}_n \equiv$ 

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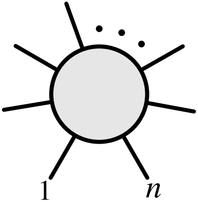

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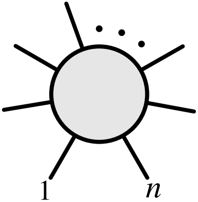
$\mathcal{A}_n \equiv$ 

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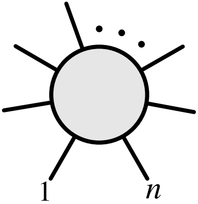

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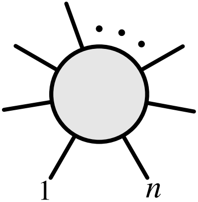

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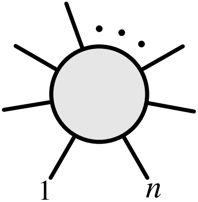

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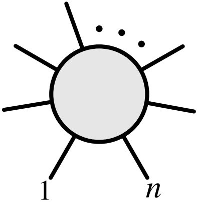
$\mathcal{A}_n \equiv$ 

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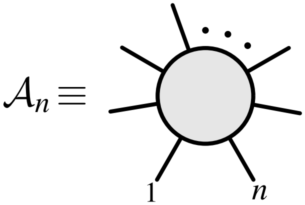

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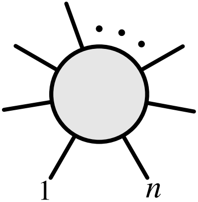

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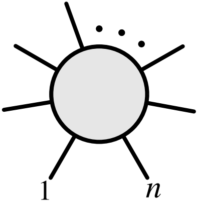

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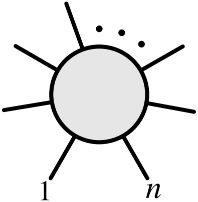

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- q_a all the *non-kinematical* quantum numbers of a (color, flavor, . . .)

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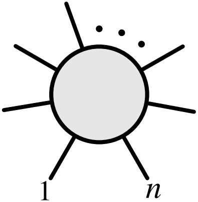

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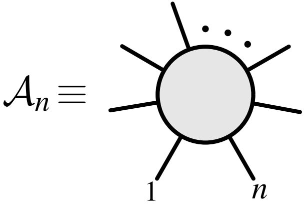

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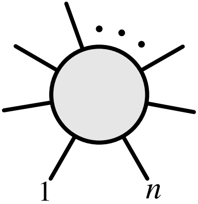
$\mathcal{A}_n \equiv$ 

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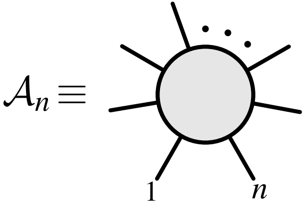

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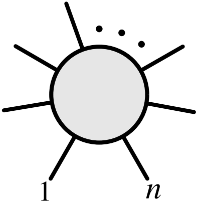
$\mathcal{A}_n \equiv$ 

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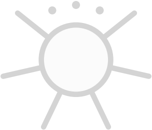

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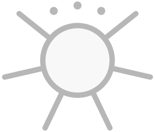

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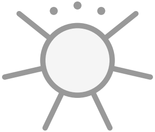

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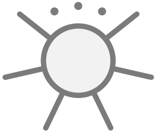

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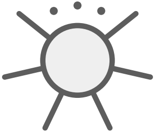

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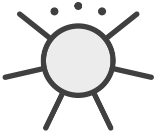

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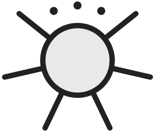

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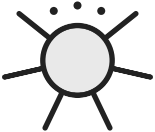

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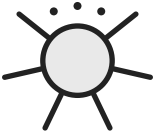

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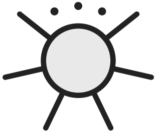

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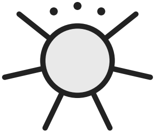

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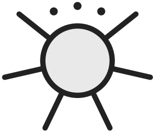

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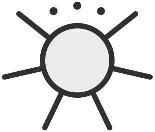

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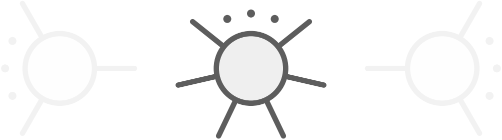

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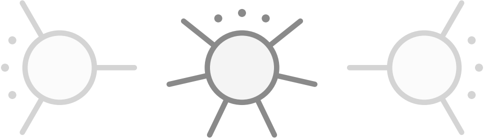

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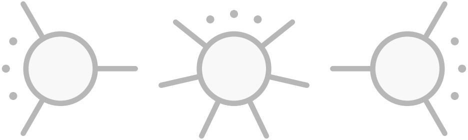

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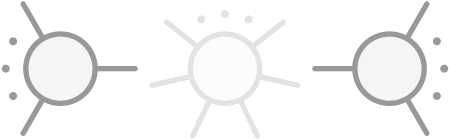

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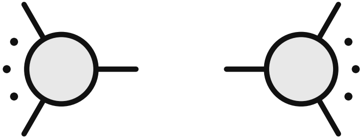

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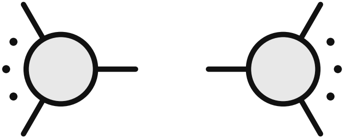

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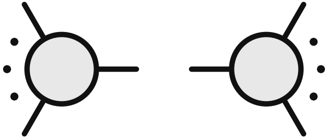

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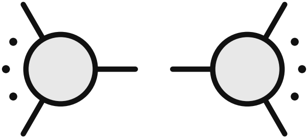

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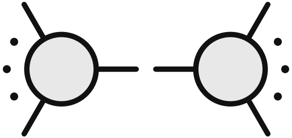

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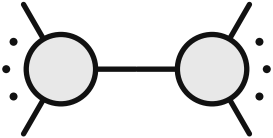

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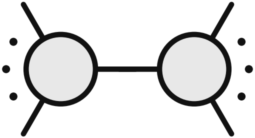

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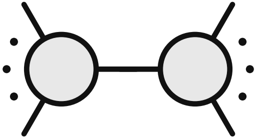

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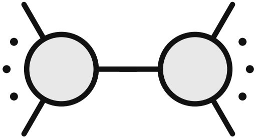

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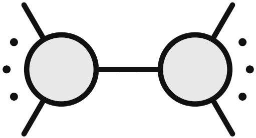

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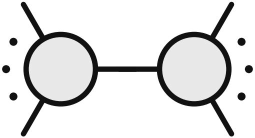

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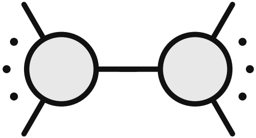

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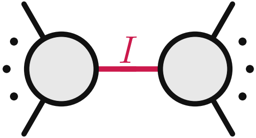

Internal Particles:

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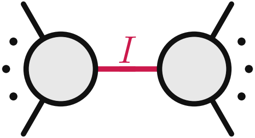

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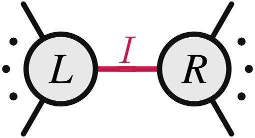

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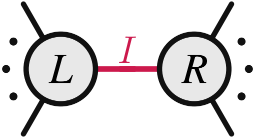

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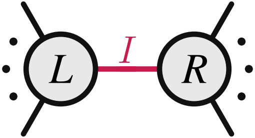

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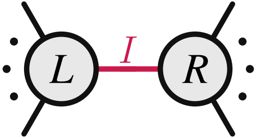

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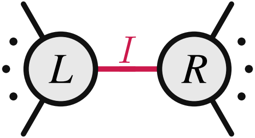

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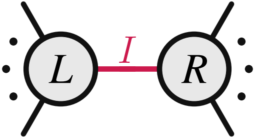

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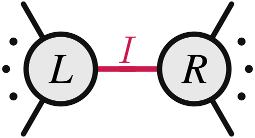

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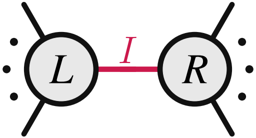

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Broadening the Class of Physically Meaningful Functions

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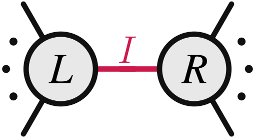

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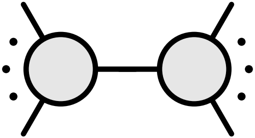

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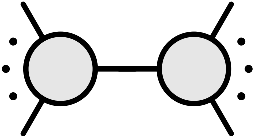

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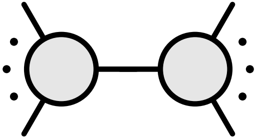

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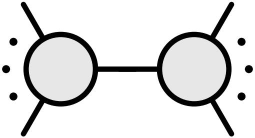

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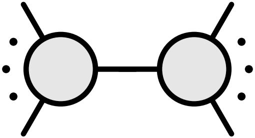

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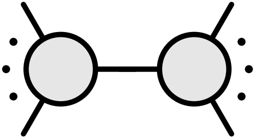

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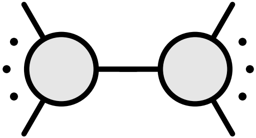

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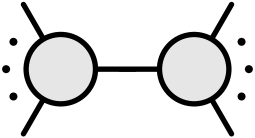

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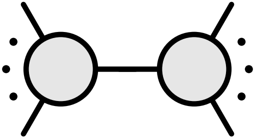

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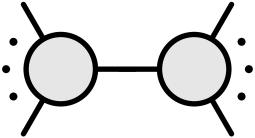

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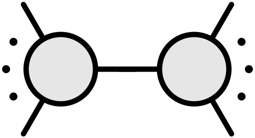

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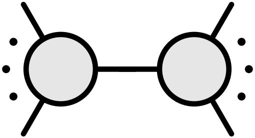

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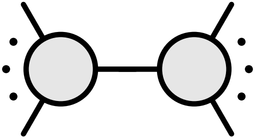

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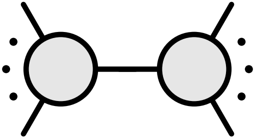

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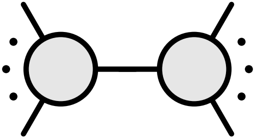

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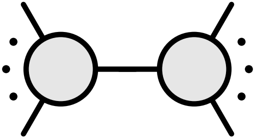

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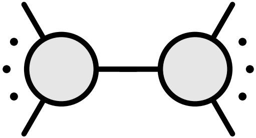

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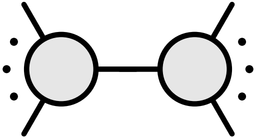

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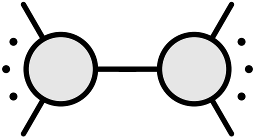

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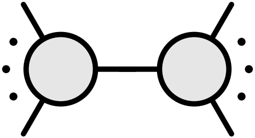

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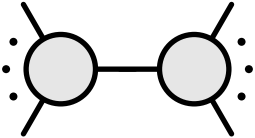

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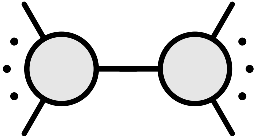

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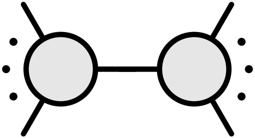

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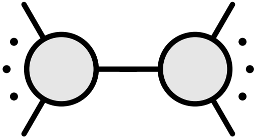

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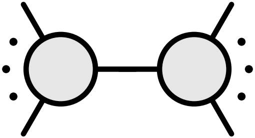

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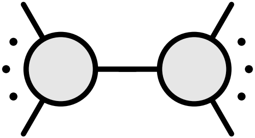

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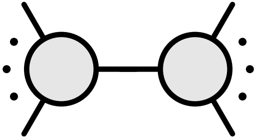

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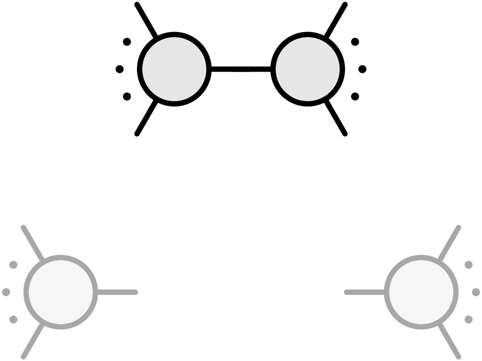

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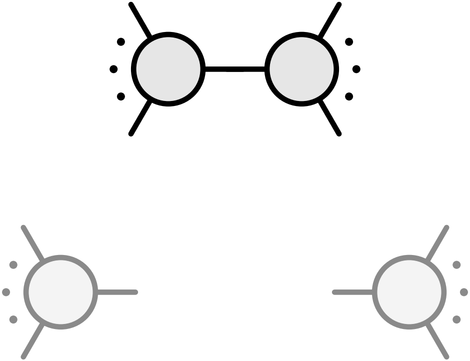

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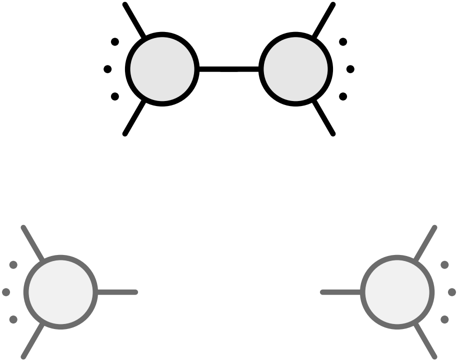

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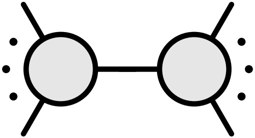

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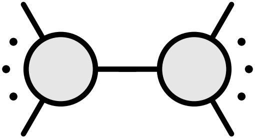

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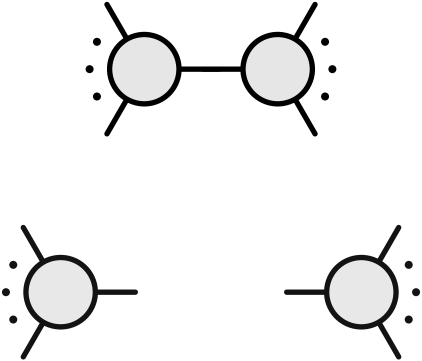

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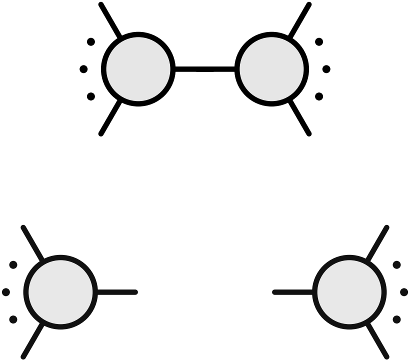

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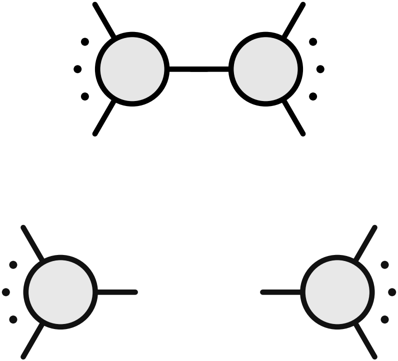

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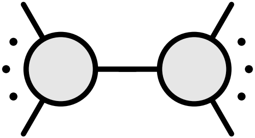

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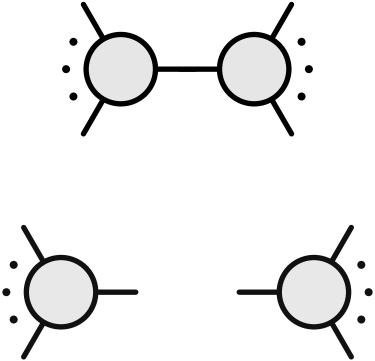

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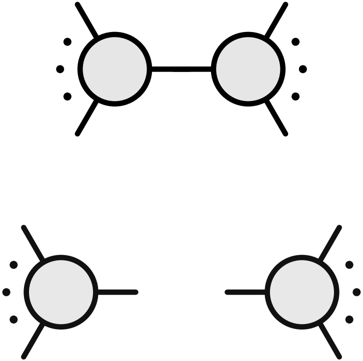

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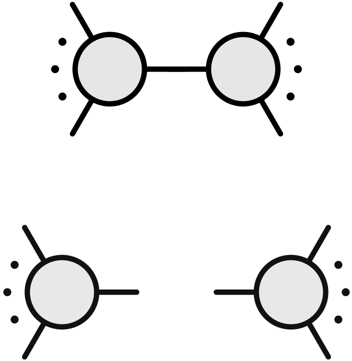

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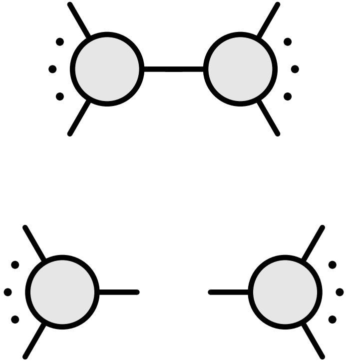

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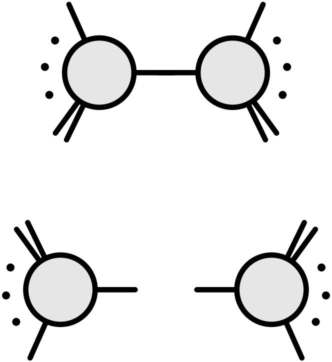

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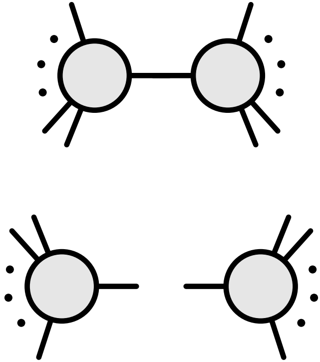

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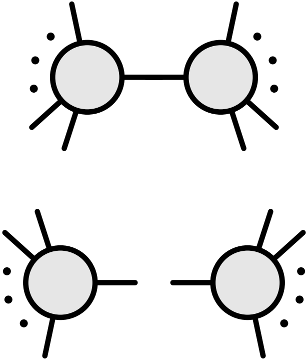

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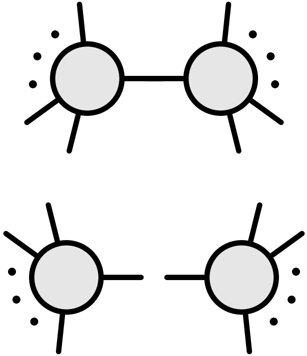

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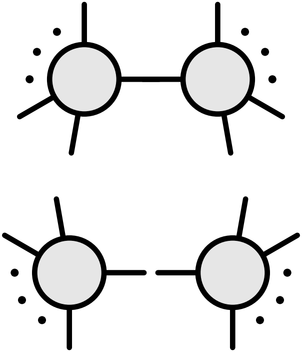

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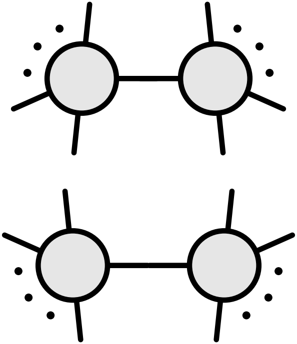

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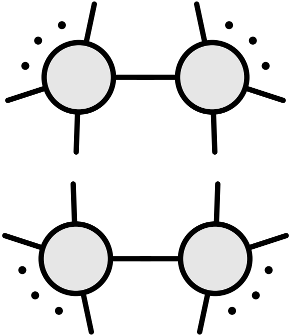

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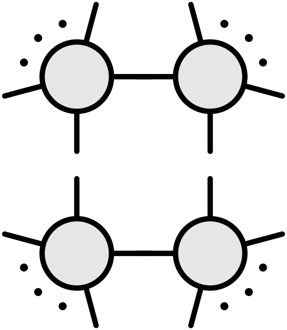

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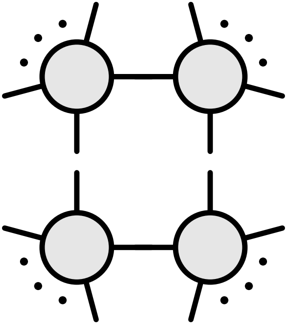

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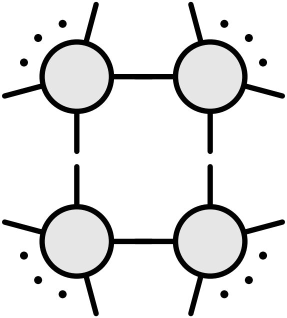

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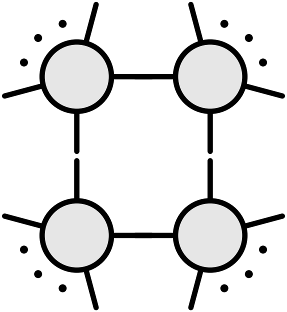

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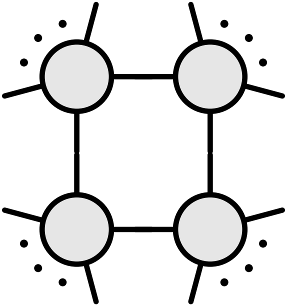

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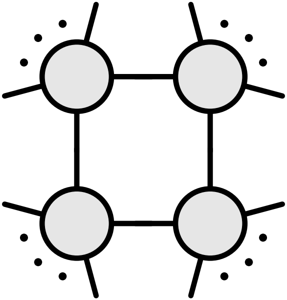

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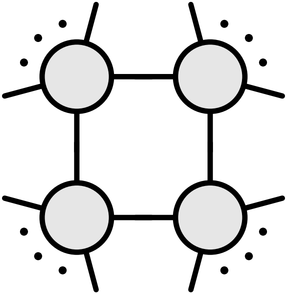

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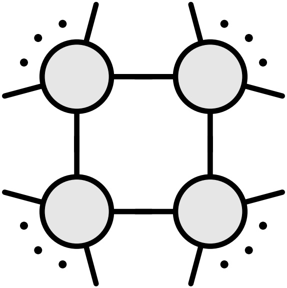

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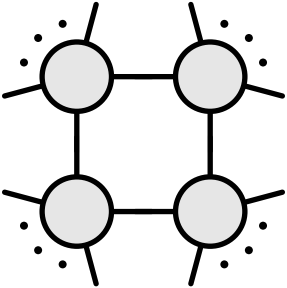

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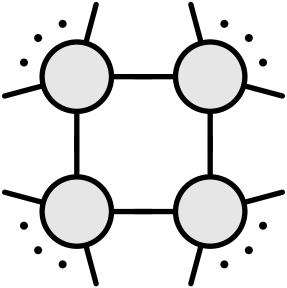

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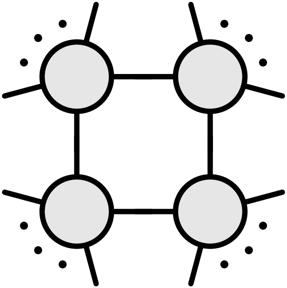

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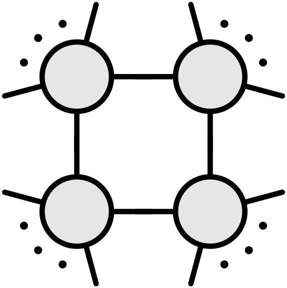

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Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

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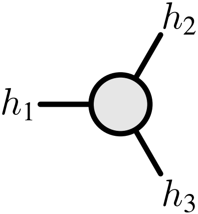

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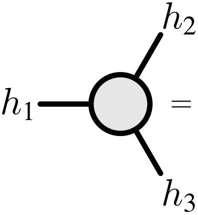

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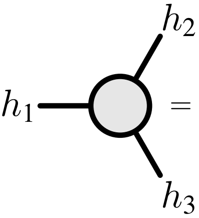

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$$\widetilde{\lambda} \equiv \begin{pmatrix} \widetilde{\lambda}_1^{\dot{1}} & \widetilde{\lambda}_2^{\dot{1}} & \widetilde{\lambda}_3^{\dot{1}} \\ \widetilde{\lambda}_1^{\dot{2}} & \widetilde{\lambda}_2^{\dot{2}} & \widetilde{\lambda}_3^{\dot{2}} \end{pmatrix}$$

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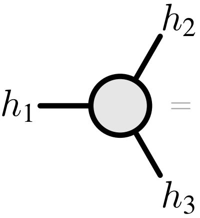
$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}$$



$$= f(\lambda_1\widetilde{\lambda}_1, \lambda_2\widetilde{\lambda}_2, \lambda_3\widetilde{\lambda}_3)\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})$$

$$\widetilde{\lambda} \equiv \begin{pmatrix} \widetilde{\lambda}_1^{\dot{1}} & \widetilde{\lambda}_2^{\dot{1}} & \widetilde{\lambda}_3^{\dot{1}} \\ \widetilde{\lambda}_1^{\dot{2}} & \widetilde{\lambda}_2^{\dot{2}} & \widetilde{\lambda}_3^{\dot{2}} \end{pmatrix}$$

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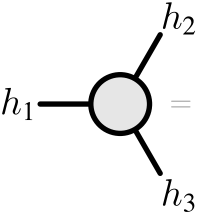

$$= f(\lambda_1\widetilde{\lambda}_1, \lambda_2\widetilde{\lambda}_2, \lambda_3\widetilde{\lambda}_3)\delta^{2\times2}(\lambda\cdot\widetilde{\lambda}) \Rightarrow \begin{cases} \lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \\ \\ \widetilde{\lambda} \equiv \begin{pmatrix} \widetilde{\lambda}_1^{\dot{1}} & \widetilde{\lambda}_2^{\dot{1}} & \widetilde{\lambda}_3^{\dot{1}} \\ \widetilde{\lambda}_1^{\dot{2}} & \widetilde{\lambda}_2^{\dot{2}} & \widetilde{\lambda}_3^{\dot{2}} \end{pmatrix} \end{cases}$$

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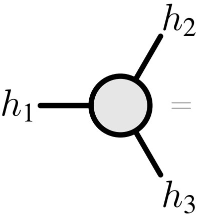

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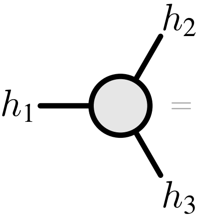

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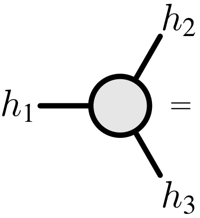

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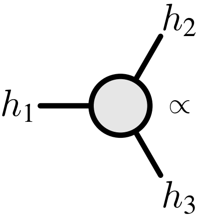

$$\propto \begin{cases} f(\lambda_1, \lambda_2, \lambda_3) \\ \\ f(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3) \end{cases}$$

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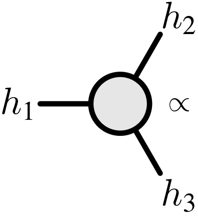

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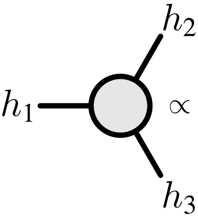

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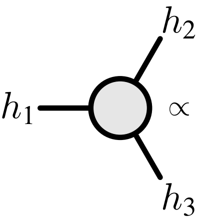

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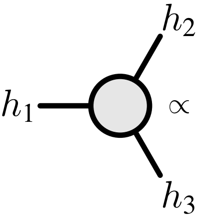

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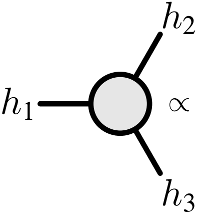

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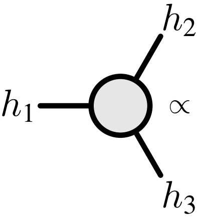

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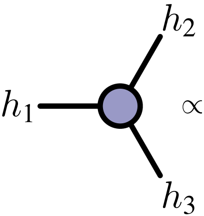

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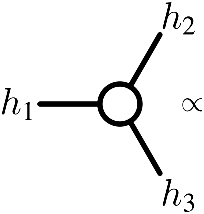

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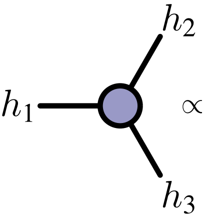

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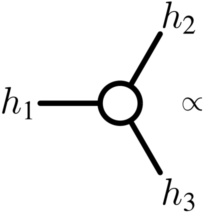

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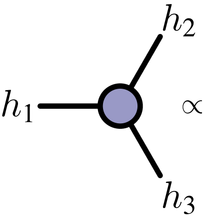

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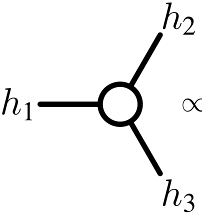

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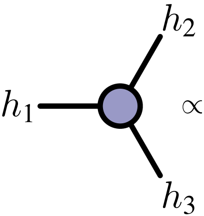

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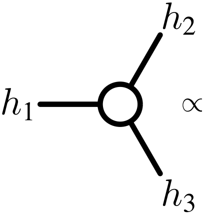

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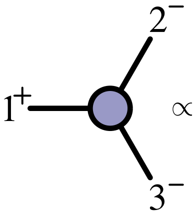

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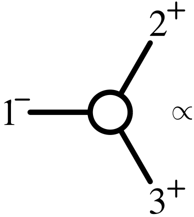

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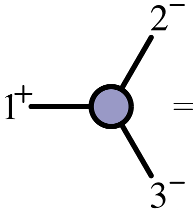
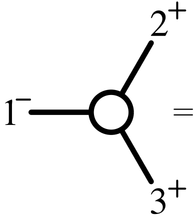


$$= \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})$$



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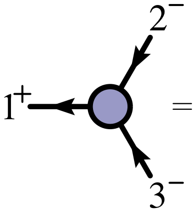

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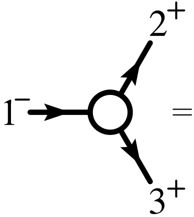

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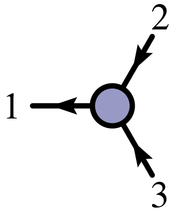

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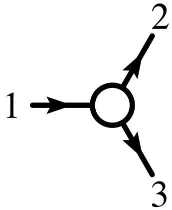

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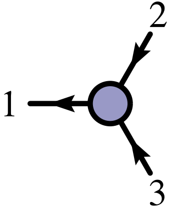

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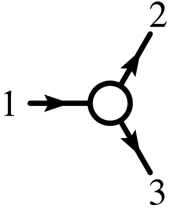

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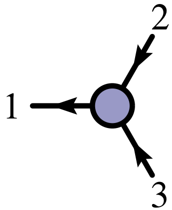

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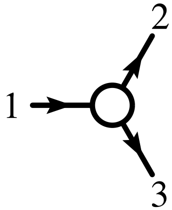

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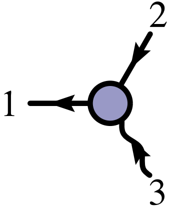

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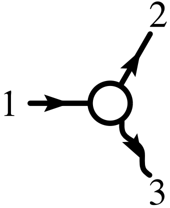

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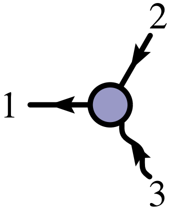

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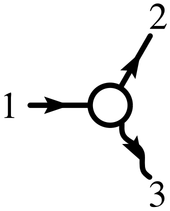

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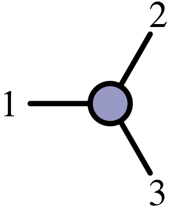

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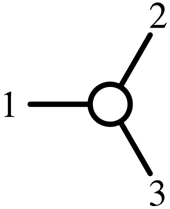

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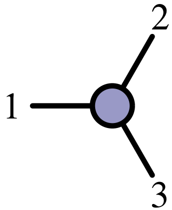

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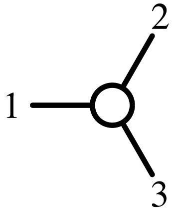

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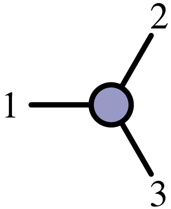

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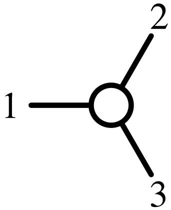

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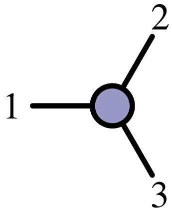

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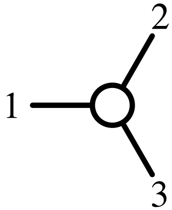

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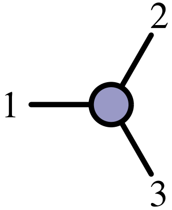

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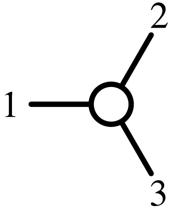

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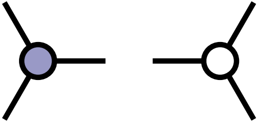

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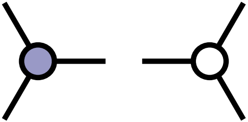

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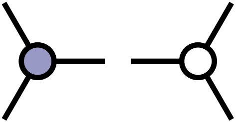

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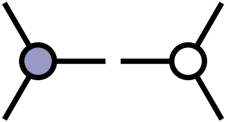

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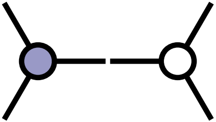

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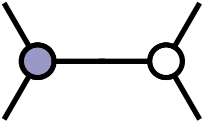

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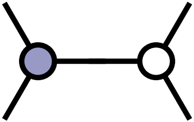

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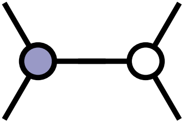

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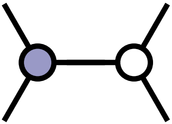

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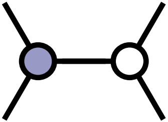

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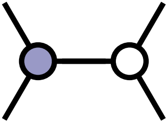

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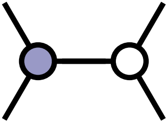

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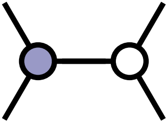

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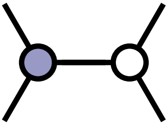

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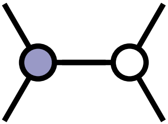

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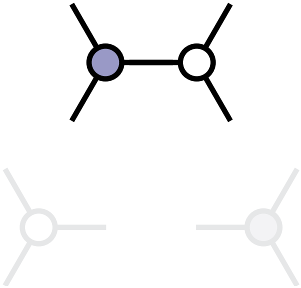

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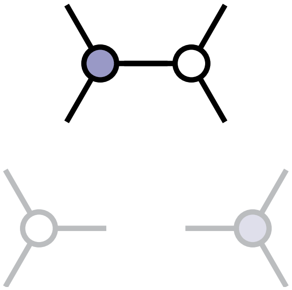

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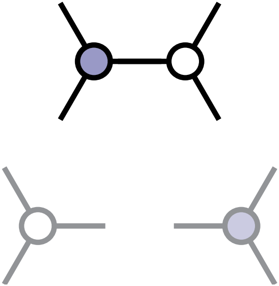

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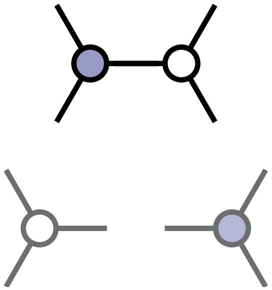

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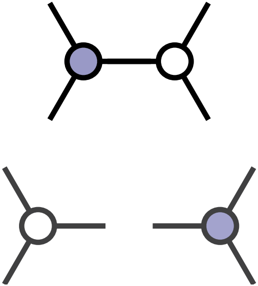

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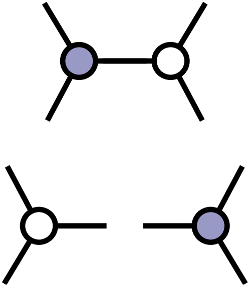

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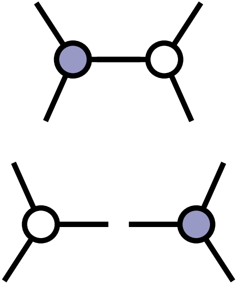

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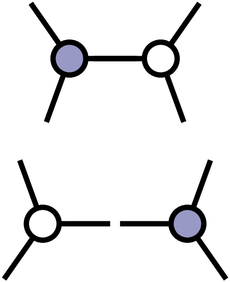

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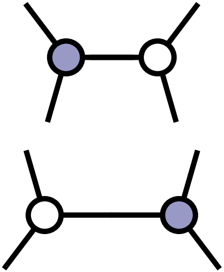

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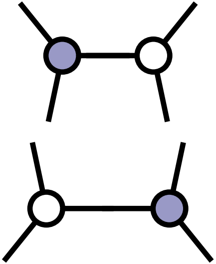

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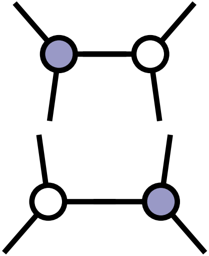

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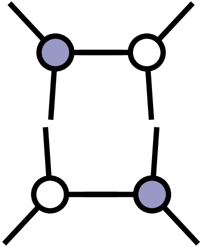

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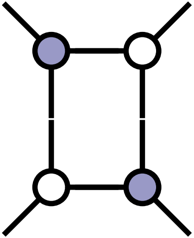

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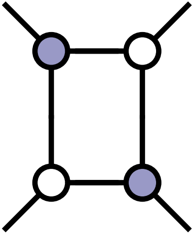

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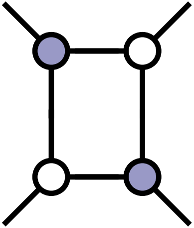

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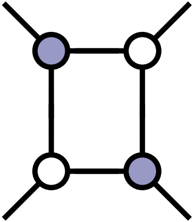

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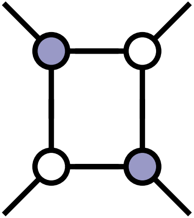

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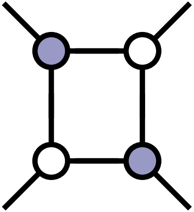

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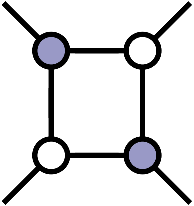

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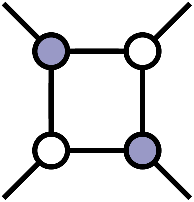

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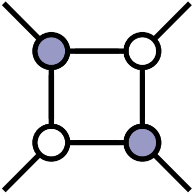

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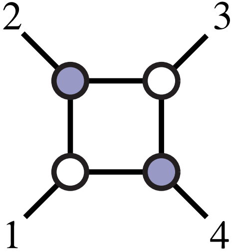

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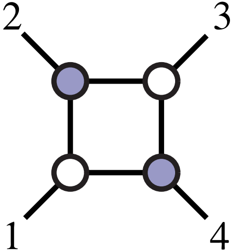

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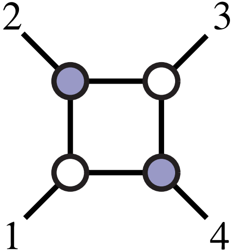

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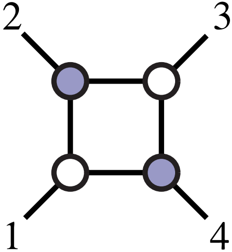

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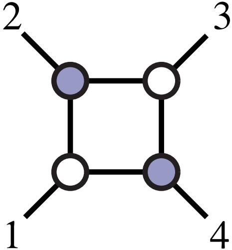

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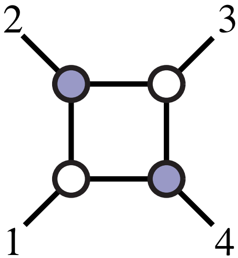

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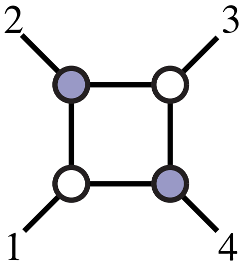

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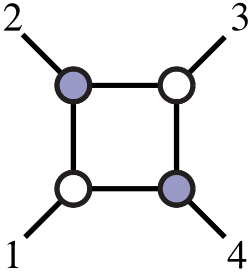

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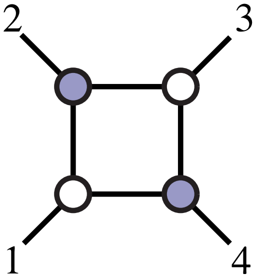

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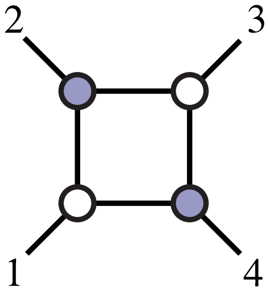

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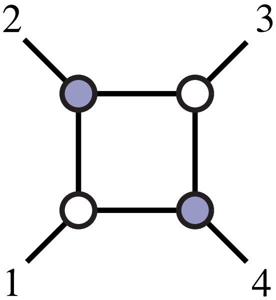

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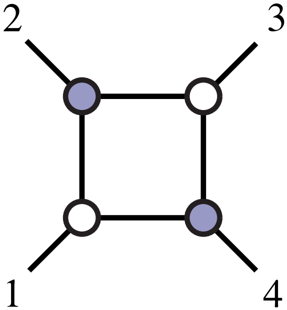

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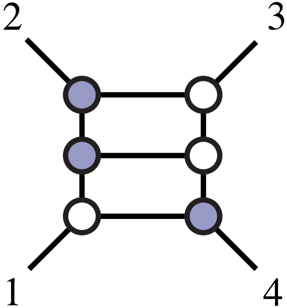

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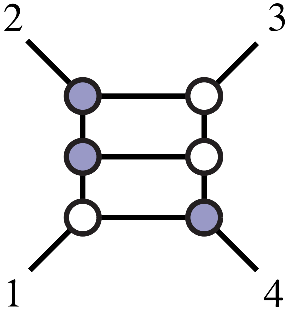

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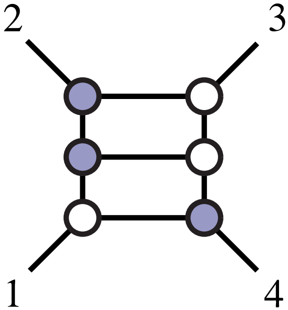

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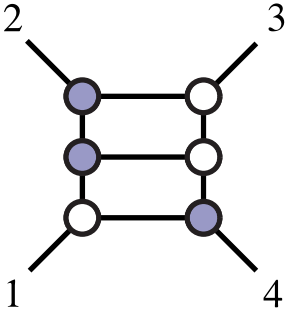

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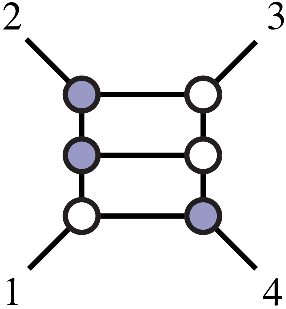

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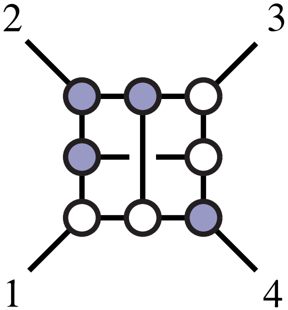

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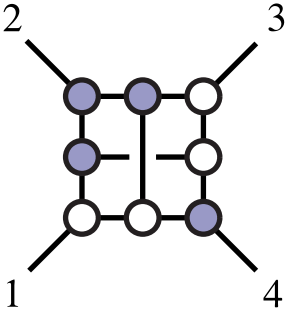

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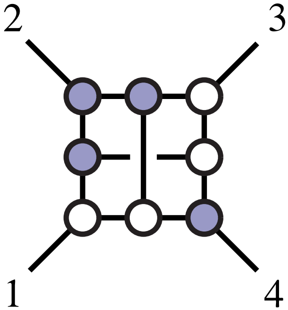

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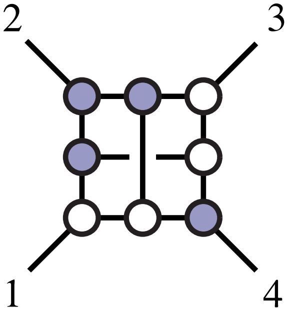

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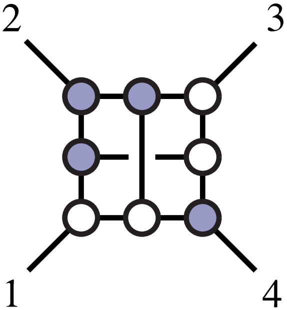

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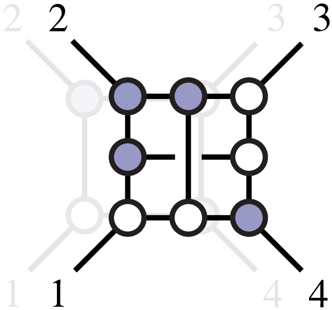

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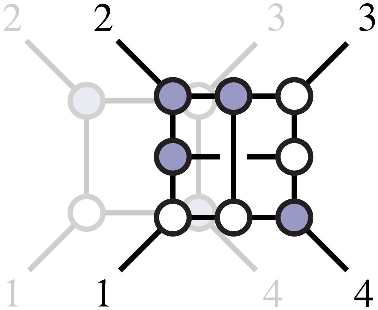

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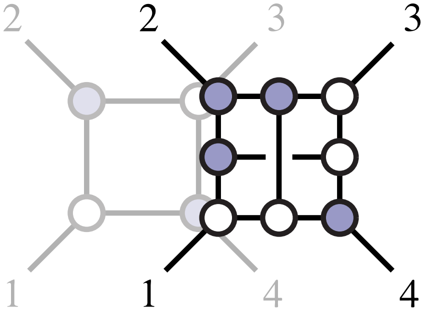

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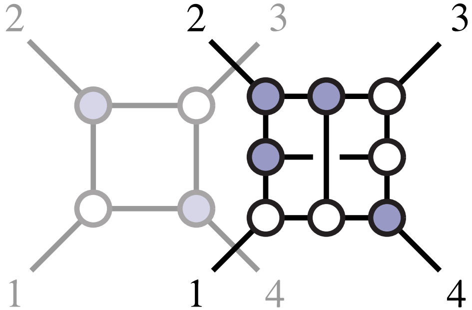

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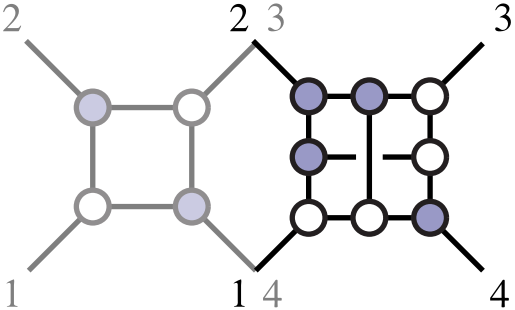

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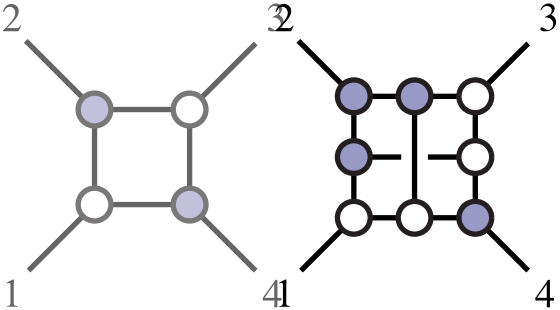

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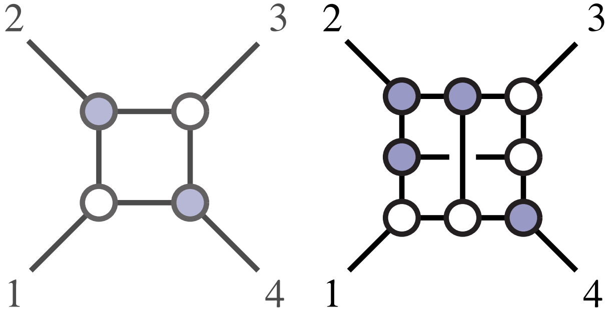

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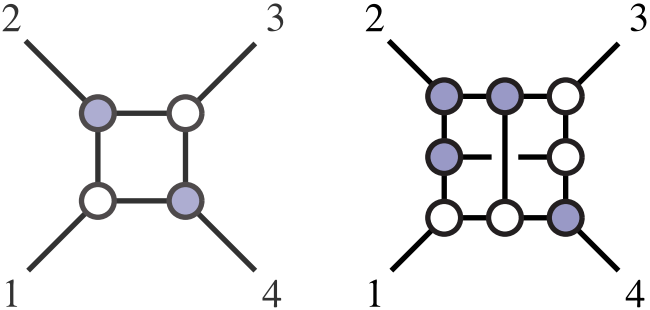

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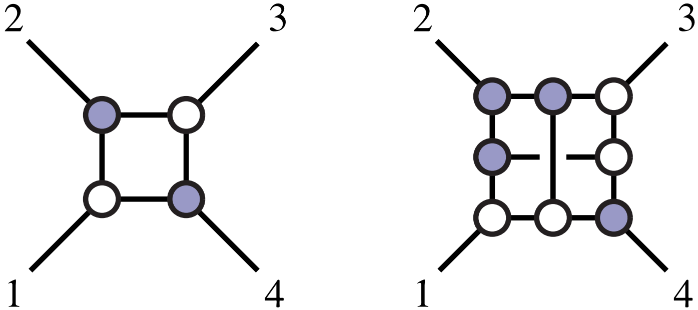

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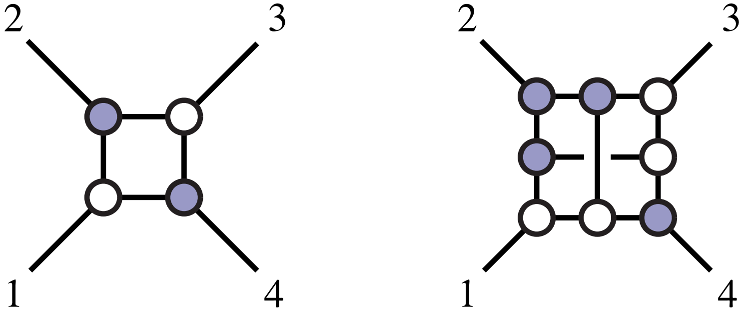

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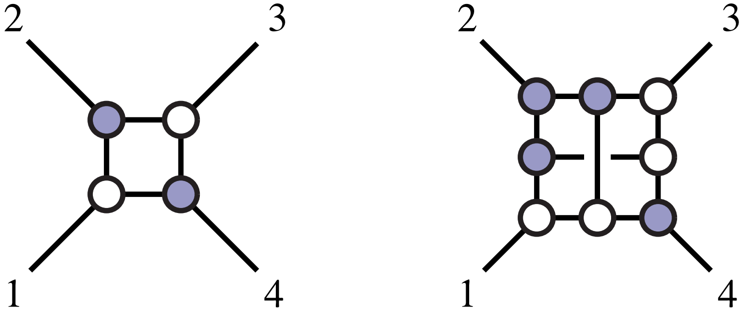

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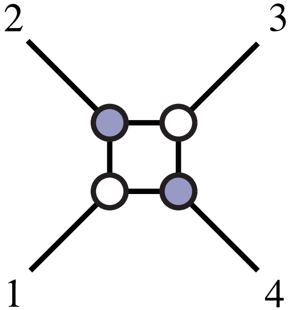

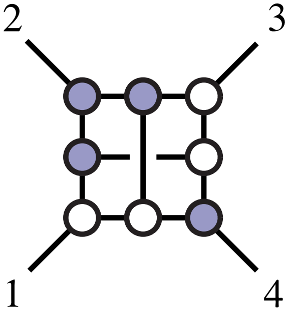

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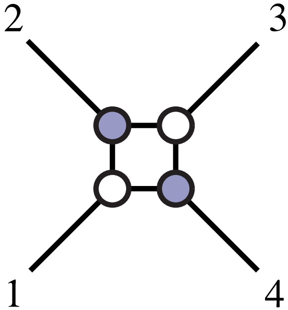

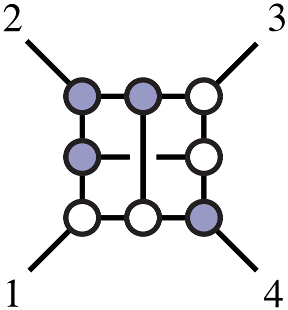

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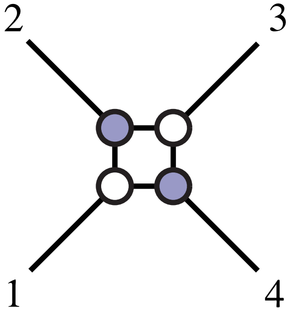

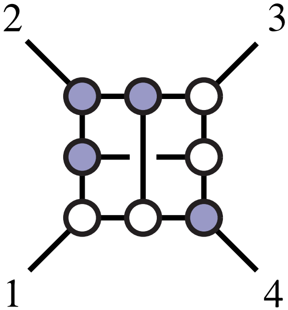

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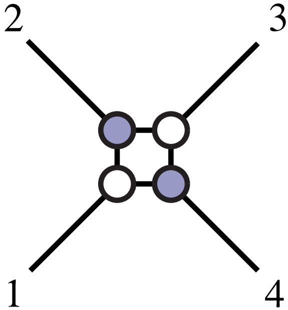

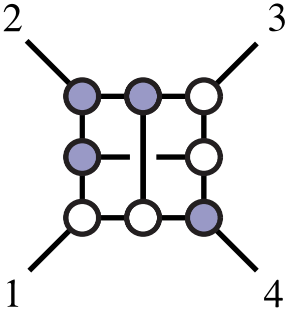

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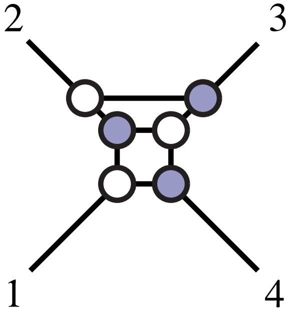

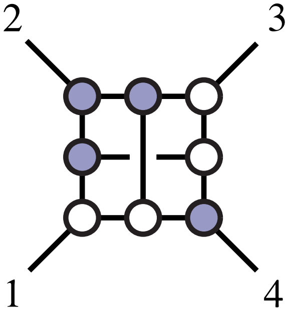

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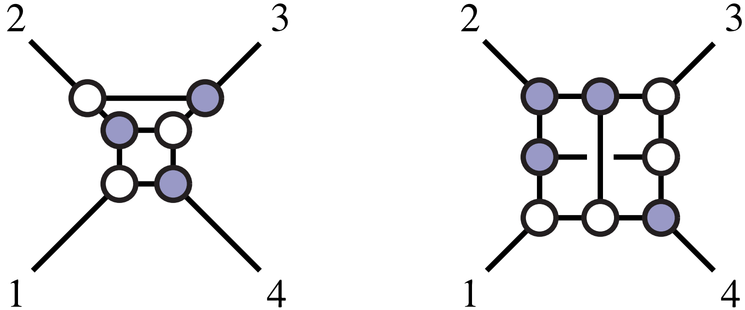

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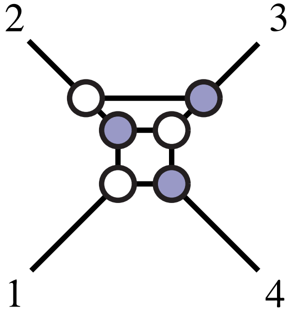

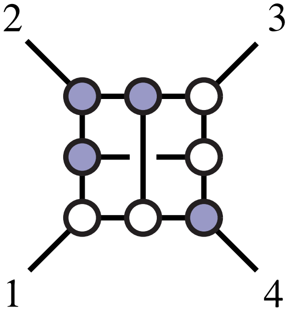

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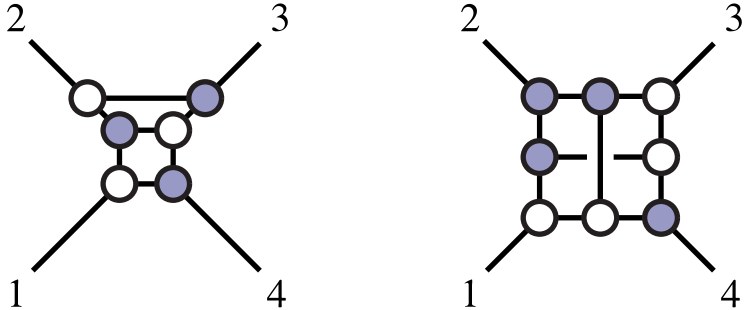

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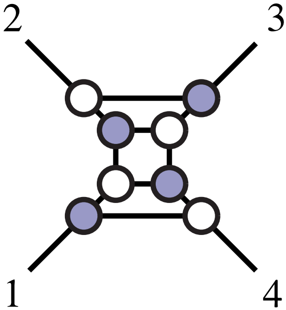

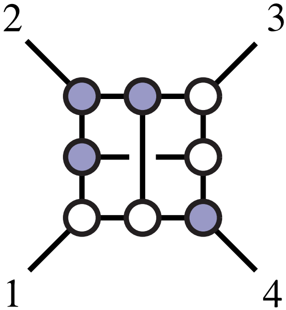

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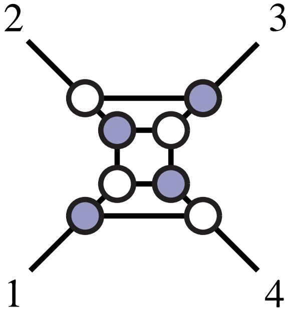

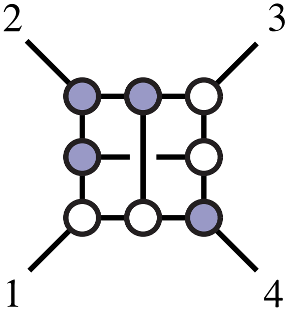

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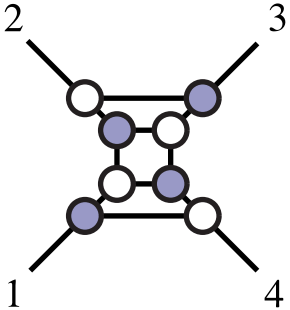

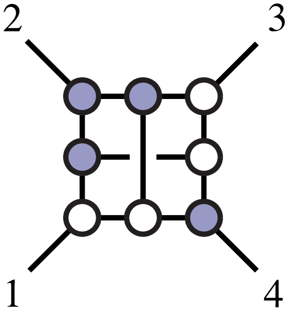

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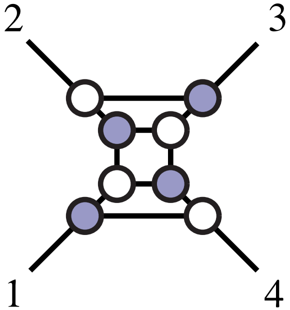

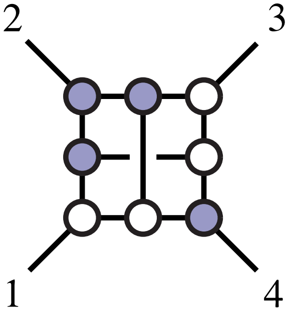

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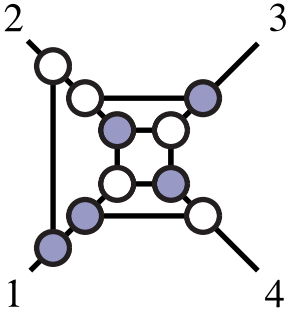

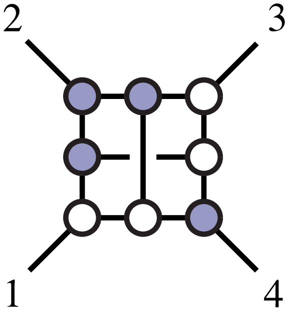

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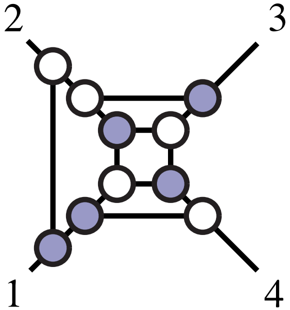

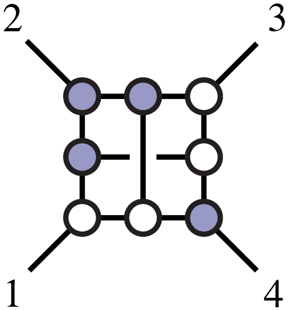

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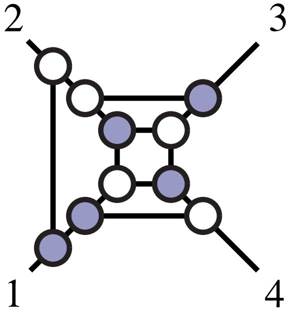

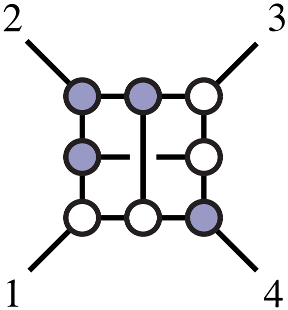

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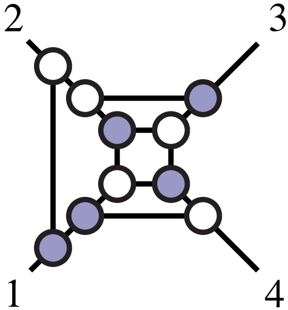

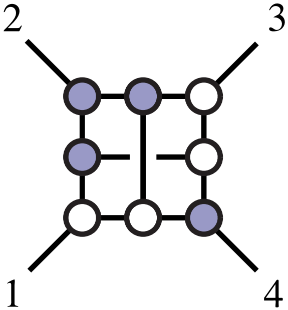

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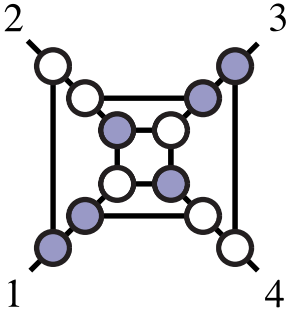

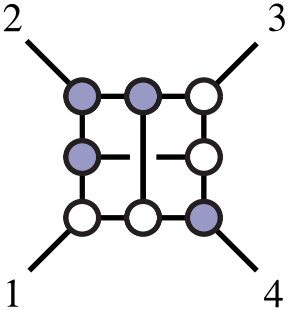

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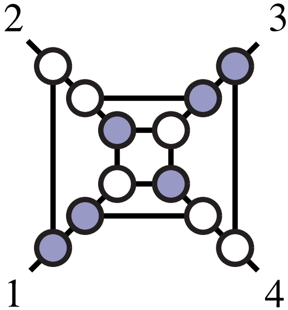

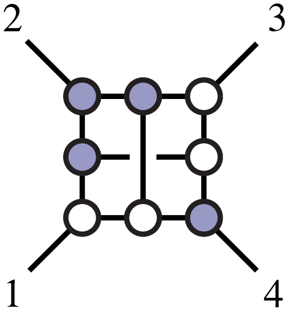

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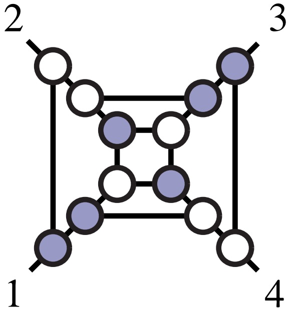

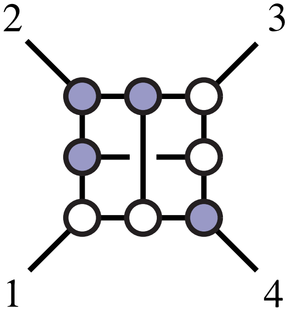

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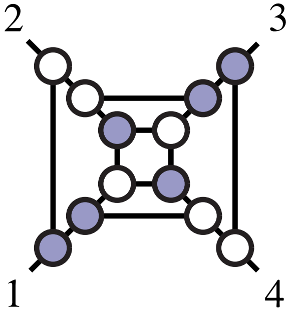

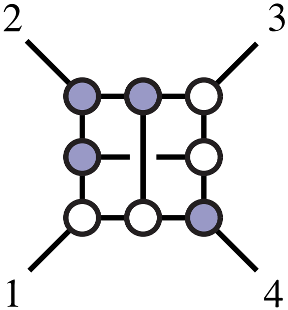

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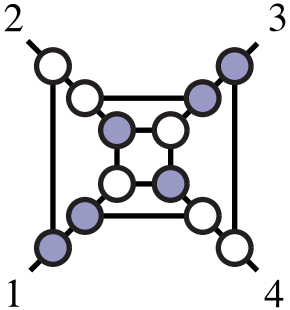

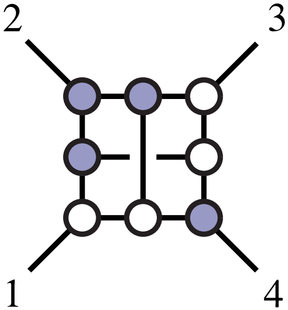

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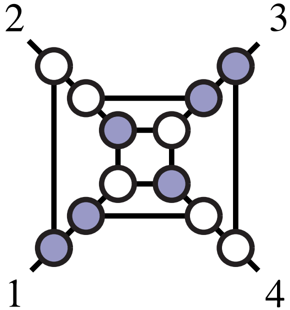

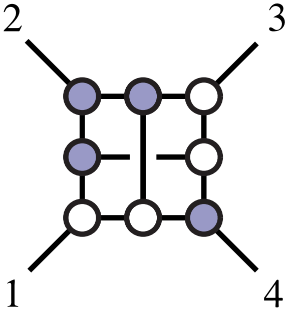

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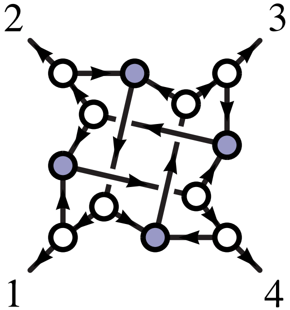

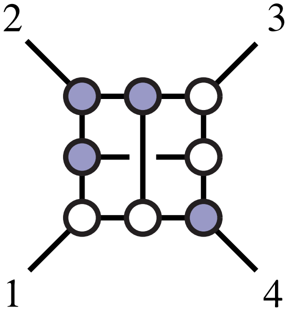

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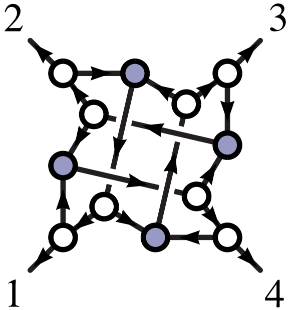

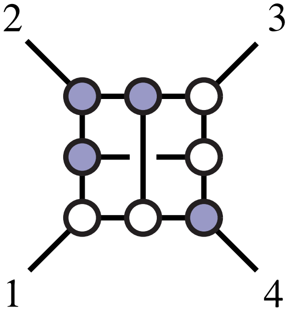

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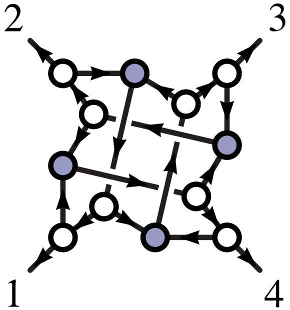

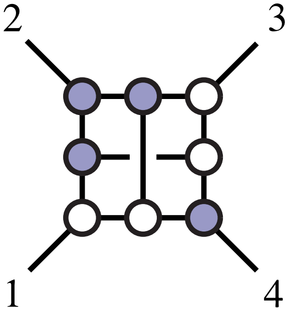

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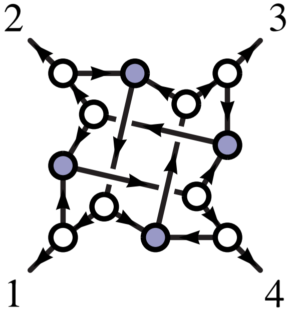

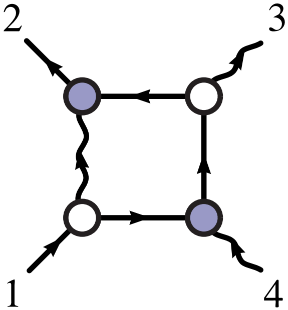

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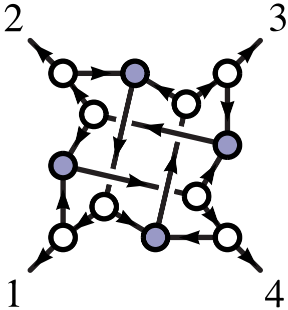

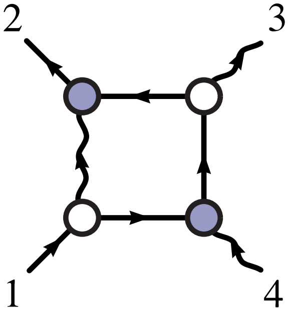

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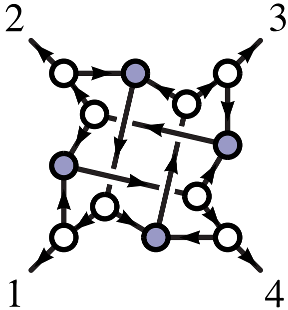

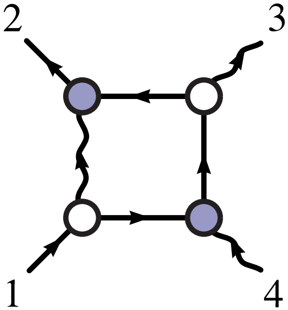

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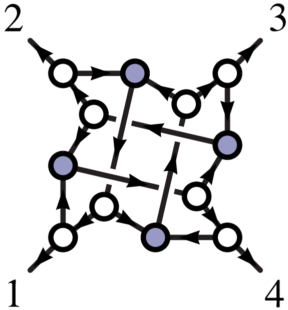

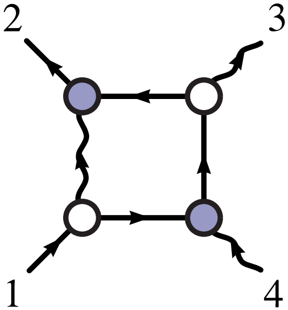

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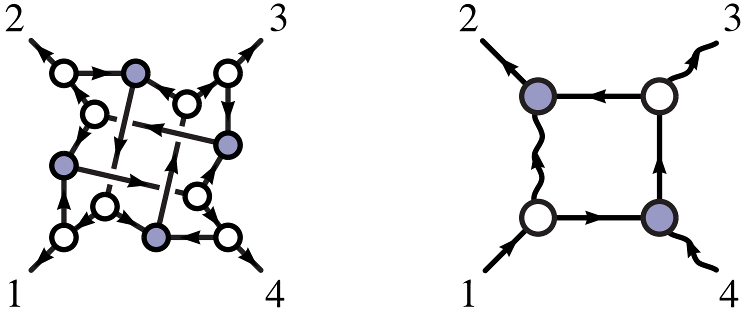

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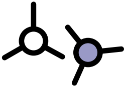

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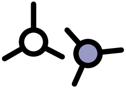

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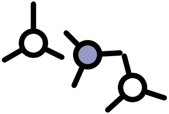

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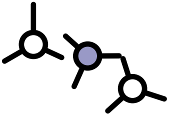

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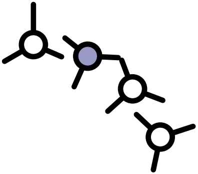

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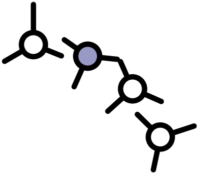

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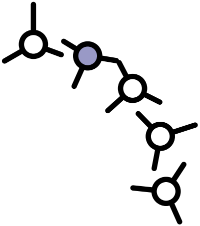

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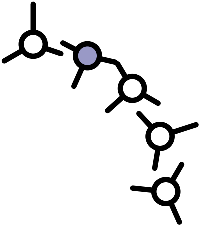

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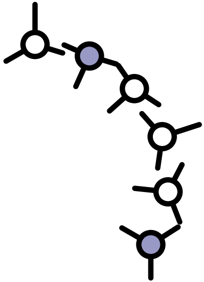

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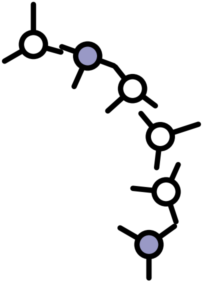

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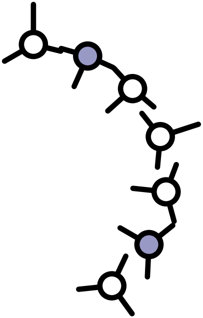

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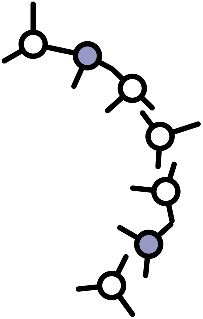

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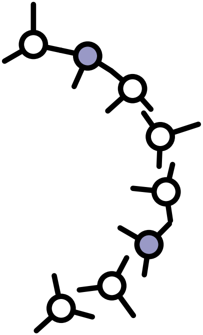

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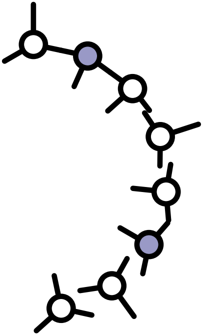

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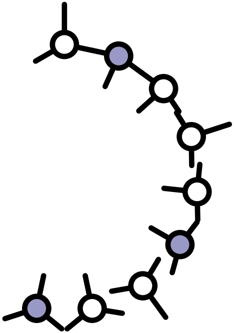

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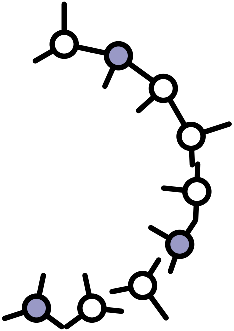

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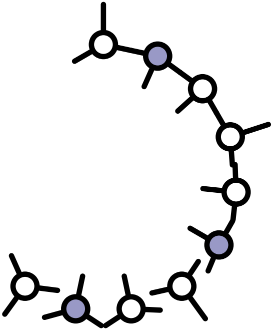

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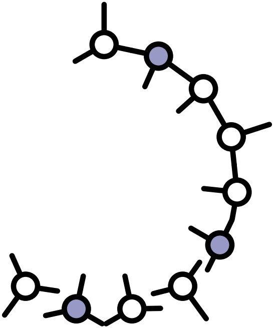

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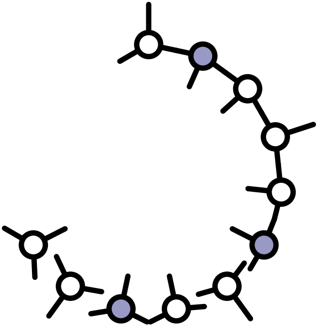

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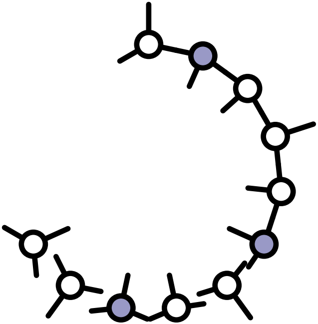

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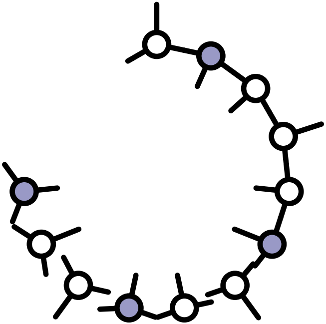

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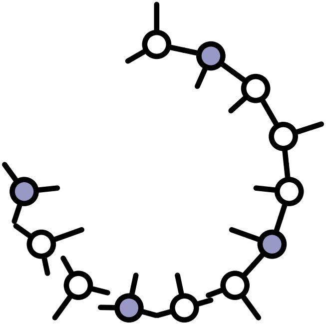

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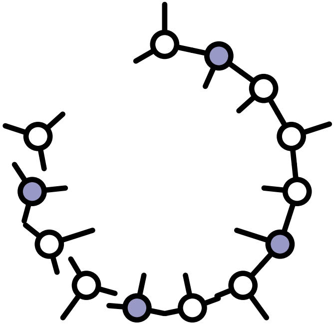

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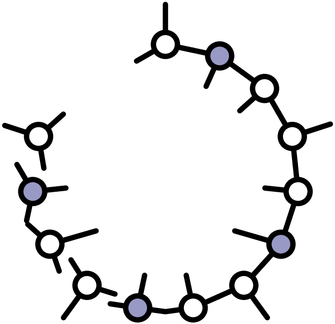

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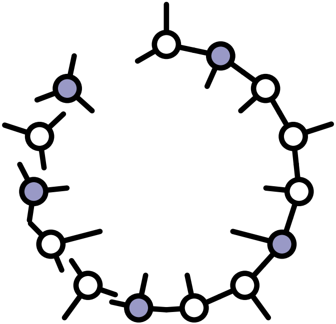

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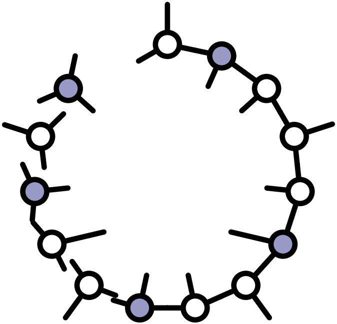

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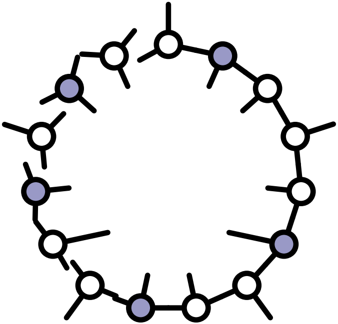

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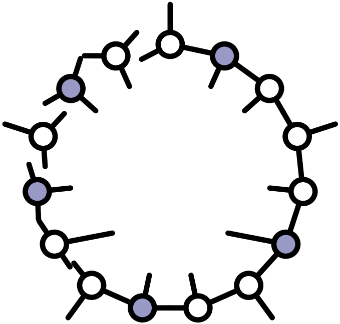

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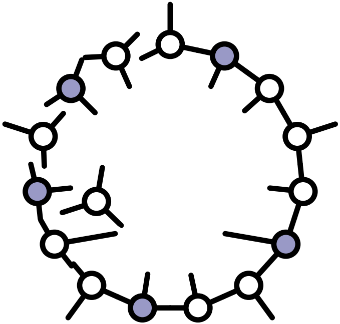

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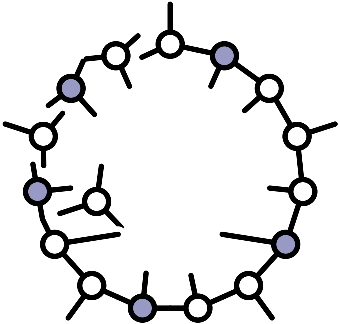

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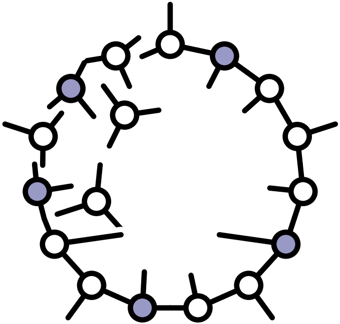

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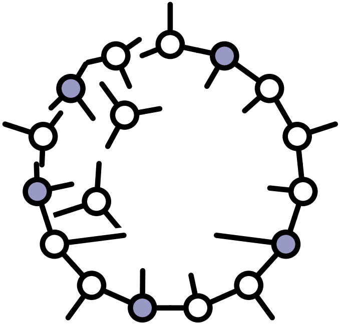

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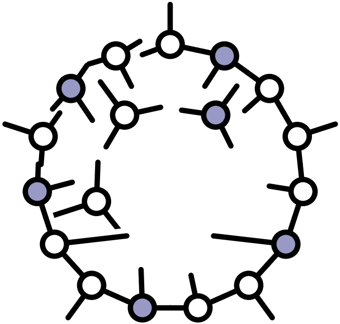

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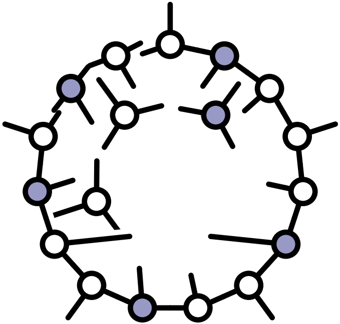

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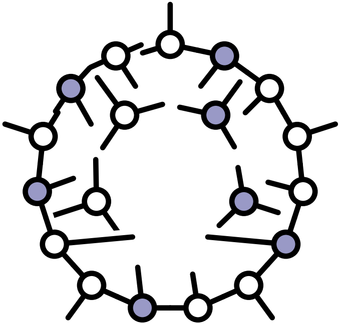

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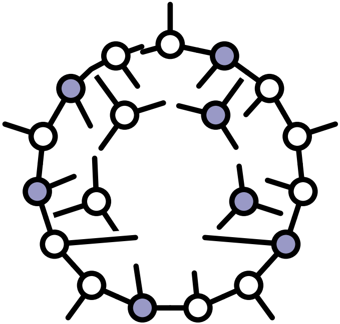

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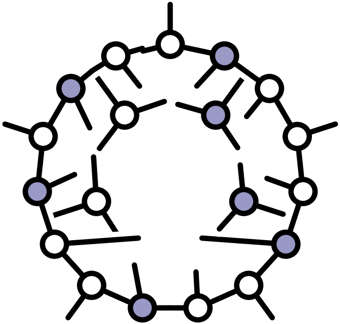

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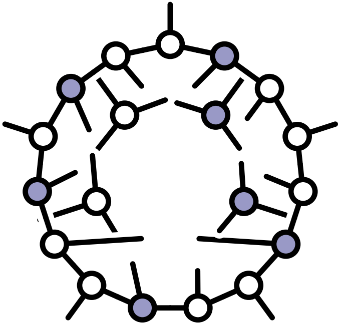

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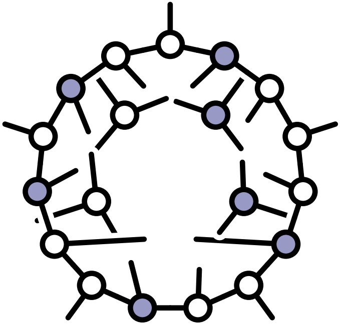

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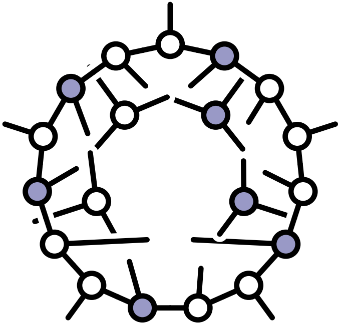

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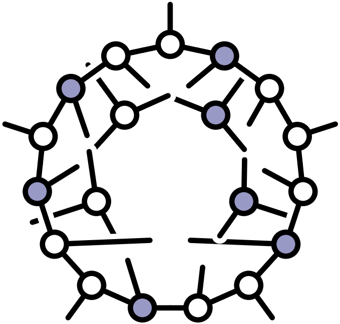

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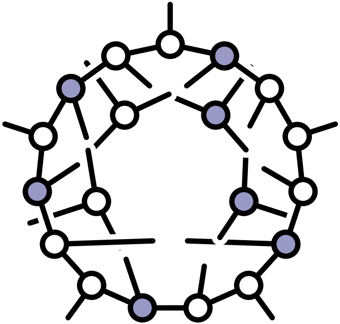

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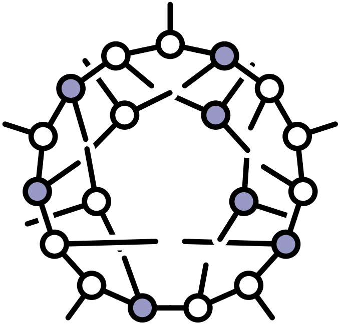

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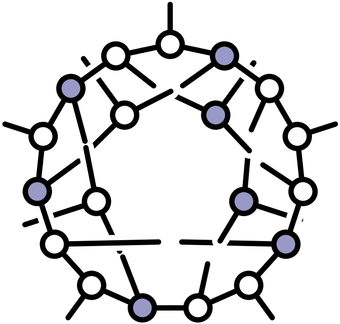

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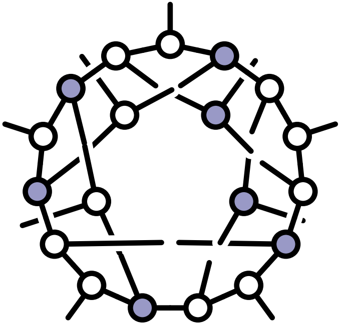

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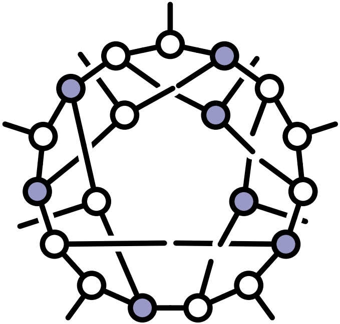

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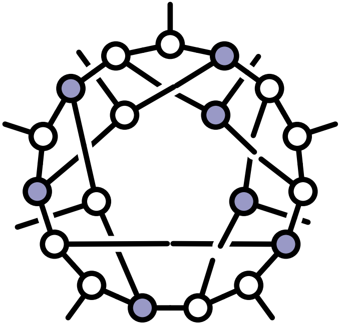

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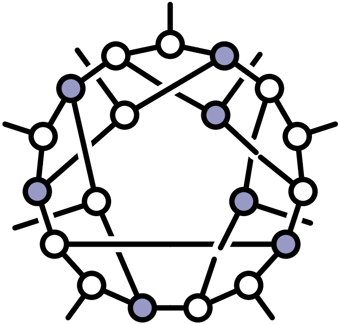

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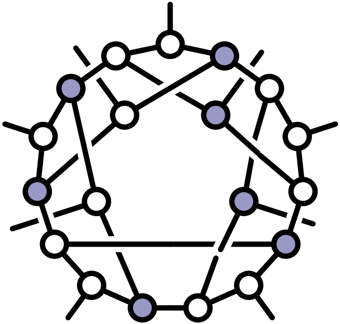

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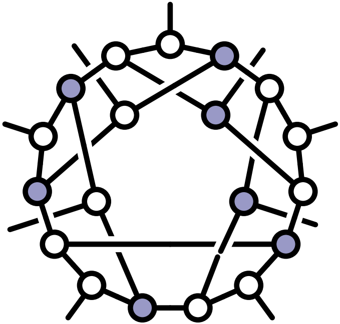

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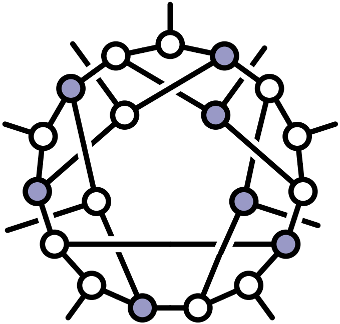

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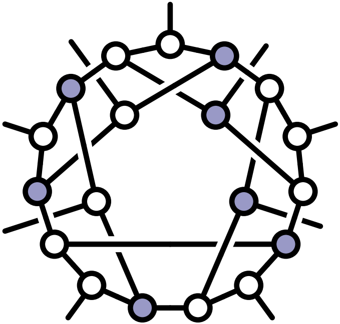

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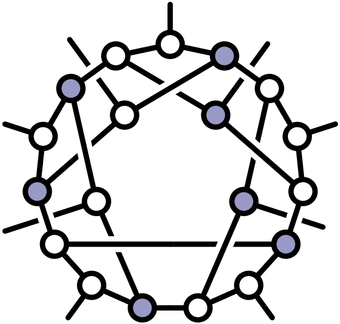

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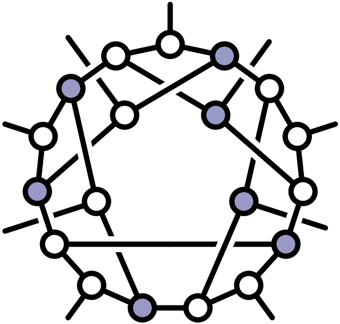

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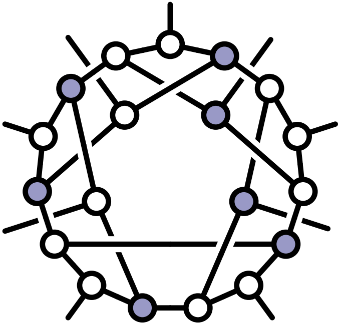

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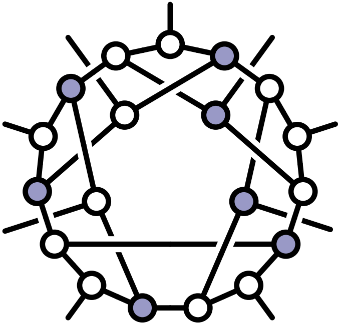

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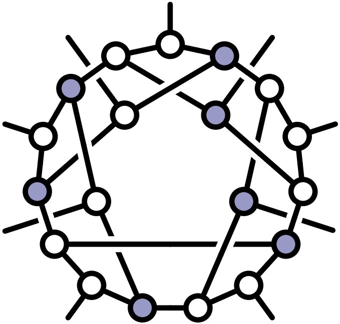

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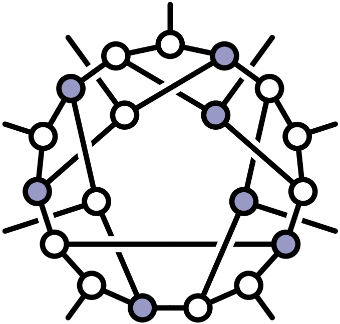

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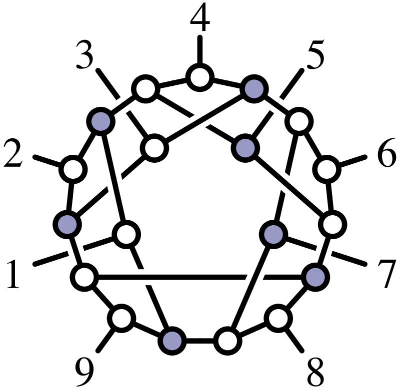

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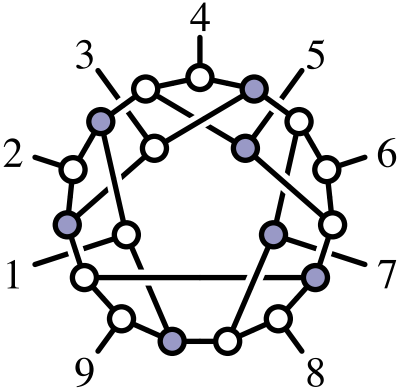

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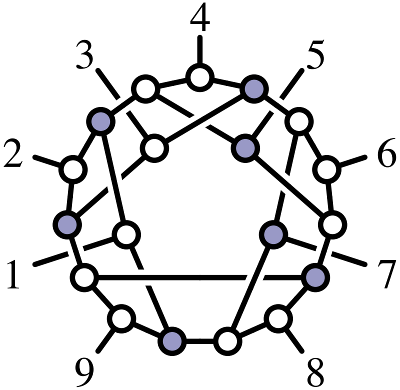

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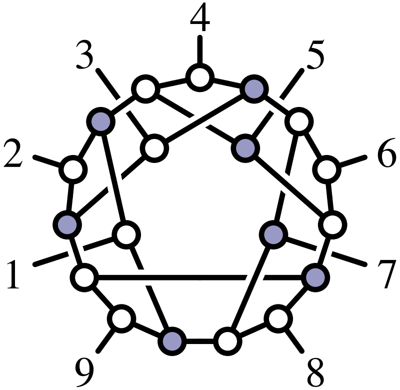

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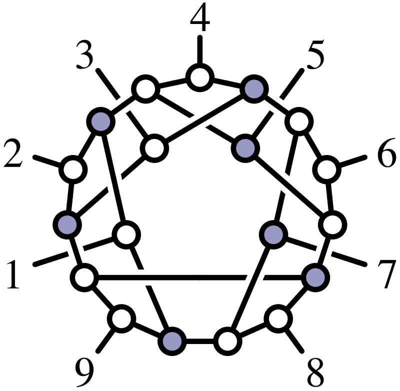

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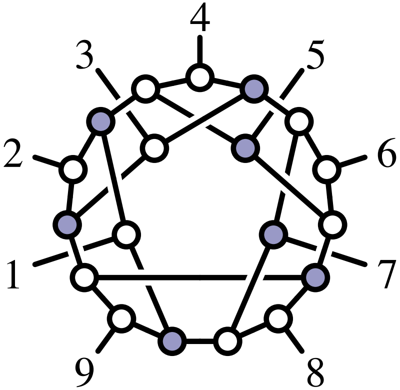

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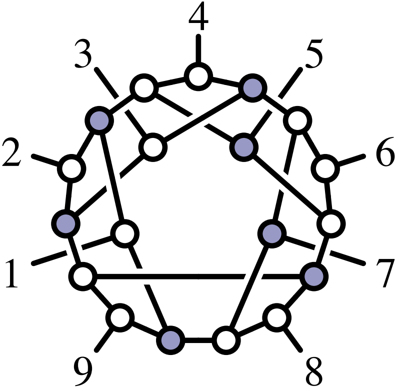

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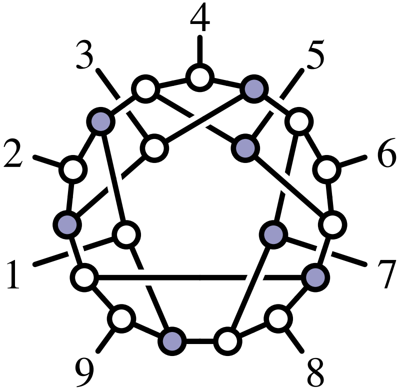

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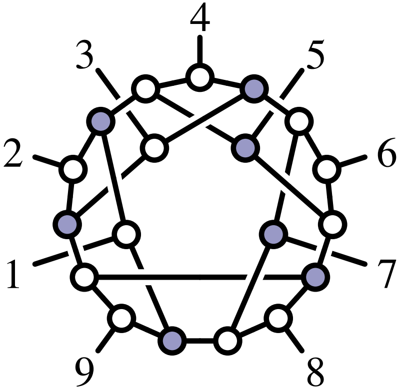

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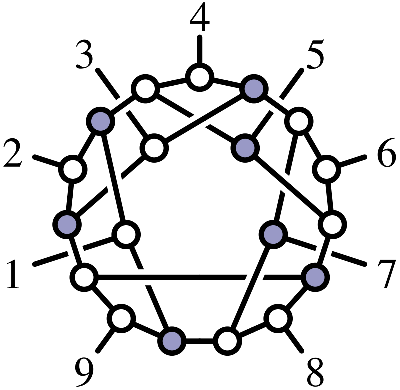

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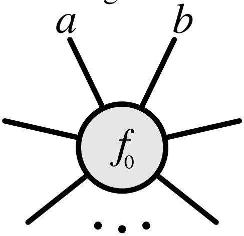

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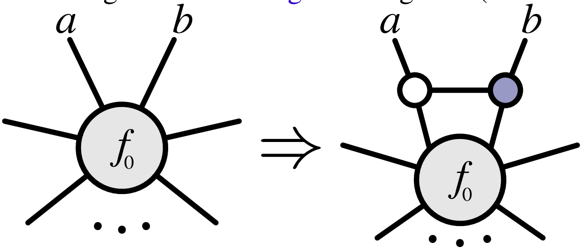

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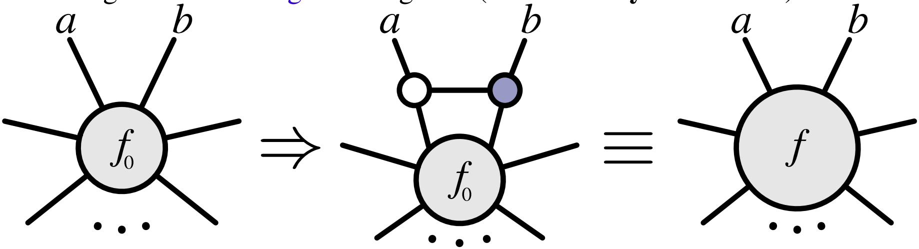

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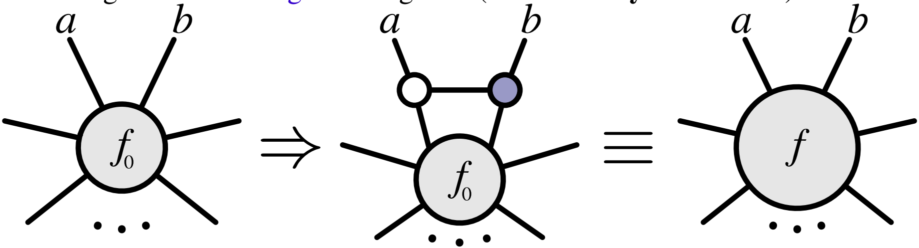

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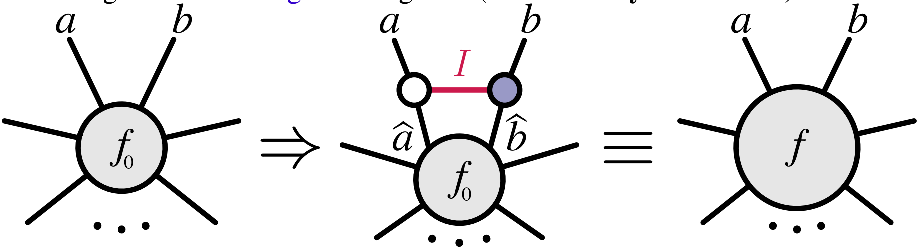

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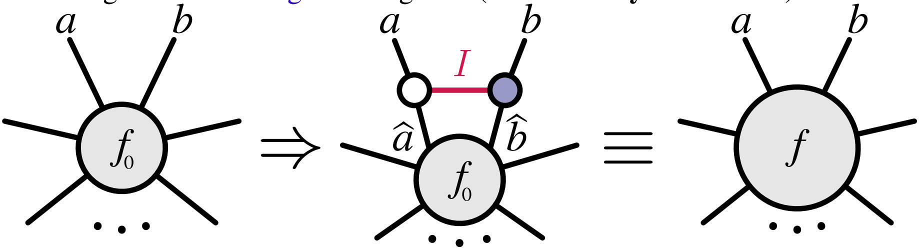

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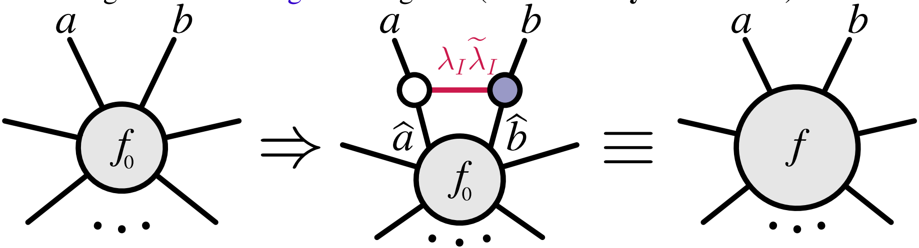

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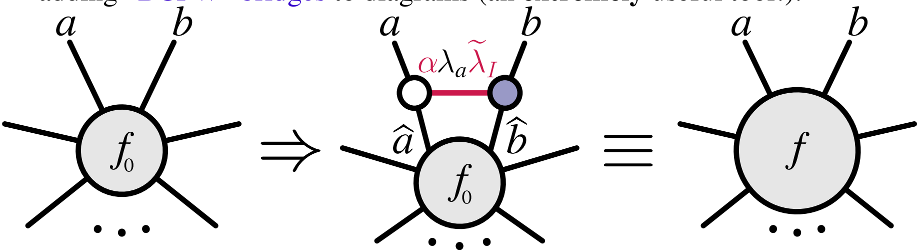

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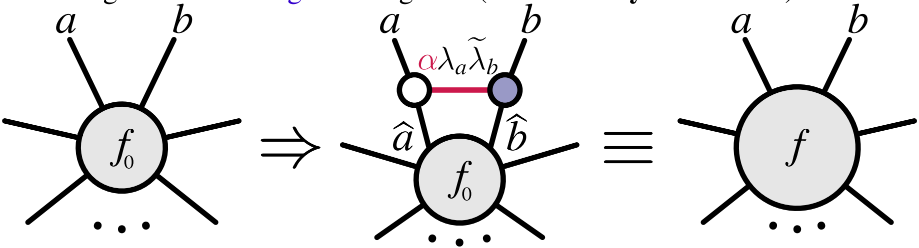

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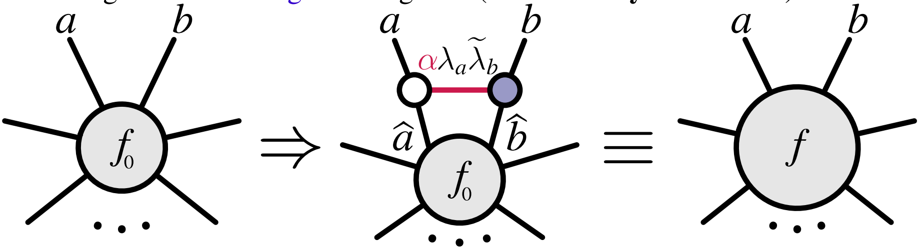

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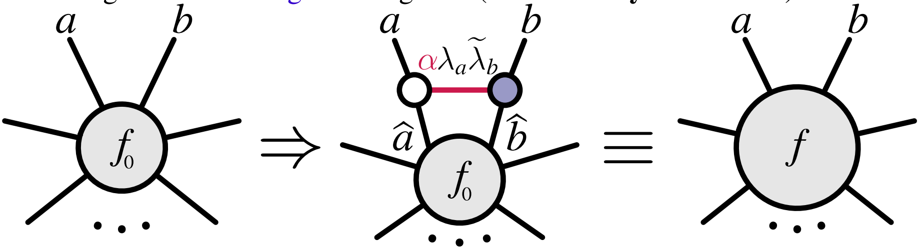

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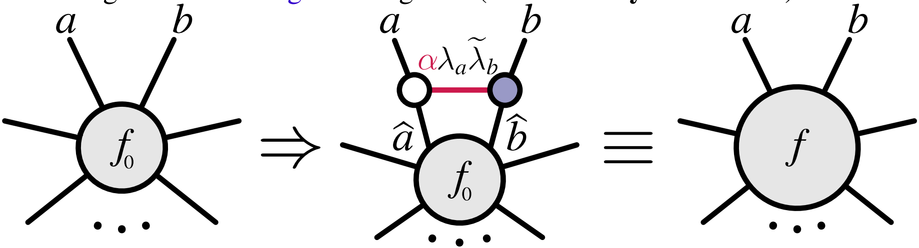

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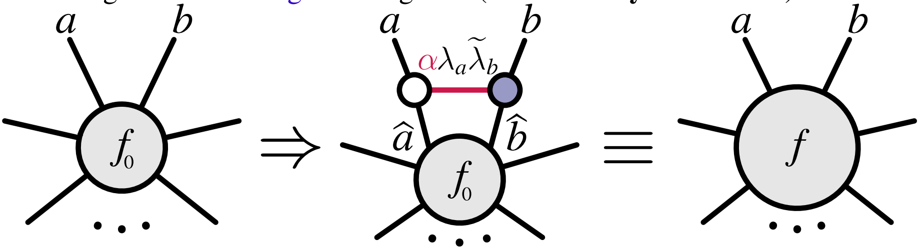

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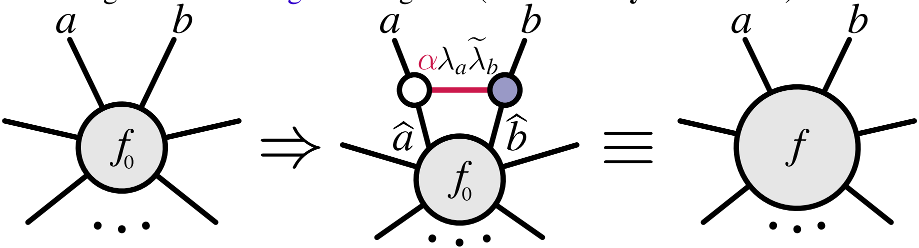

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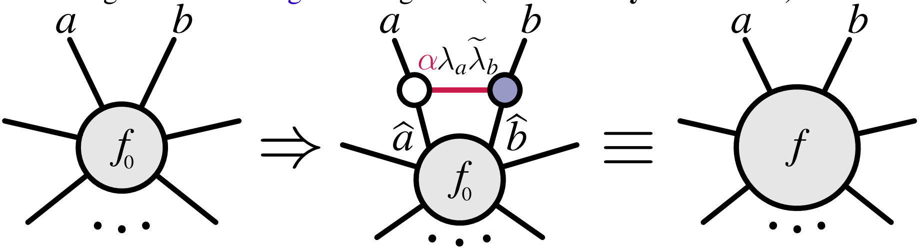

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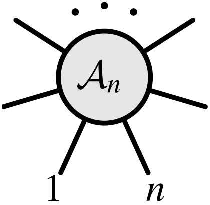

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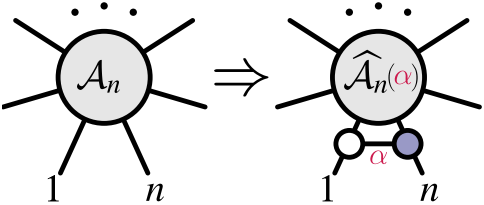

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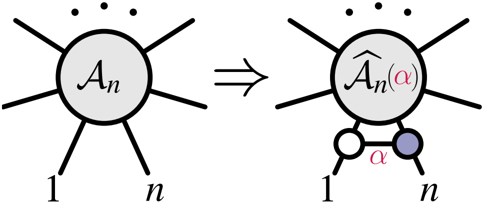

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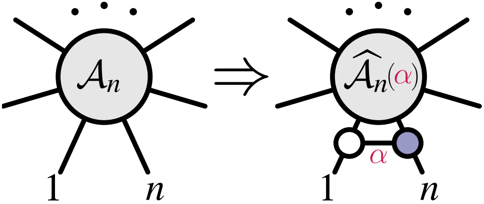

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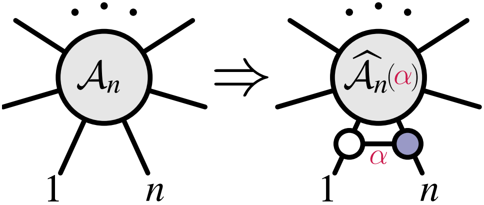

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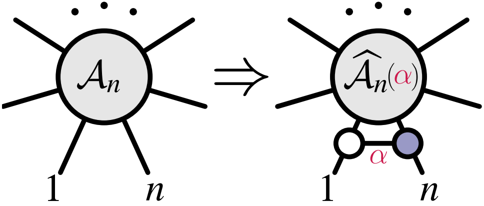

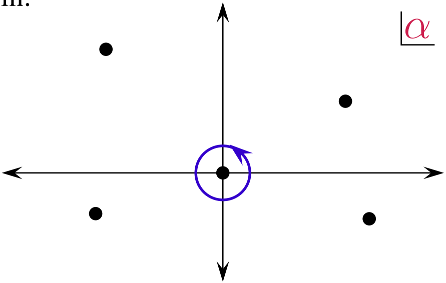

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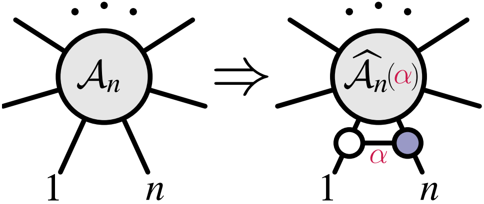

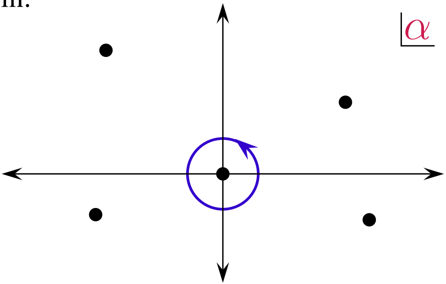

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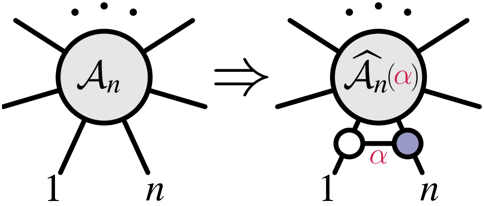

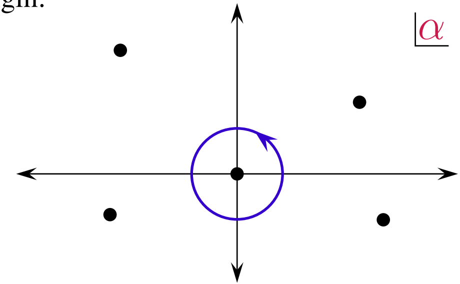

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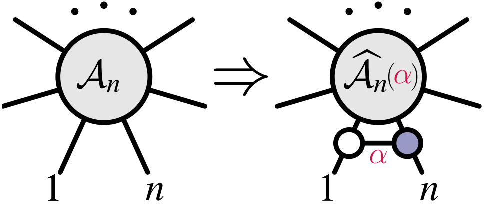

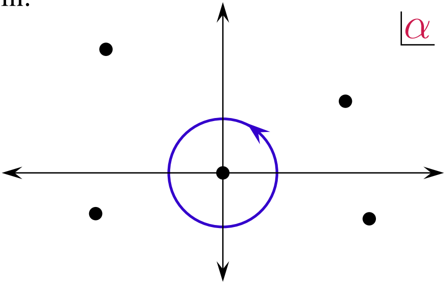

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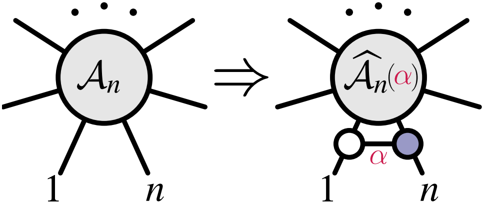

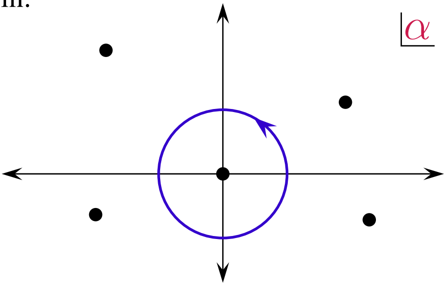

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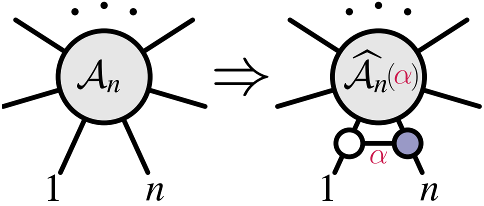

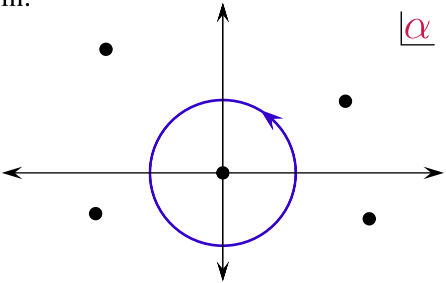

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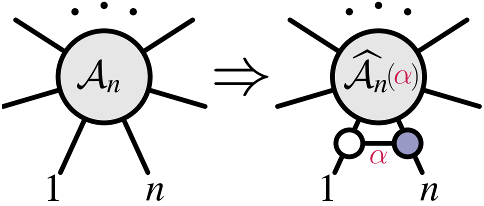

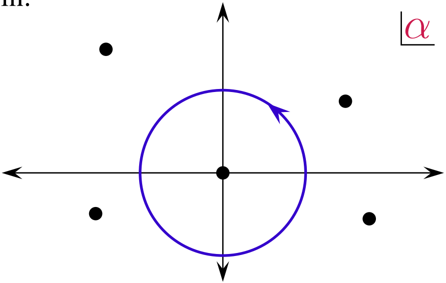

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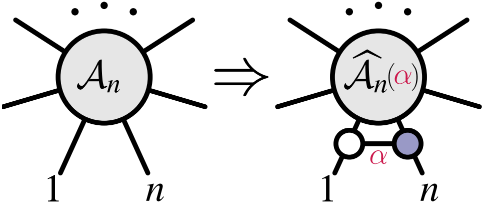

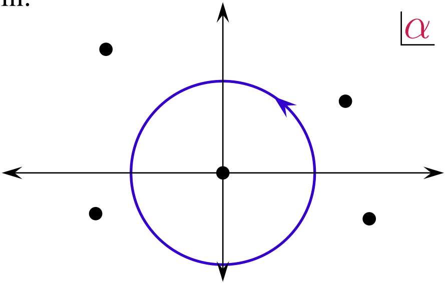

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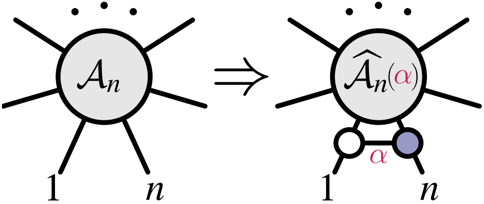

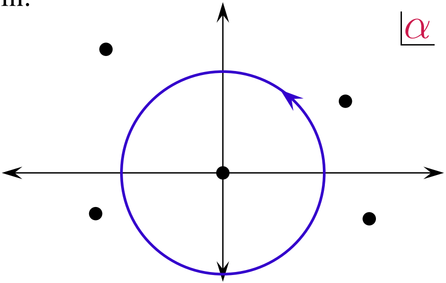

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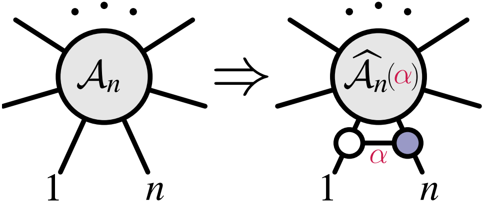

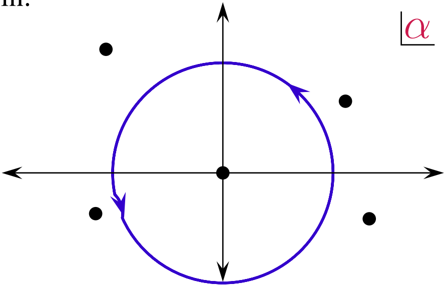

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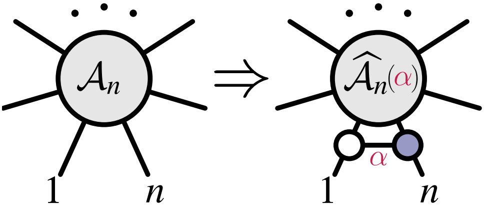

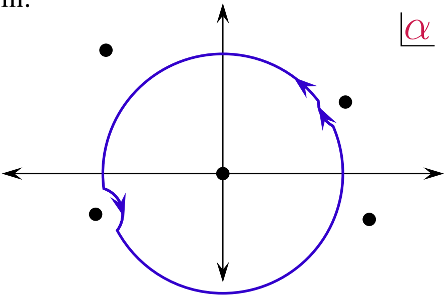

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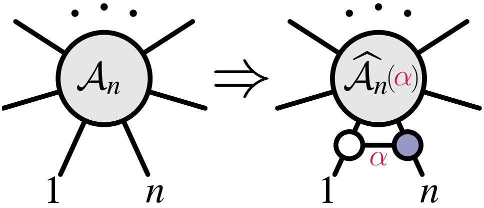

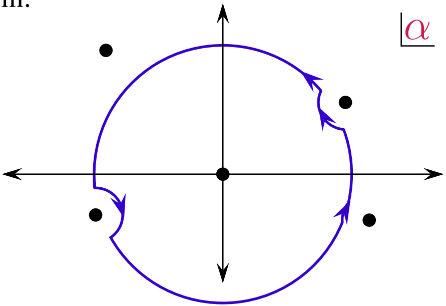

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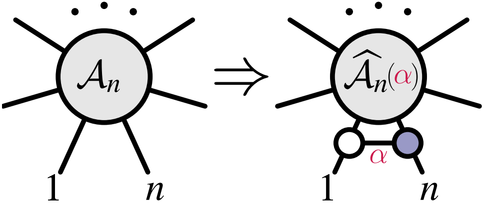

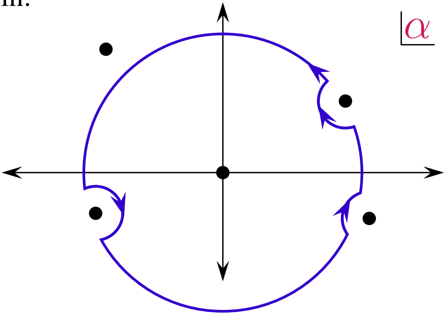

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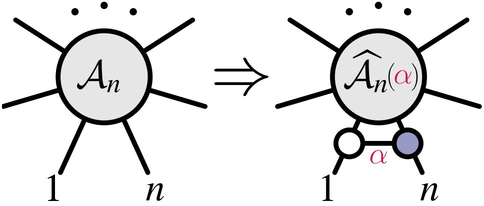

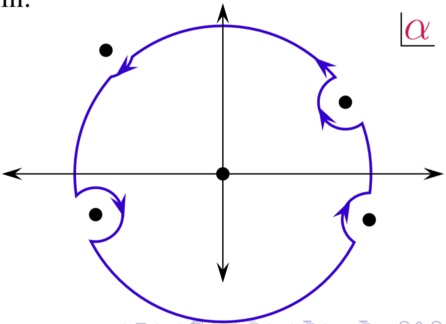

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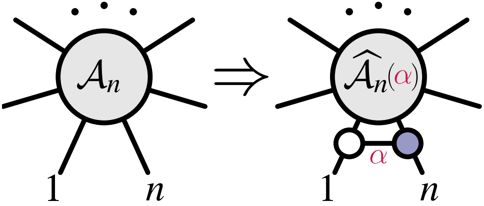

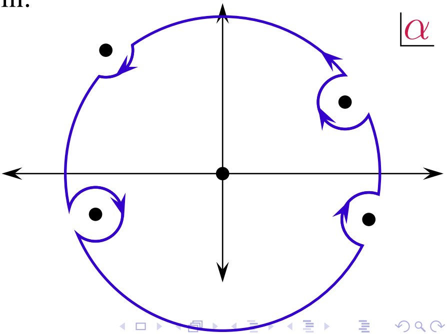

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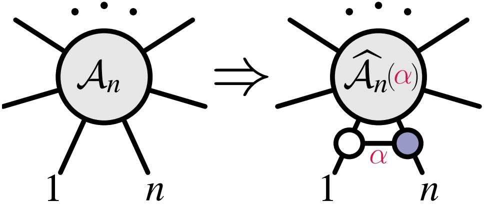

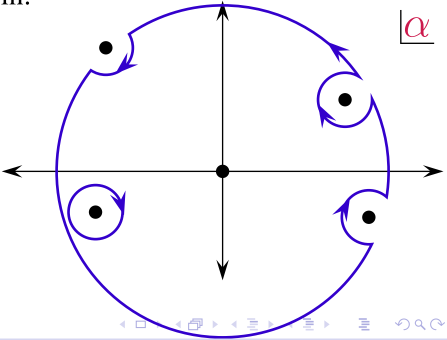

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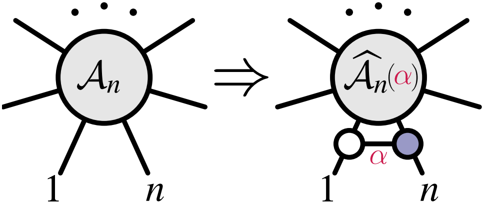

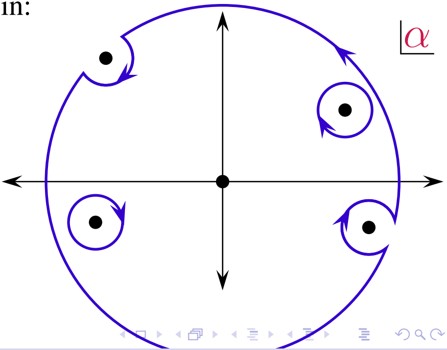

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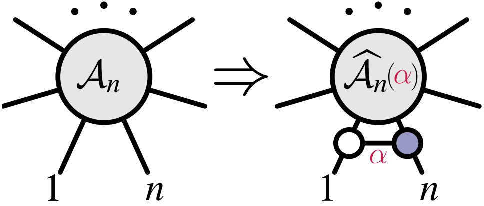

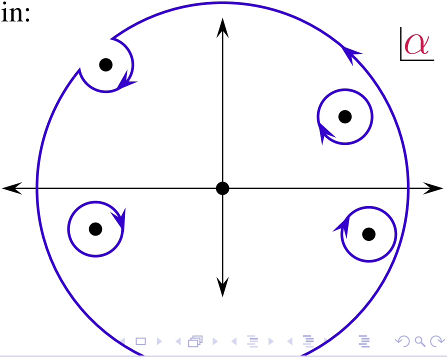

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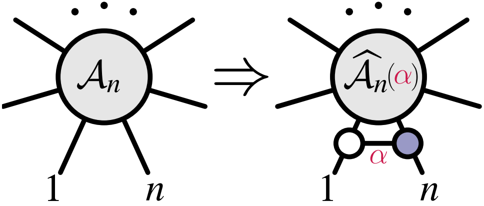

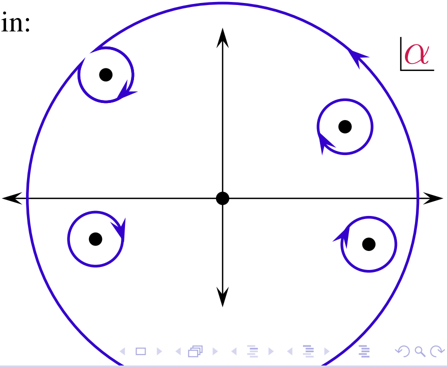

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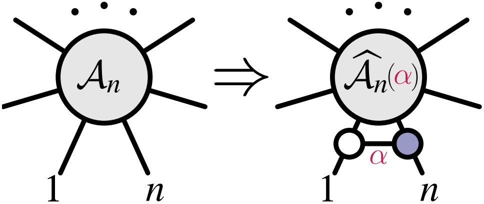

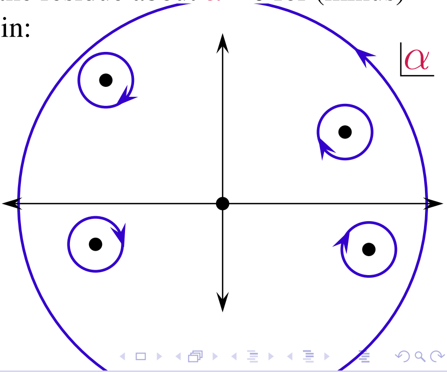

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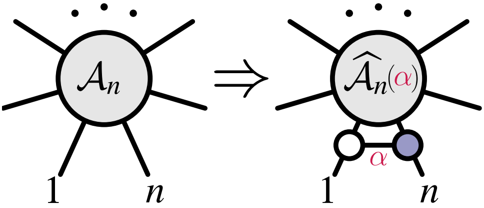

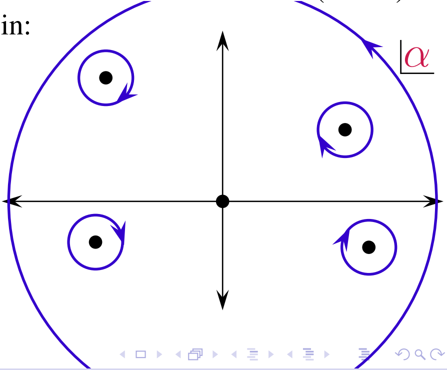

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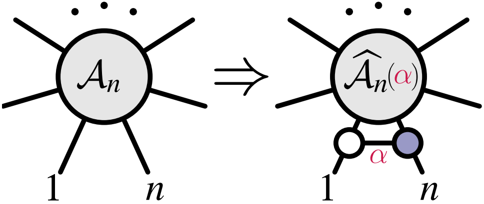

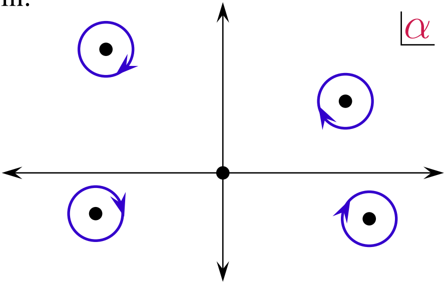

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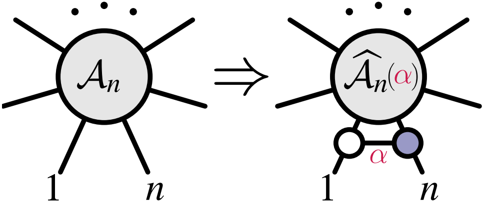

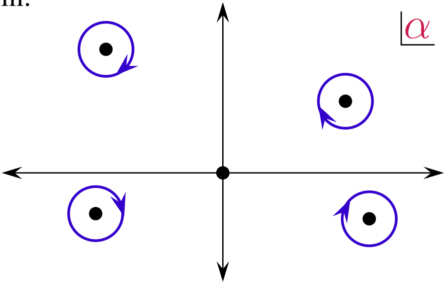

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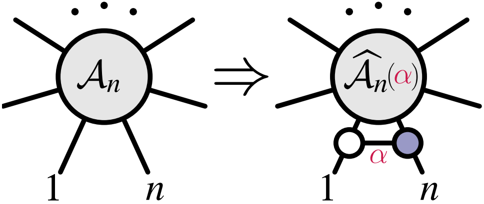

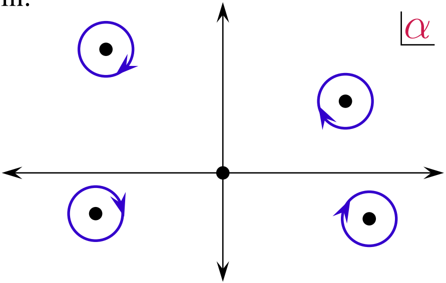

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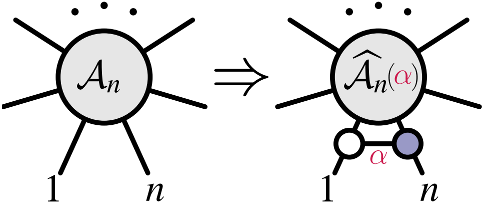

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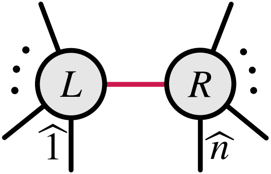

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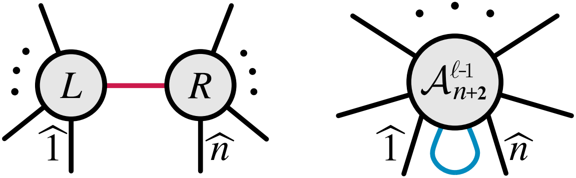

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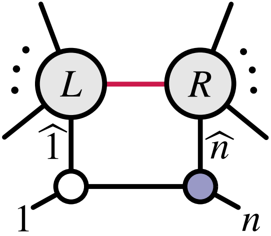

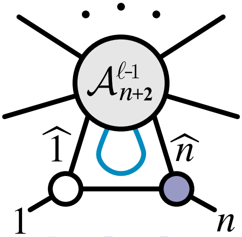

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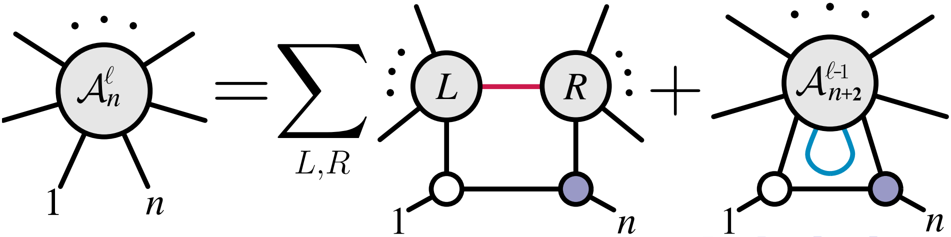

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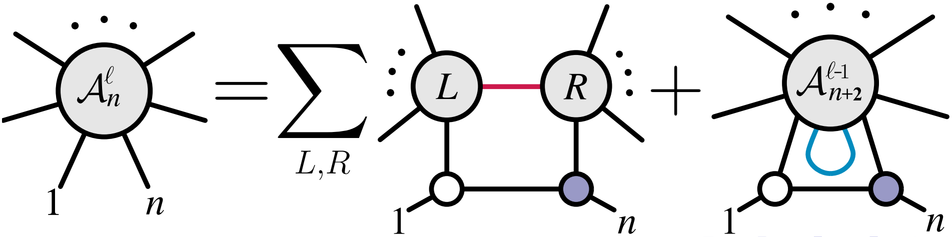

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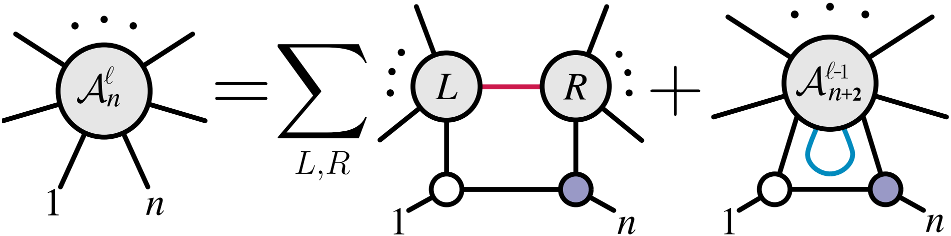

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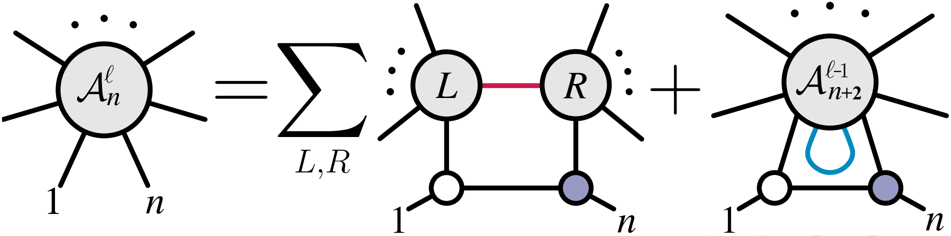

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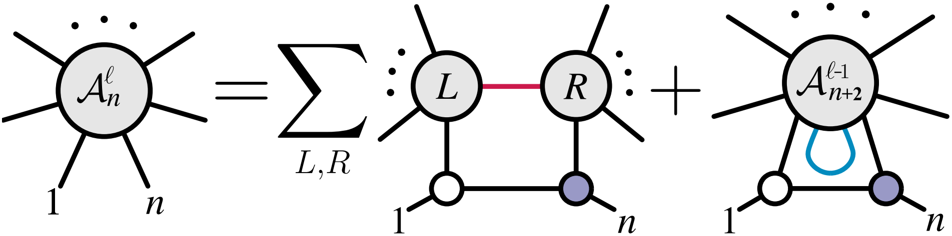

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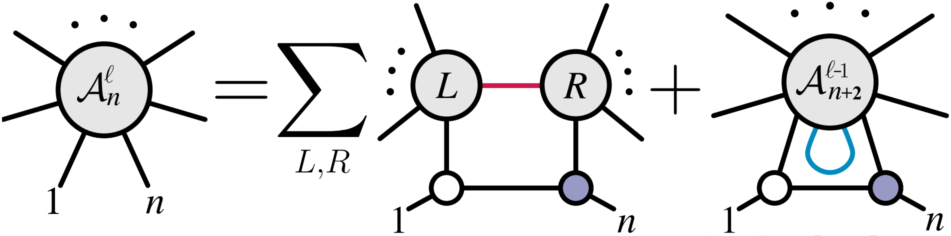

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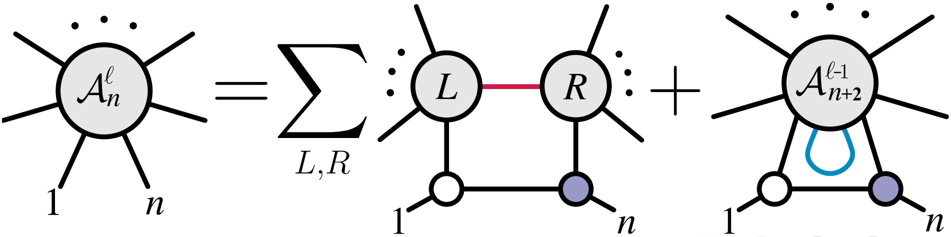

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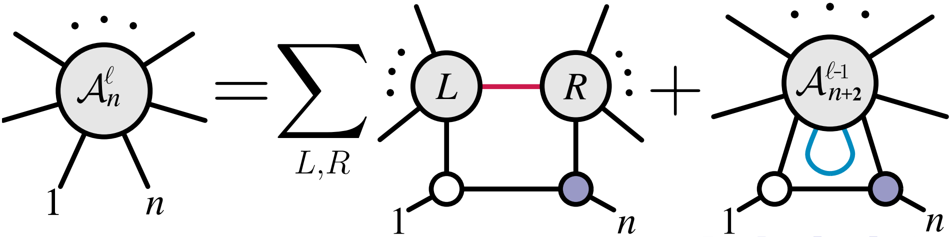

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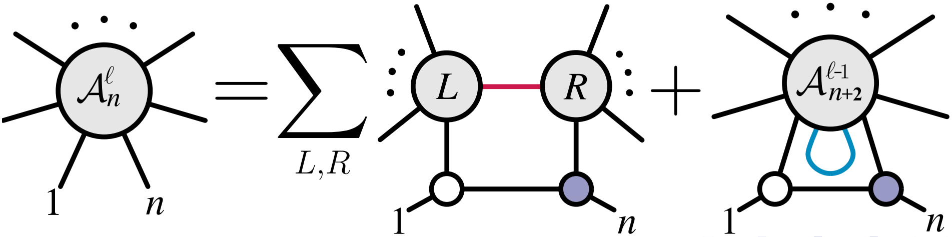

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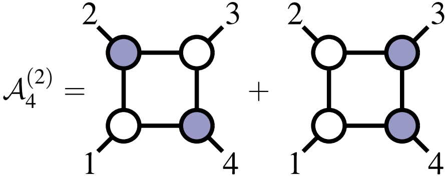

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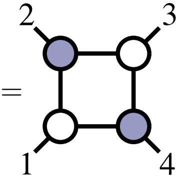
$\mathcal{A}_4^{(2)} =$ 

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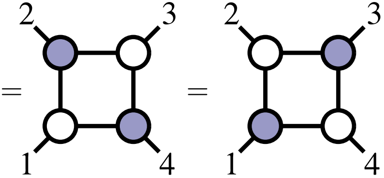

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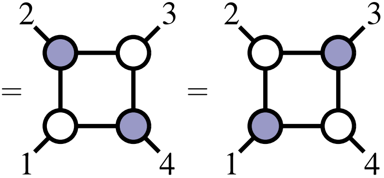

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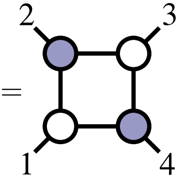

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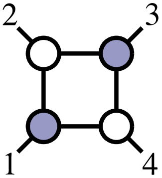

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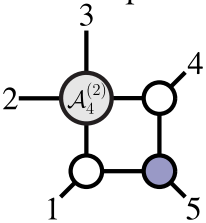

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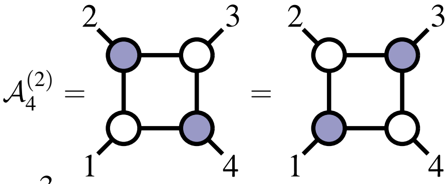

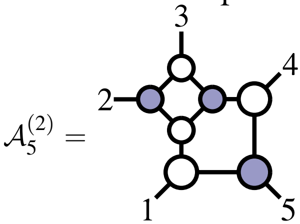

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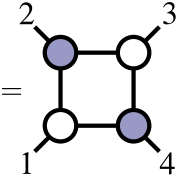
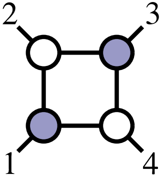
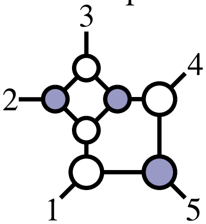
$\mathcal{A}_4^{(2)} =$  $=$ 

$\mathcal{A}_5^{(2)} =$ 

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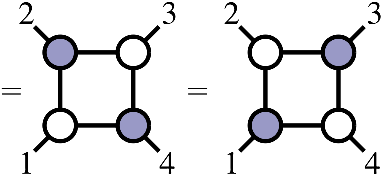

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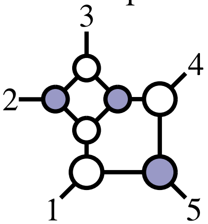

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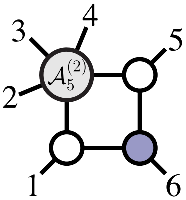

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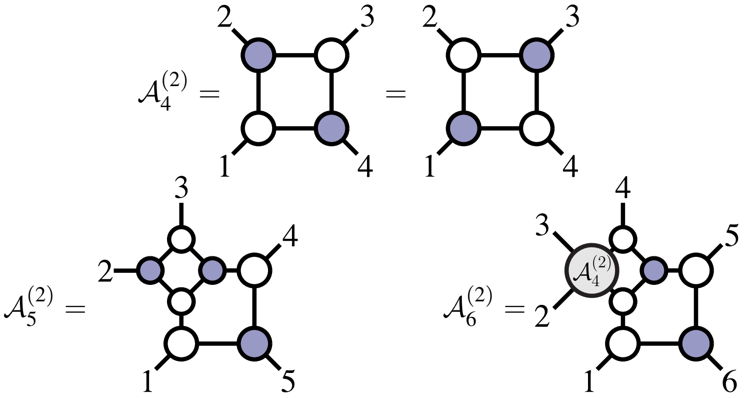

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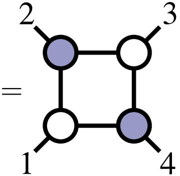

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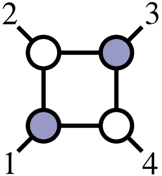

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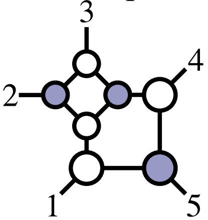

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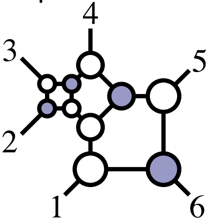

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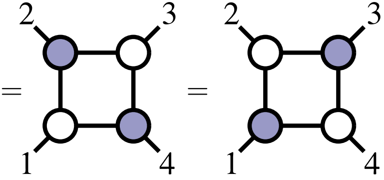
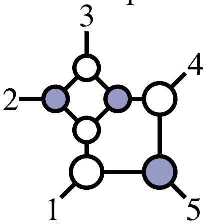
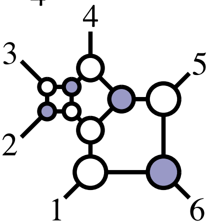
$\mathcal{A}_4^{(2)} =$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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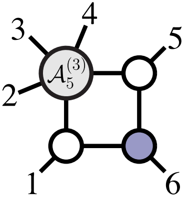

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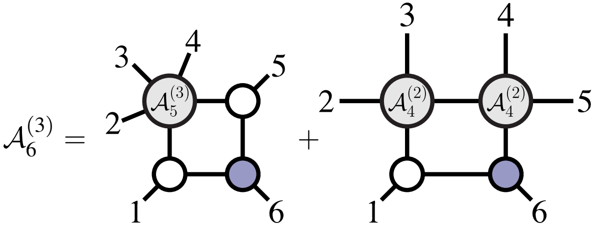

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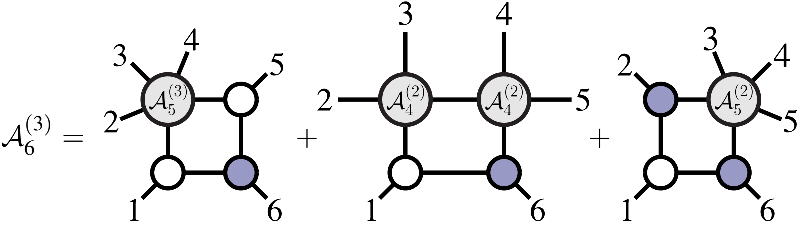

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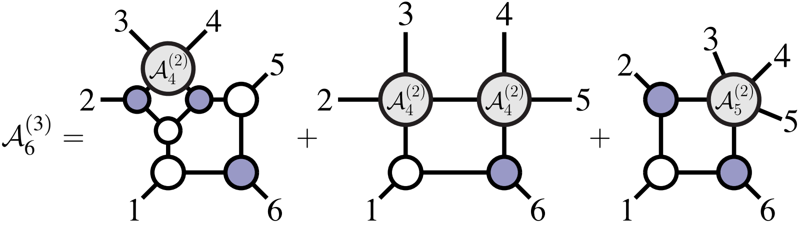

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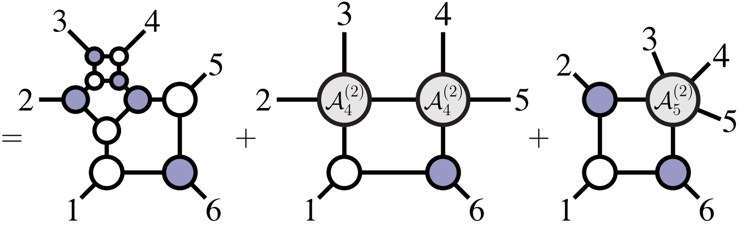

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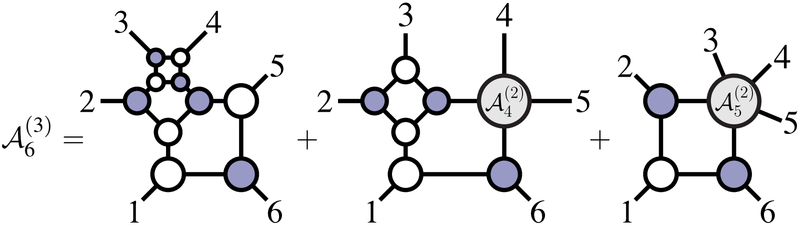

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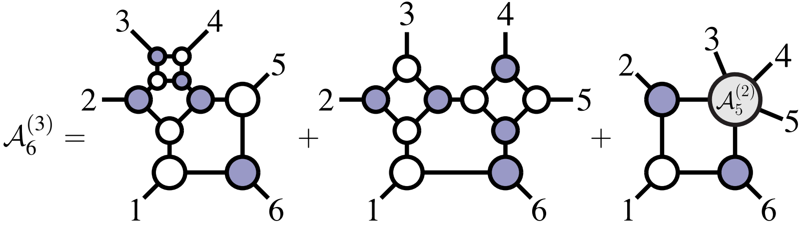

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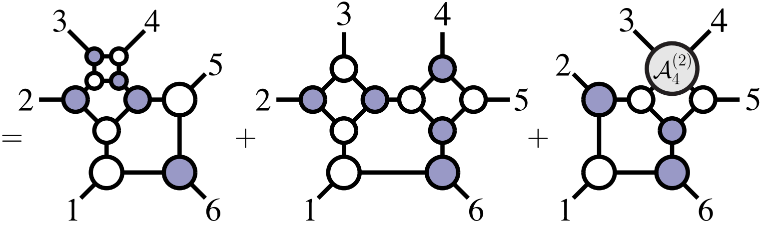

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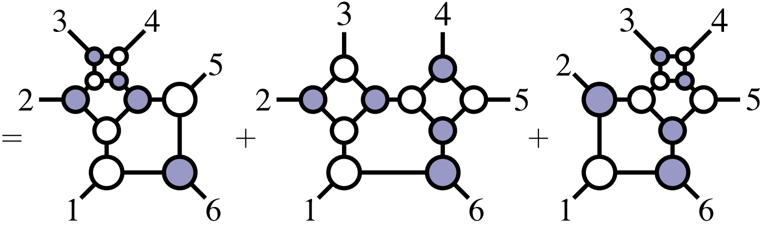

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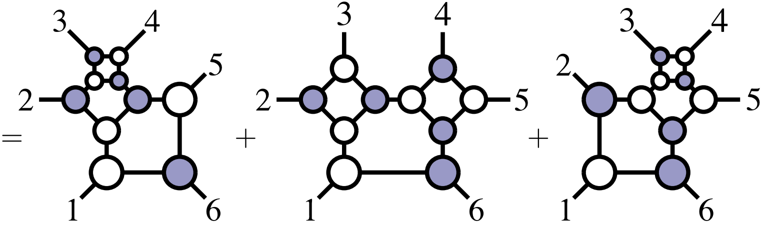
$\mathcal{A}_6^{(3)} =$ 

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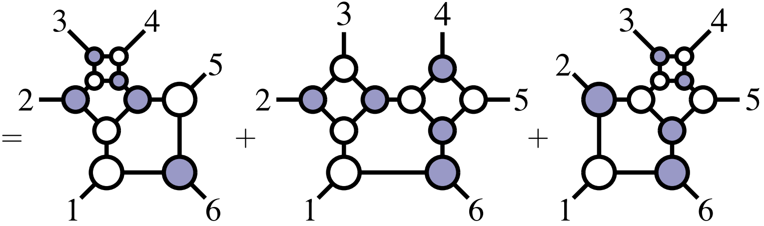

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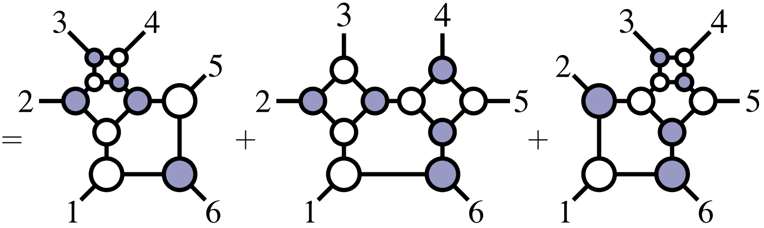

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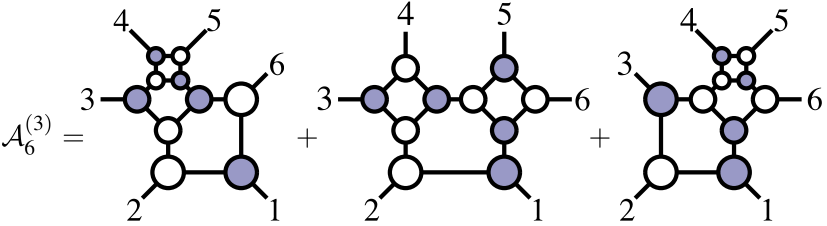

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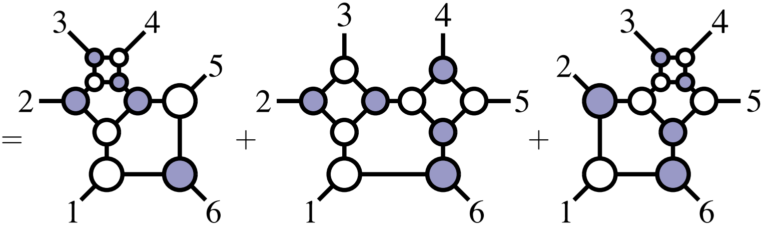

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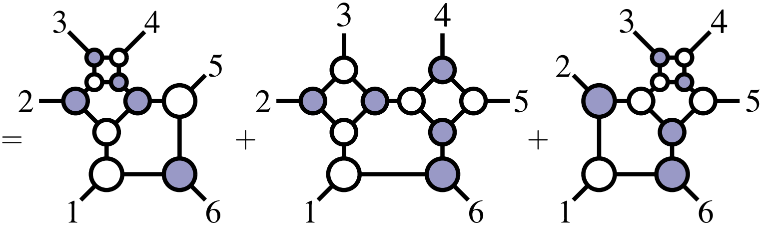

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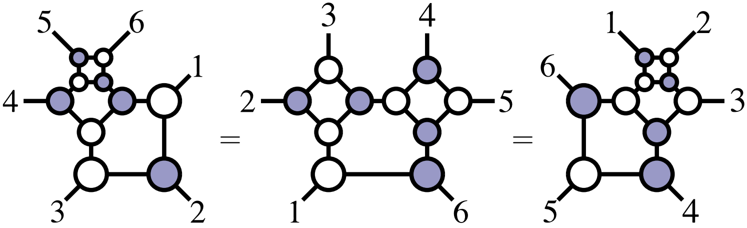

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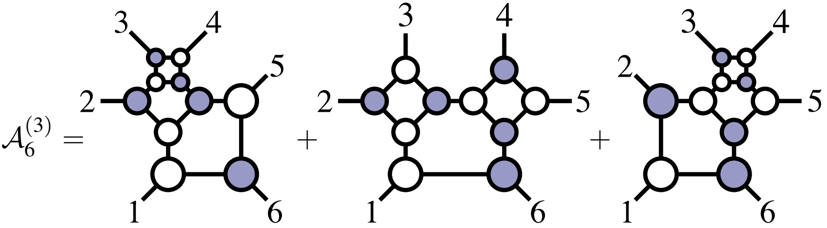

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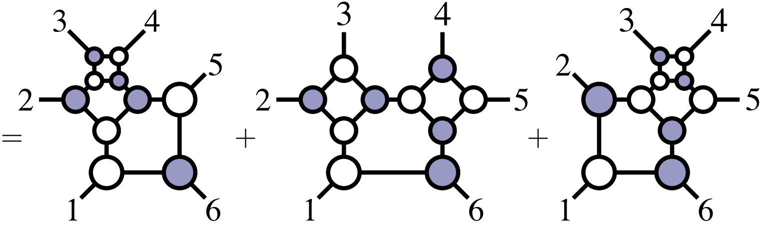

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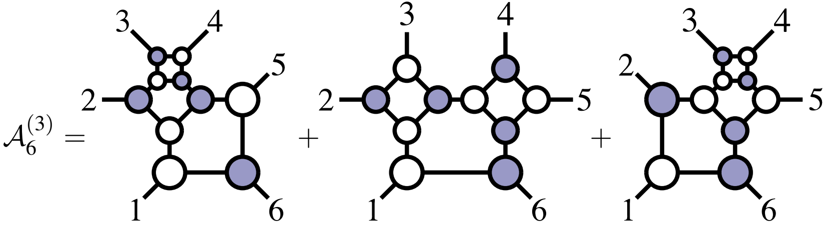

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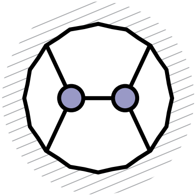

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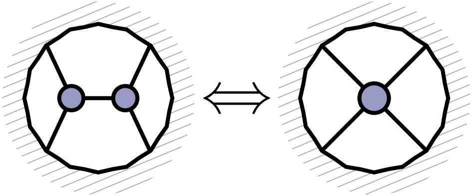

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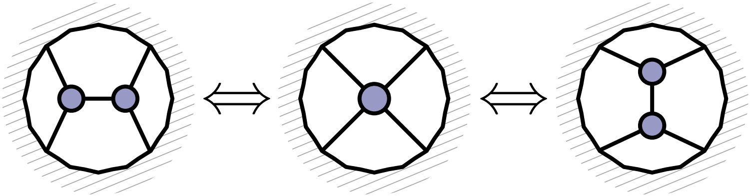

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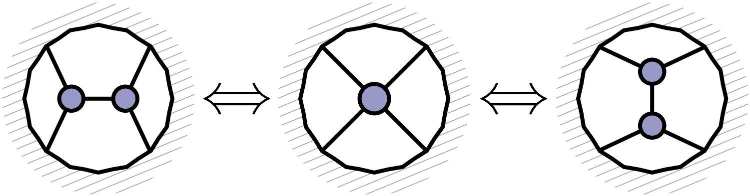

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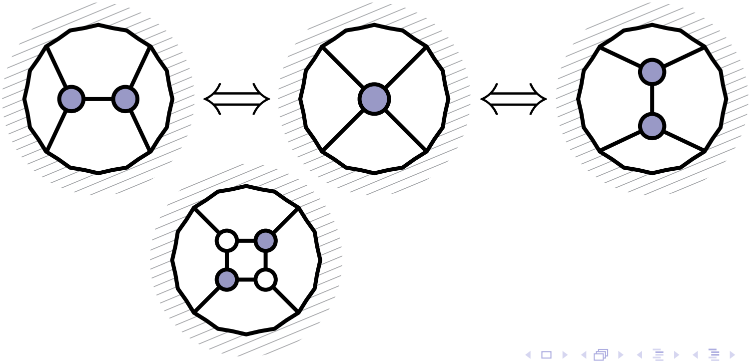

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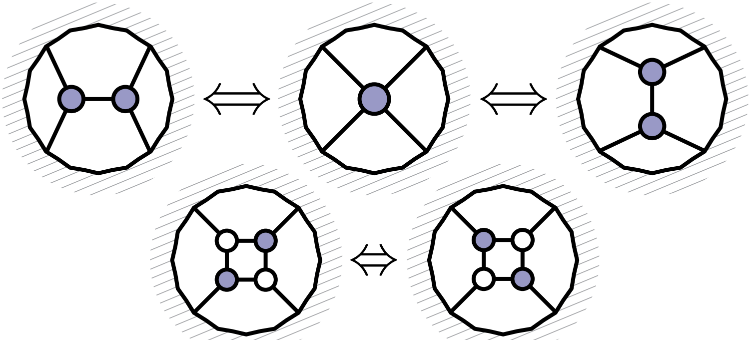

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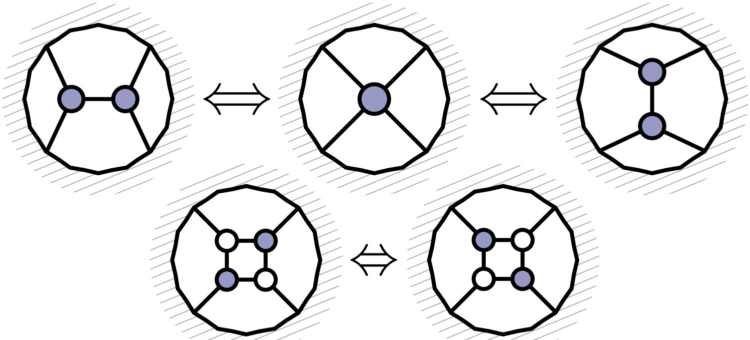

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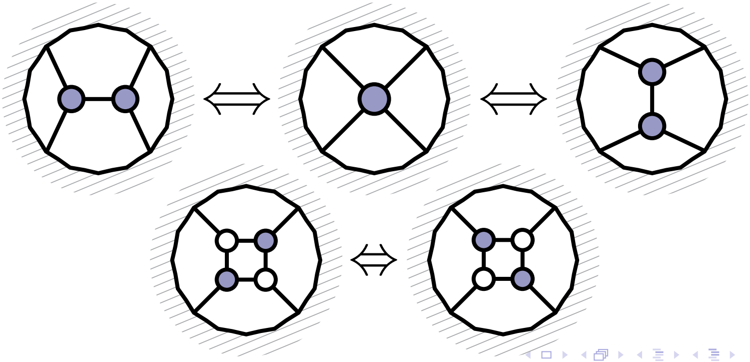

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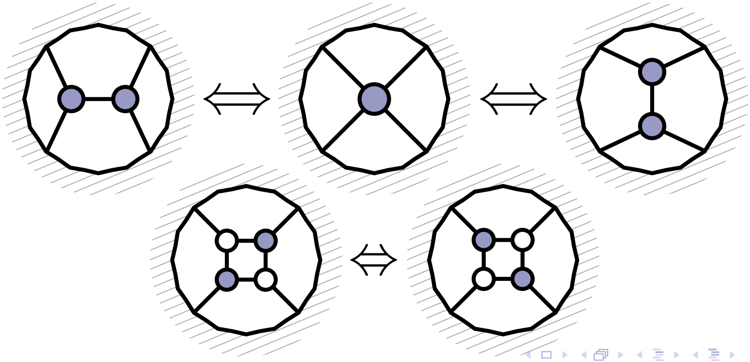

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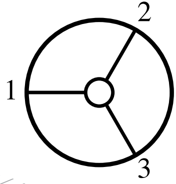

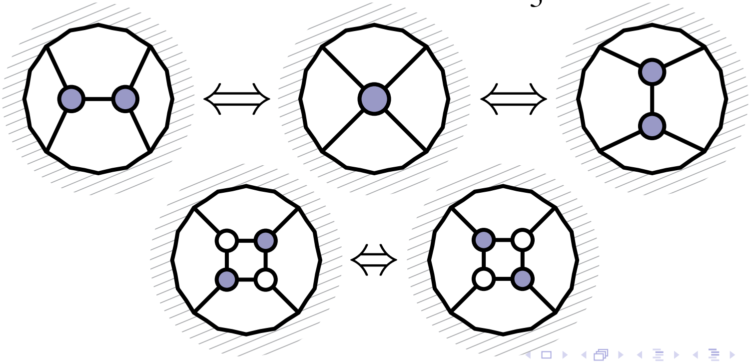

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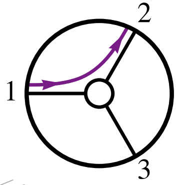

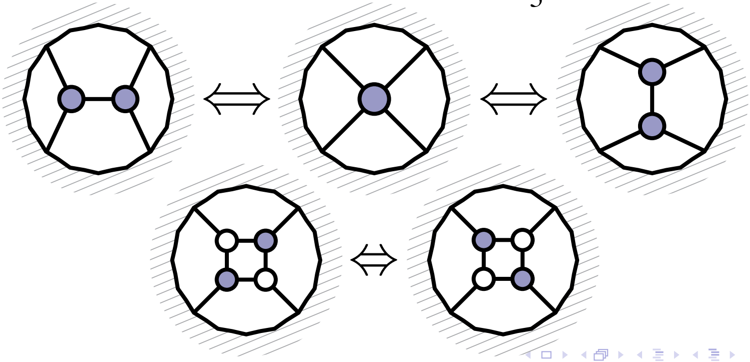

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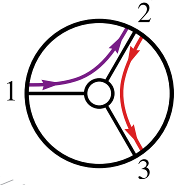

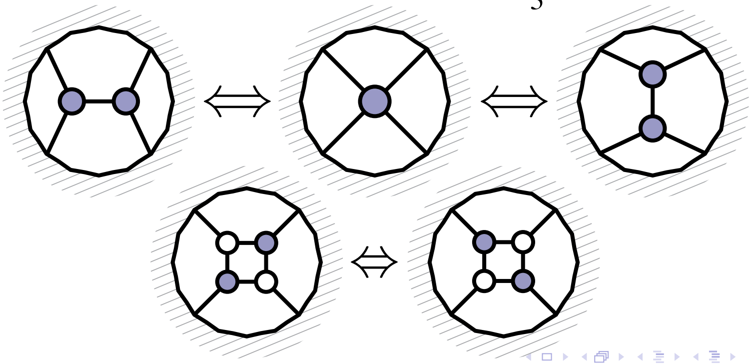

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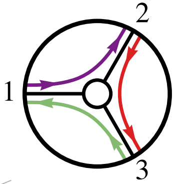

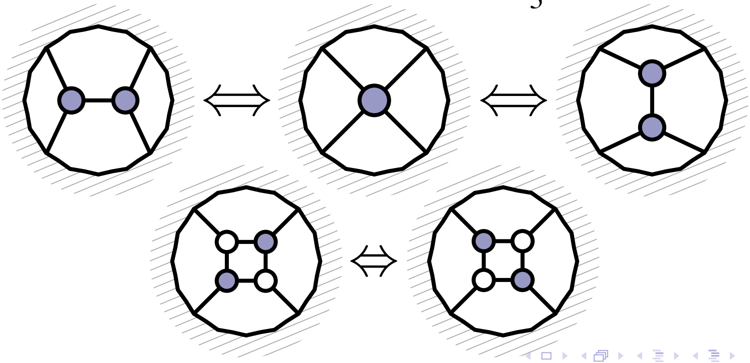

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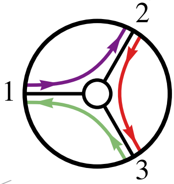

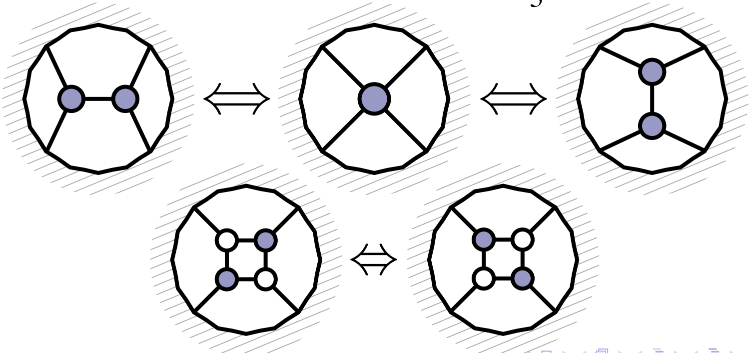

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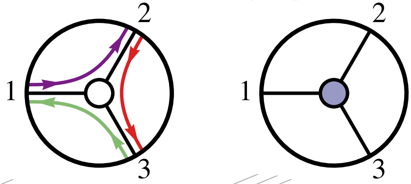

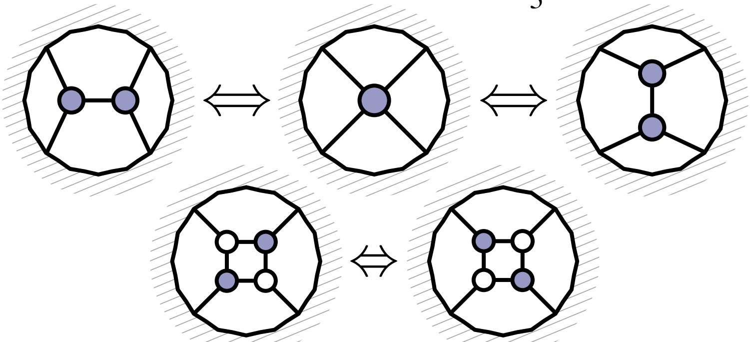

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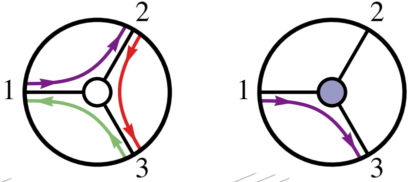

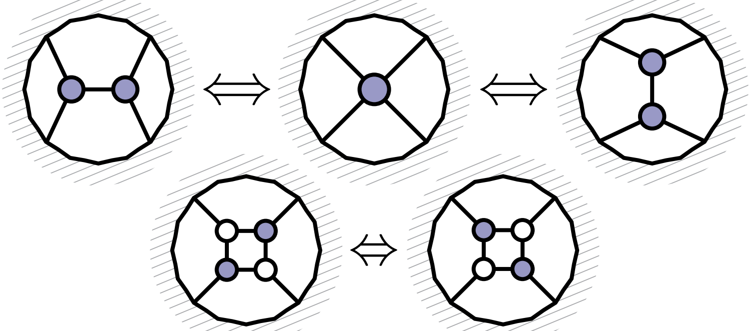

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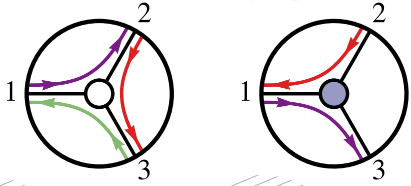

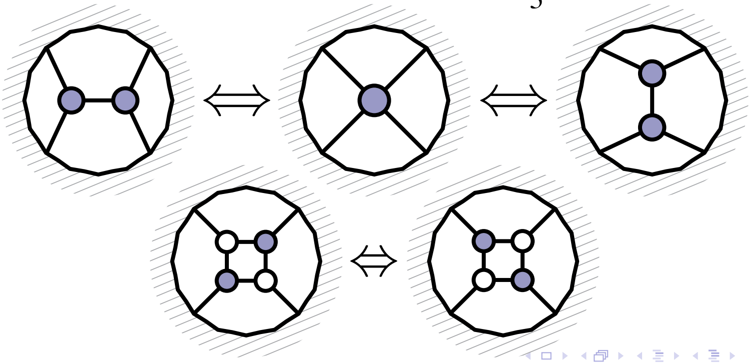

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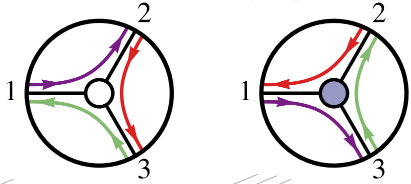

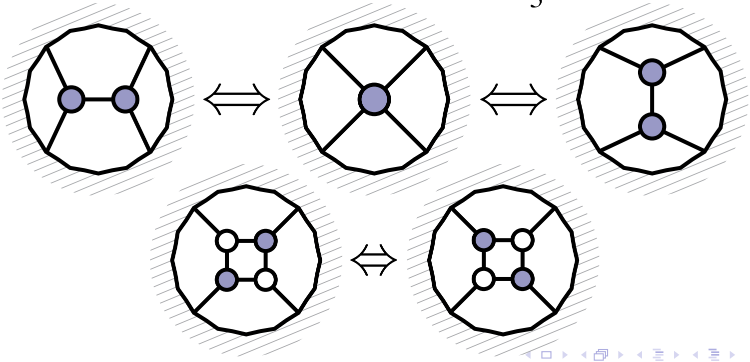

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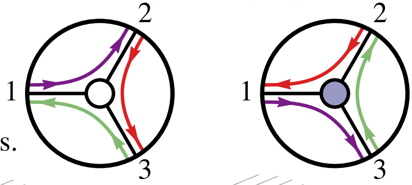

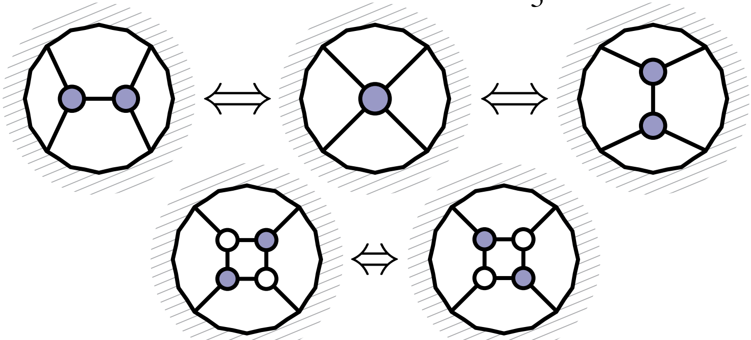

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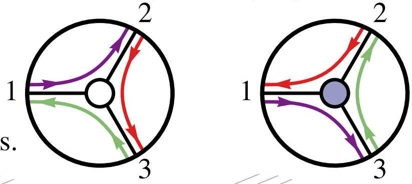

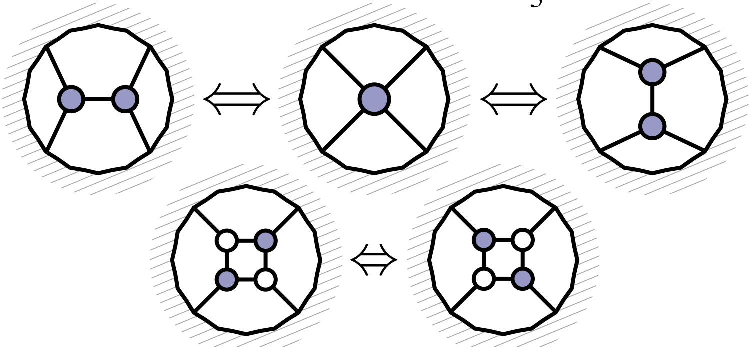

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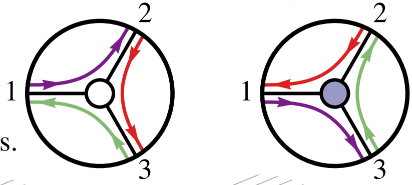

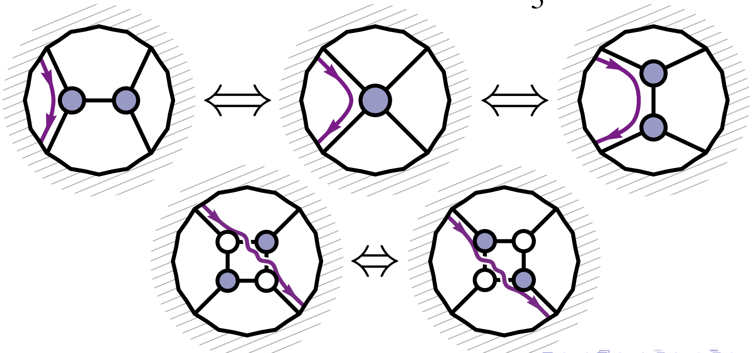

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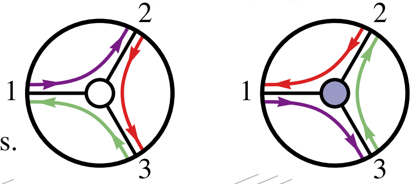

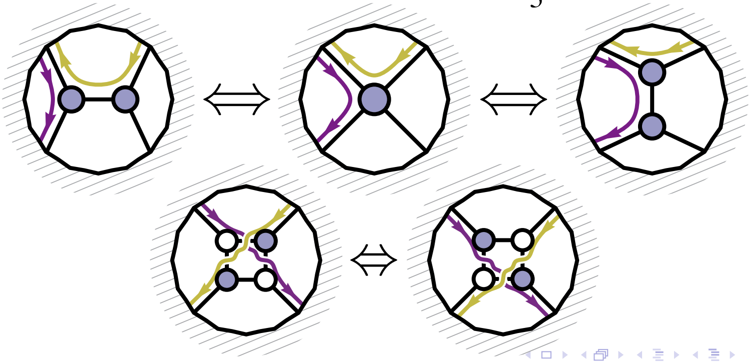

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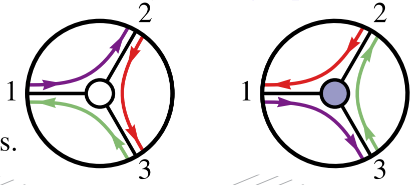

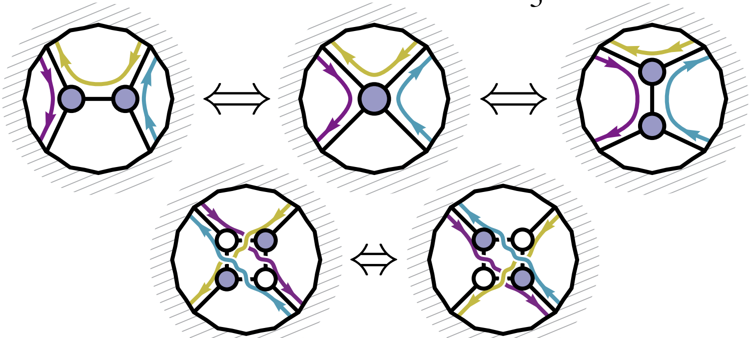

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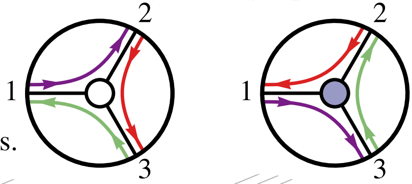

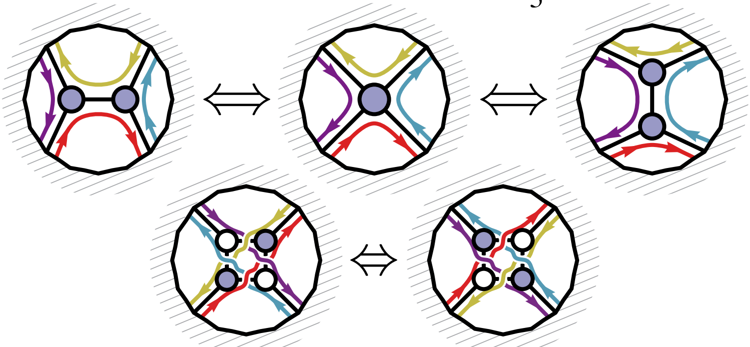

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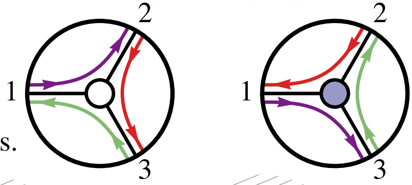

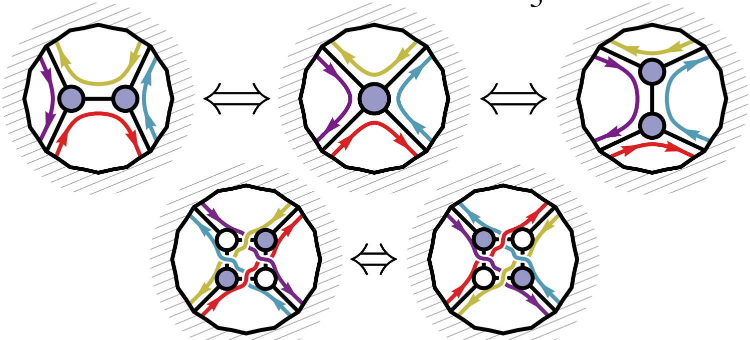

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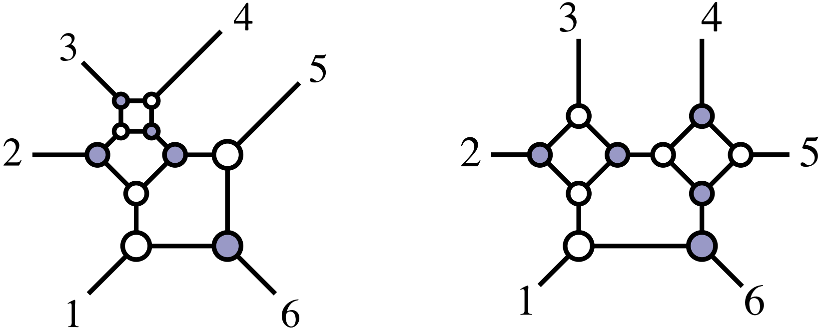

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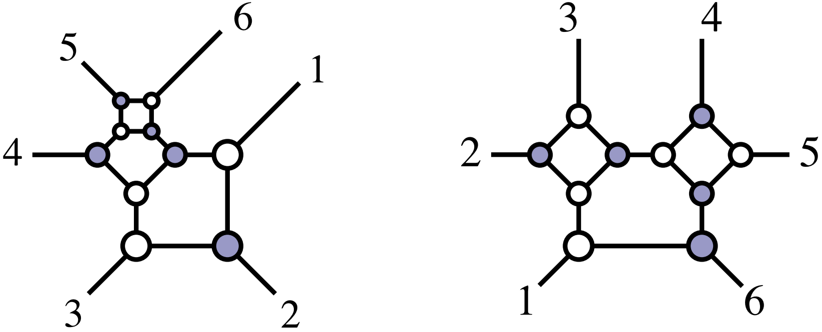

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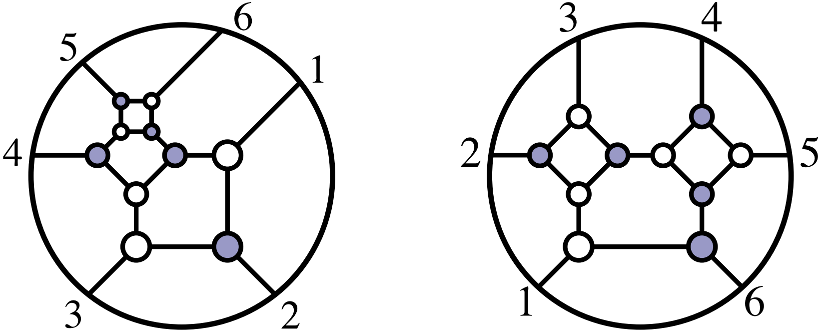

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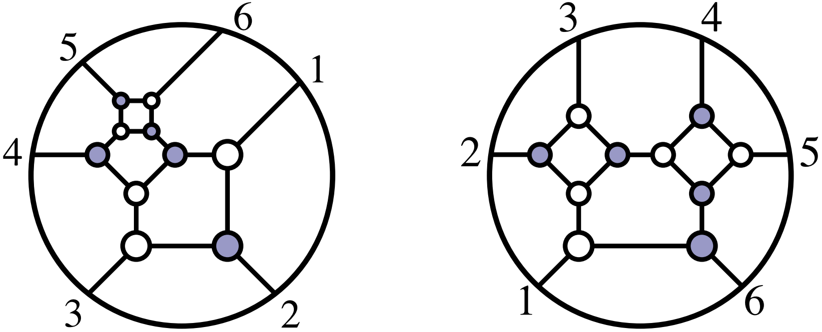

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$$\sigma : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & & & & & \end{pmatrix}$$

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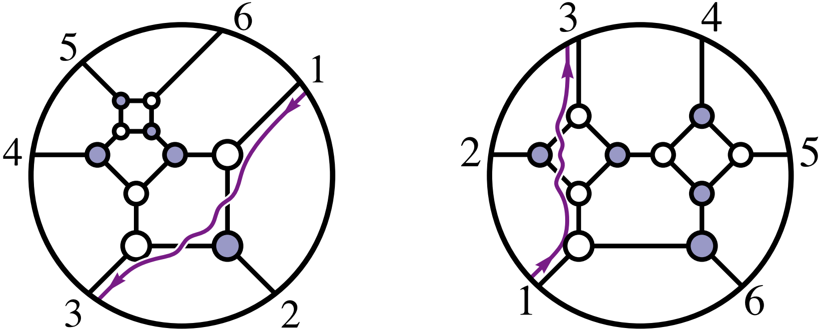

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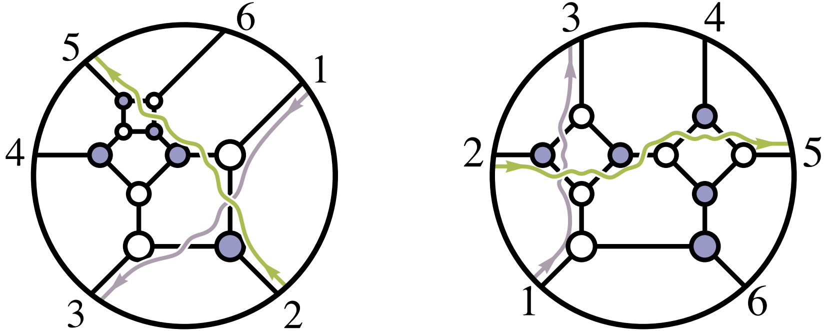

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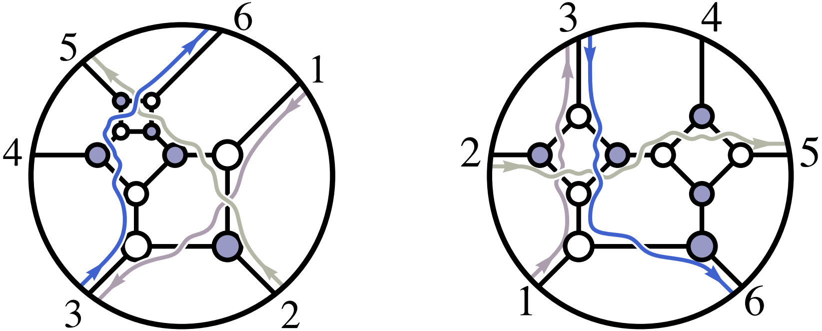

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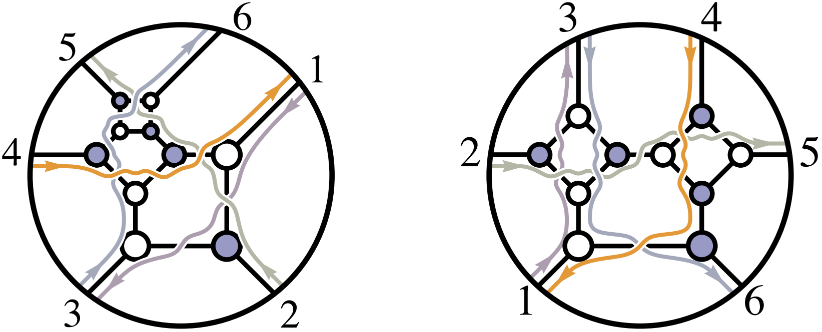

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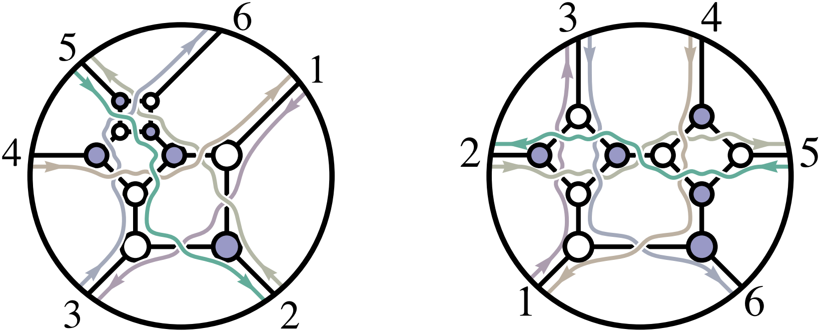

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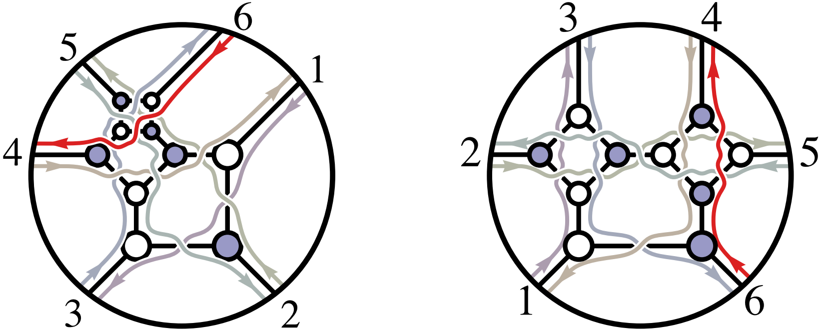

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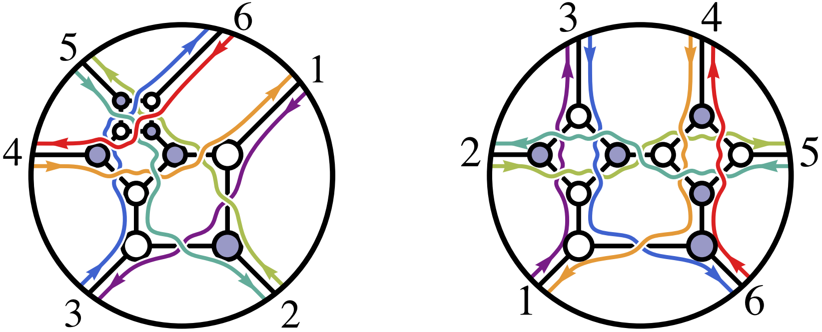

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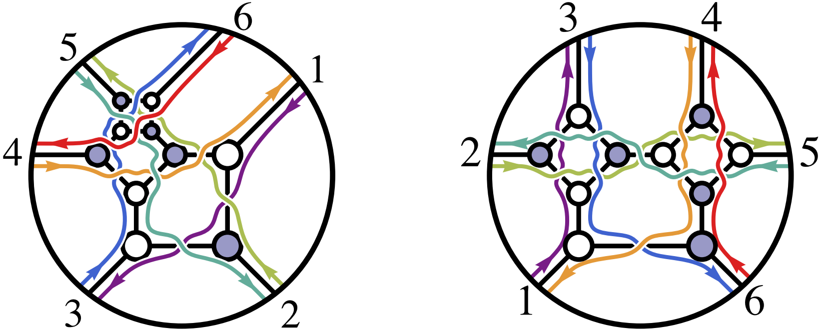

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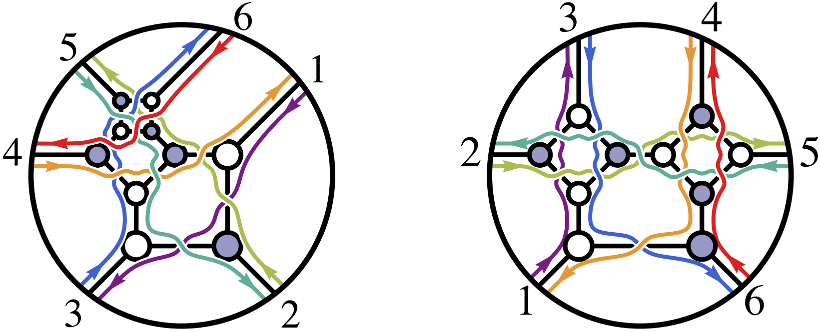

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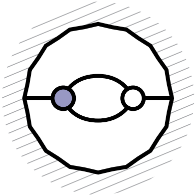

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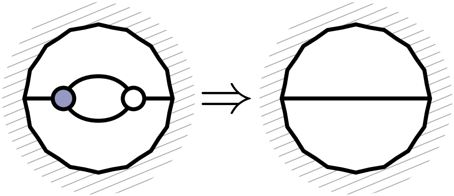

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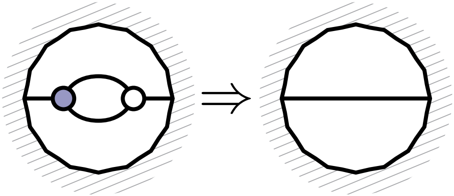

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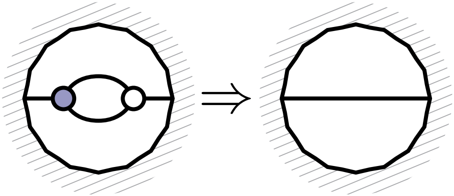

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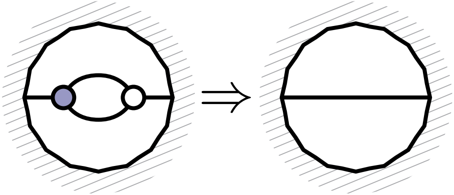

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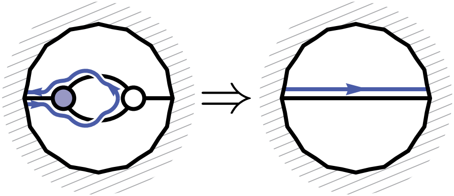

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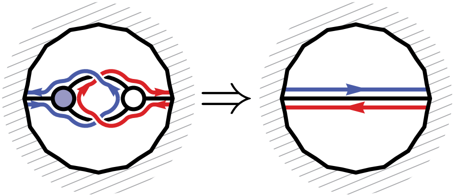

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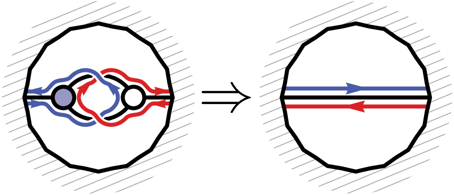
Such factors of $d\alpha/\alpha$ arising from bubble deletion encode loop integrands!



Canonical Coordinates for Computing On-Shell Functions

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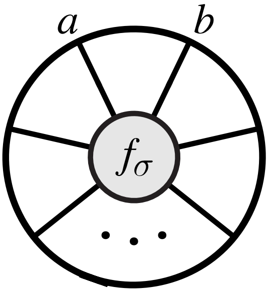

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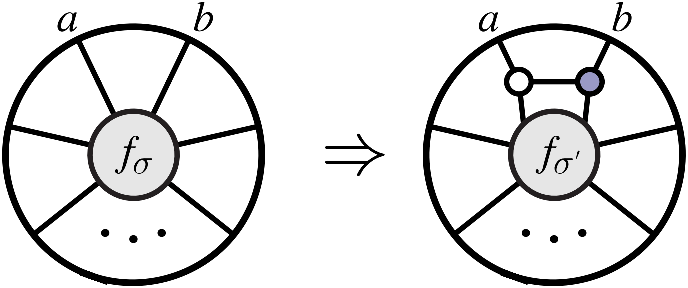

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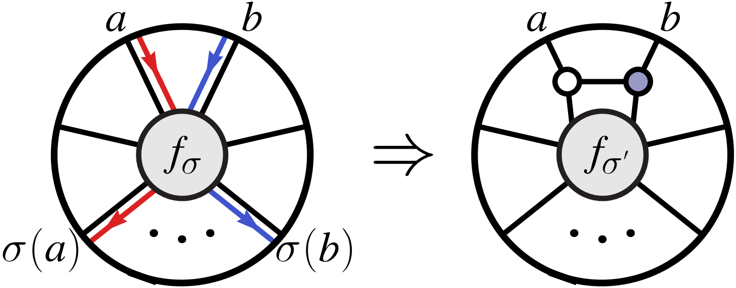

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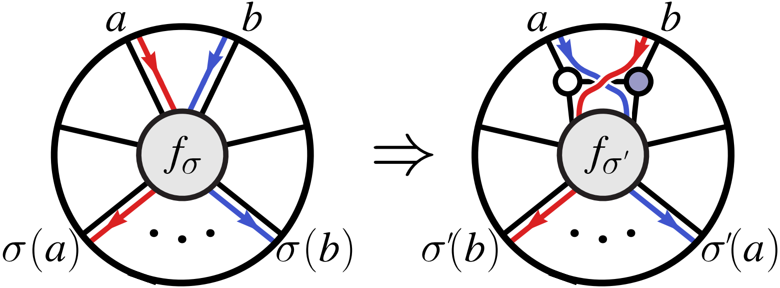

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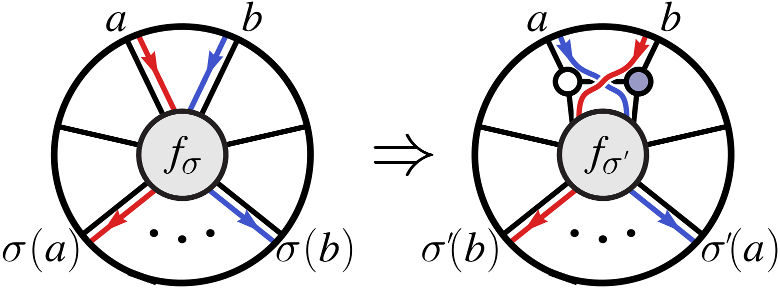

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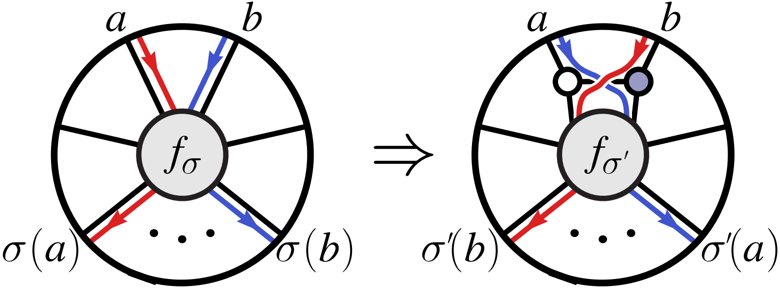

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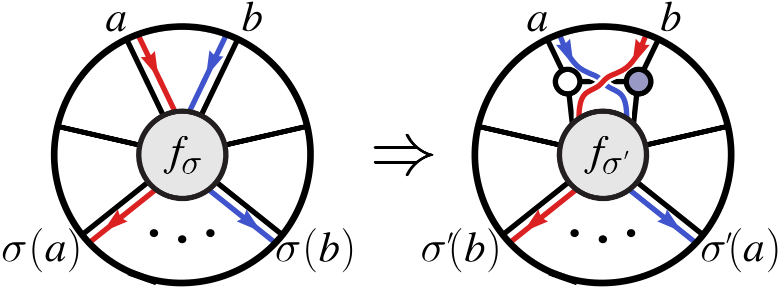

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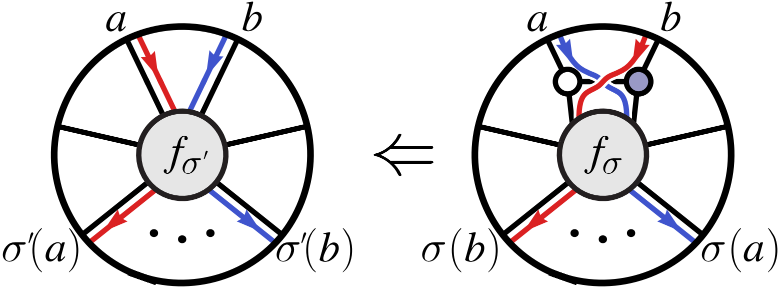

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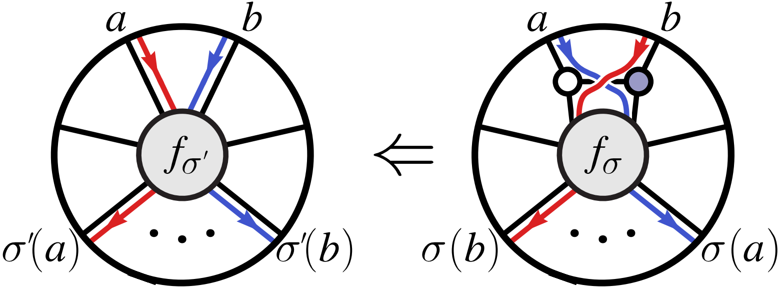
Read the other way, we can 'peel-off' bridges and thereby decompose a permutation into transpositions according to $\sigma = (a\,b) \circ \sigma'$



Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions

'Bridge' Decomposition

$$\sigma:\begin{pmatrix}1&2&3&4&5&6\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\ 3&5&6&7&8&10\end{pmatrix}$$

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'Bridge' Decomposition

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ f_{\sigma} & \{3 & 5 & 6 & 7 & 8 & 10\} \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

'Bridge' Decomposition

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ f_\sigma & \{3 & 5 & 6 & 7 & 8 & 10\} \end{array}$$

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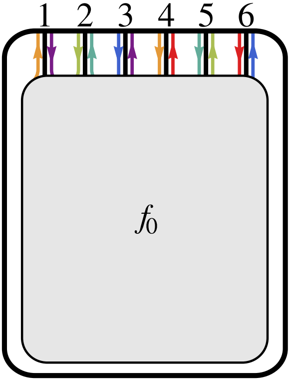

'Bridge' Decomposition

$$\begin{array}{ccccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_0 & \{ & 3 & 5 & 6 & 7 & 8 & 10\} & \end{array}$$

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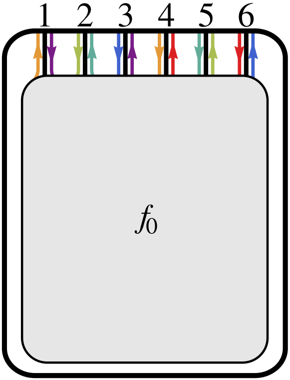

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$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & (1\,2) \end{array}$$

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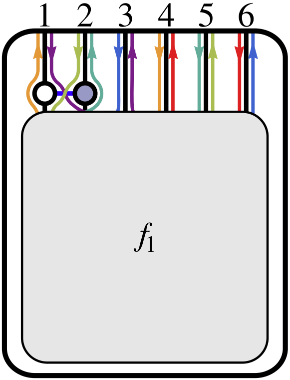

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$$\begin{array}{lccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & (1\,2) \\
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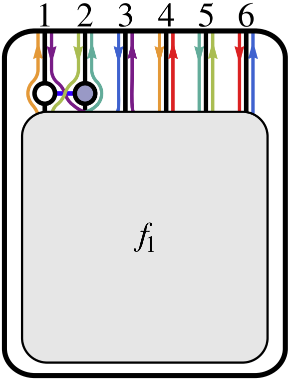

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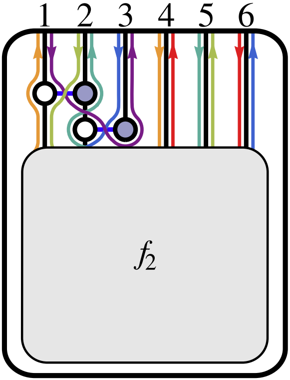

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$$\begin{array}{rccccccl}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & \\
 & & & & & & & (1\,2) \\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & \\
 & & & & & & & (2\,3) \\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} &
\end{array}$$

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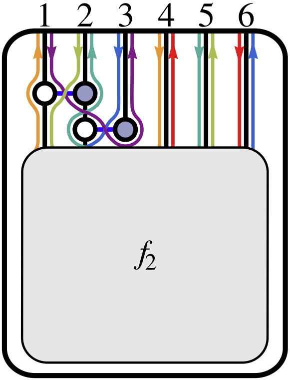

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)

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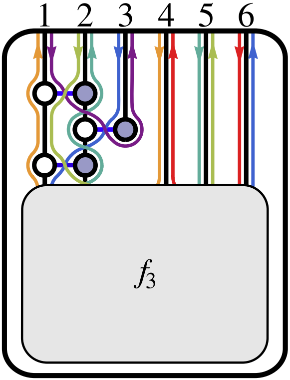

'Bridge' Decomposition

$$\begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & (1\,2) \\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & (2\,3) \\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} & (1\,2) \\
f_3 & \{6 & 5 & 3 & 7 & 8 & 10\} &
\end{array}$$

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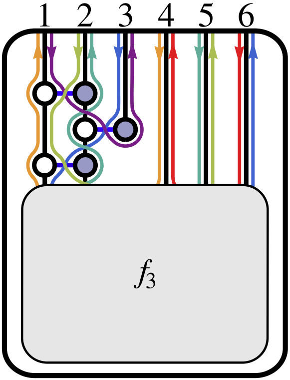

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
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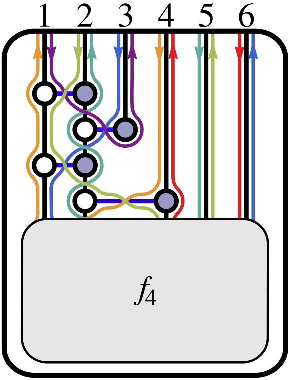

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	

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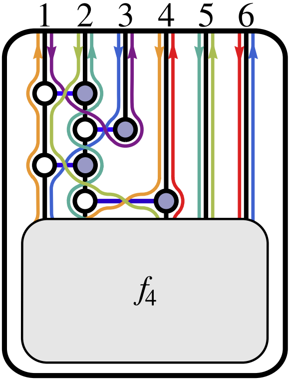

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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)

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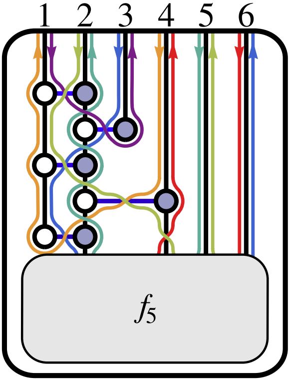

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	

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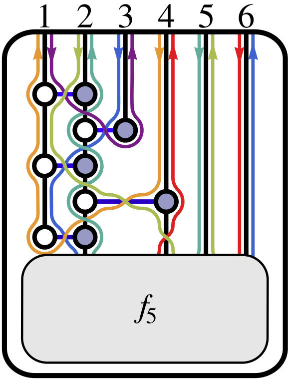

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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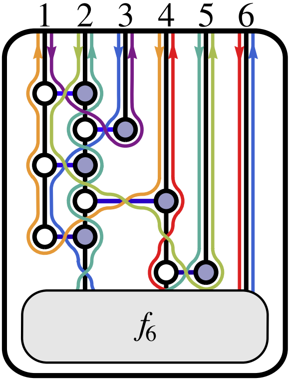

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	

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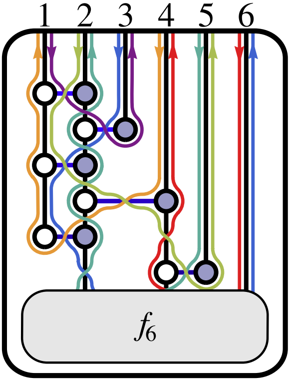

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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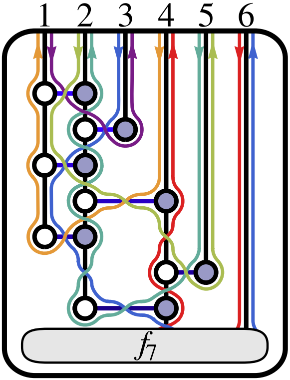

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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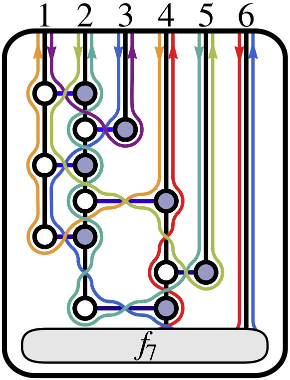

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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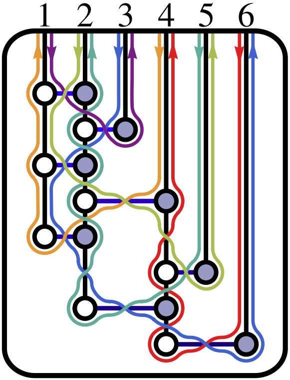

‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0 {	3	5	6	7	8	10}	(1 2)
f_1 {	5	3	6	7	8	10}	(2 3)
f_2 {	5	6	3	7	8	10}	(1 2)
f_3 {	6	5	3	7	8	10}	(2 4)
f_4 {	6	7	3	5	8	10}	(1 2)
f_5 {	7	6	3	5	8	10}	(4 5)
f_6 {	7	6	3	8	5	10}	(2 4)
f_7 {	7	8	3	6	5	10}	(4 6)
f_8 {	7	8	3	10	5	6 }	

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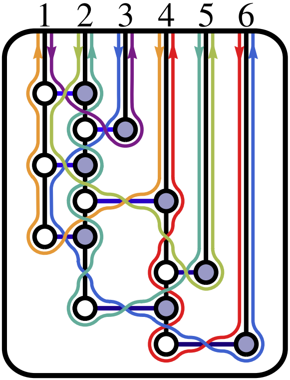

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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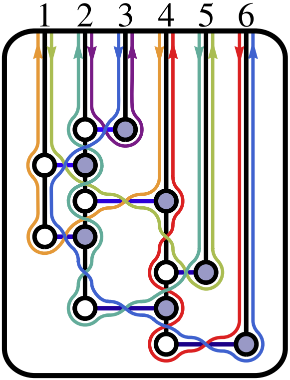

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6 }	

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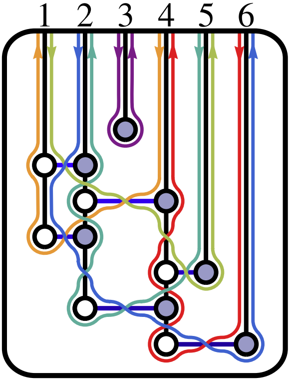

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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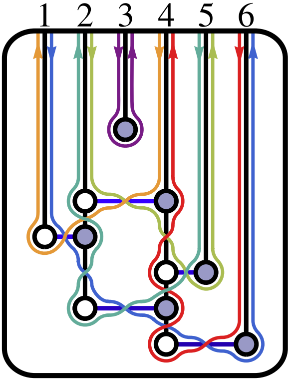

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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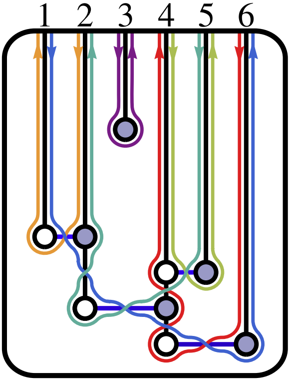

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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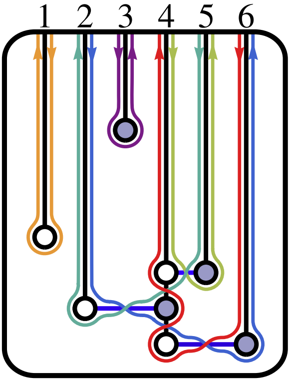

‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_5 {	7	6	3	5	8	10 }	$(4\,5)$
f_6 {	7	6	3	8	5	10 }	$(2\,4)$
f_7 {	7	8	3	6	5	10 }	$(4\,6)$
f_8 {	7	8	3	10	5	6 }	

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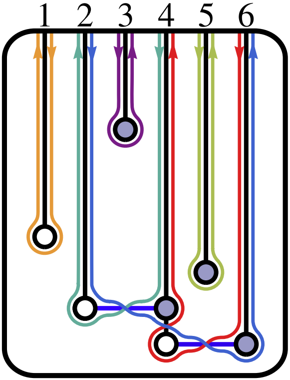

‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
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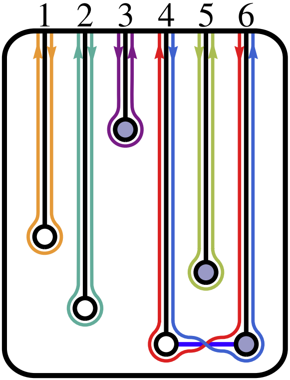

‘Bridge’ Decomposition

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \\ f_7\,\{7 & 8 & 3 & 6 & 5 & 10\} & & (4\,6) \\ f_8\,\{7 & 8 & 3 & 10 & 5 & 6\} & & \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

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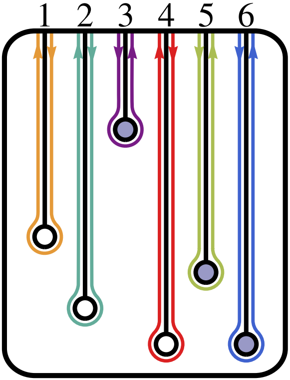

'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

$$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$$

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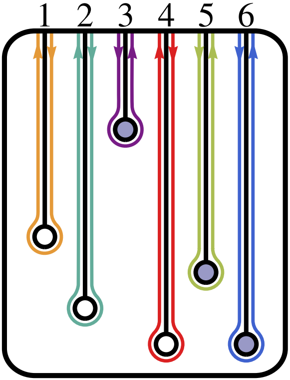

'Bridge' Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

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$$f_8 = \prod_{a=\sigma(a)+n}\Big(\delta^4(\widetilde{\eta}_a)\,\delta^2(\widetilde{\lambda}_a)\Big)\prod_{b=\sigma(b)}\Big(\delta^2(\lambda_b)\Big)$$

‘Bridge’ Decomposition

$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \end{array}$$

$$f_8\;\{7\;\;8\;\;3\;\;10\;\;5\;\;6\;\}$$

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$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

‘Bridge’ Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

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$$f_8 = \delta^{3\times 4}(C\cdot\widetilde{\eta})\,\delta^{3\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times 3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

'Bridge' Decomposition

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \qquad \tau$$

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$$C \equiv \begin{array}{c}\\ \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

‘Bridge’ Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

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$$C \equiv \begin{array}{c}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)\end{array}$$

$$(4\,6):\quad c_6 \mapsto c_6 + \alpha_8\, c_4$$

‘Bridge’ Decomposition

$$\begin{array}{lccccccl} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\;\} & \end{array}$$

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$$f_6 = \frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,4):\quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

‘Bridge’ Decomposition

$$\begin{array}{lcccccccl} & & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_6 & \{ & 7 & 6 & 3 & 8 & 5 & 10 & \} \\ & & & & & & & & (2\,4) \\ f_7 & \{ & 7 & 8 & 3 & 6 & 5 & 10 & \} \\ & & & & & & & & (4\,6) \\ f_8 & \{ & 7 & 8 & 3 & 10 & 5 & 6 & \} \end{array}$$

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$$f_5 = \frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(4\,5):\quad c_5 \mapsto c_5 + \alpha_6\, c_4$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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$$f_4 = \frac{d\alpha_5}{\alpha_5}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_5\, c_1$$

‘Bridge’ Decomposition

$$\begin{array}{ccccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_4 & \{ & 6 & 7 & 3 & 5 & 8 & 10 & \} \\ & & & & & & & & (1\,2) \\ f_5 & \{ & 7 & 6 & 3 & 5 & 8 & 10 & \} \\ & & & & & & & & (4\,5) \\ f_6 & \{ & 7 & 6 & 3 & 8 & 5 & 10 & \} \\ & & & & & & & & (2\,4) \\ f_7 & \{ & 7 & 8 & 3 & 6 & 5 & 10 & \} \\ & & & & & & & & (4\,6) \\ f_8 & \{ & 7 & 8 & 3 & 10 & 5 & 6 & \} \end{array}$$

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$$f_3 = \frac{d\alpha_4}{\alpha_4}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(2\,4):\quad c_4 \mapsto c_4 + \alpha_4\, c_2$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_3	$\{6$	5	3	7	8	$10\}$	$(2\,4)$
f_4	$\{6$	7	3	5	8	$10\}$	$(1\,2)$
f_5	$\{7$	6	3	5	8	$10\}$	$(4\,5)$
f_6	$\{7$	6	3	8	5	$10\}$	$(2\,4)$
f_7	$\{7$	8	3	6	5	$10\}$	$(4\,6)$
f_8	$\{7$	8	3	10	5	$6\}$	

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$$f_2 = \frac{d\alpha_3}{\alpha_3} \cdots \frac{d\alpha_8}{\alpha_8} \delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)\end{array}$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_3\, c_1$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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$$f_1 = \frac{d\alpha_2}{\alpha_2} \cdots \frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\delta^{3\times2}(C\cdot\widetilde{\lambda})\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(2\,3):\quad c_3 \mapsto c_3 + \alpha_2\, c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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$$f_0 = \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_8}{\alpha_8} \delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_1\, c_1$$

‘Bridge’ Decomposition

	1 $\downarrow$	2 $\downarrow$	3 $\downarrow$	4 $\downarrow$	5 $\downarrow$	6 $\downarrow$	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
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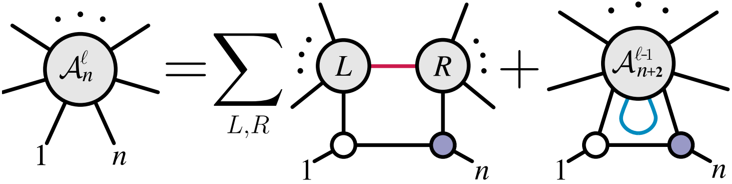

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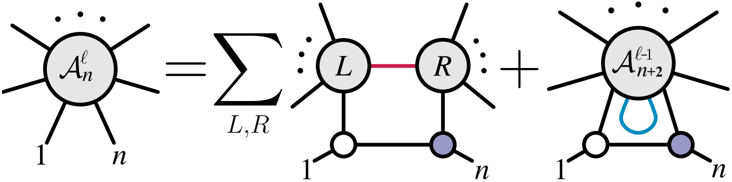

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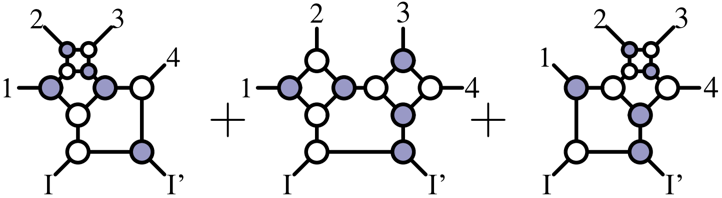

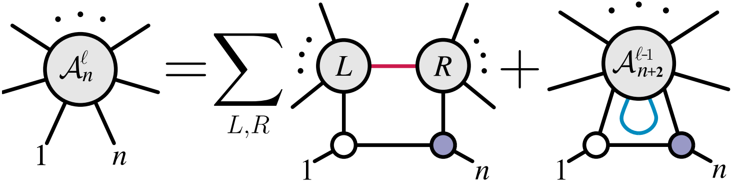

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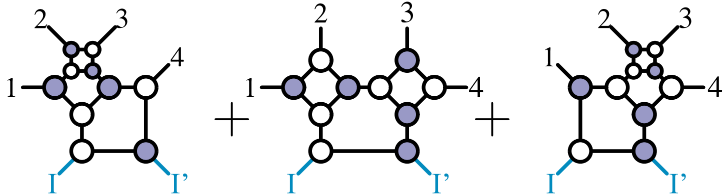

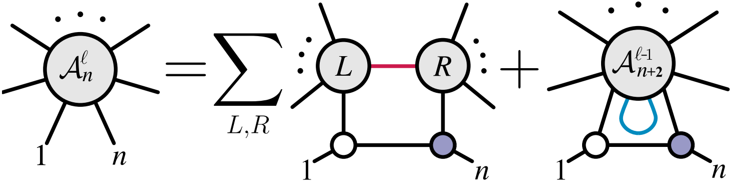

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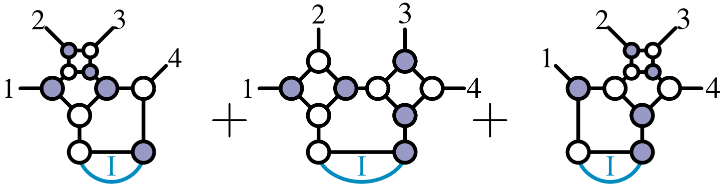

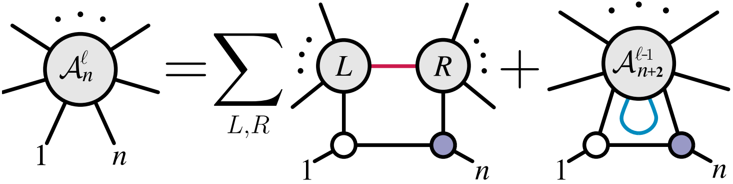

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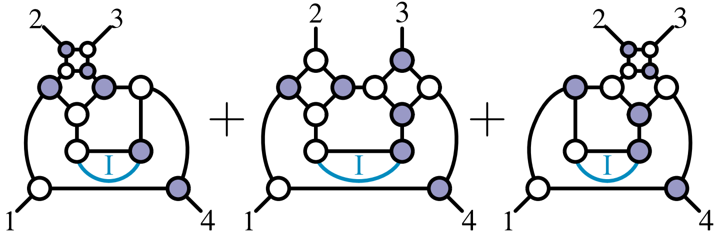

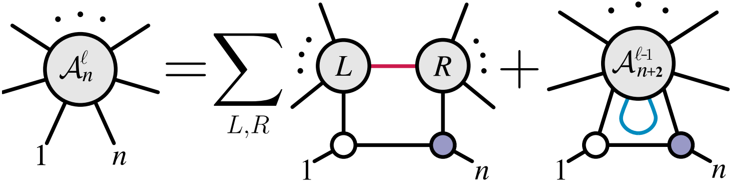

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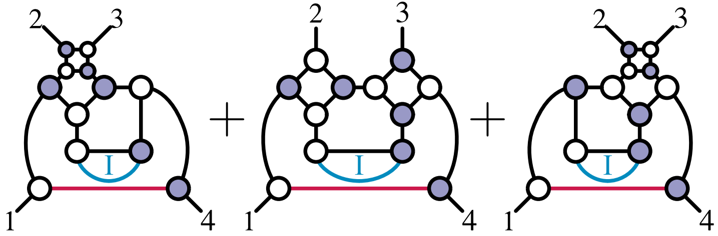

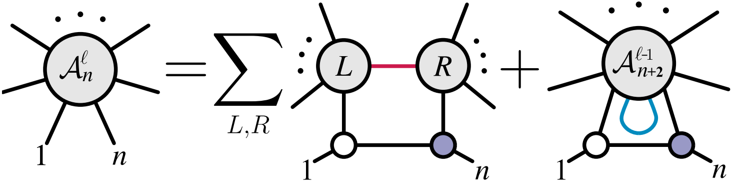

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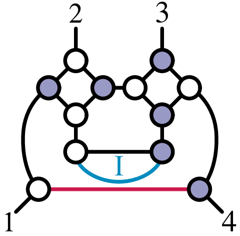

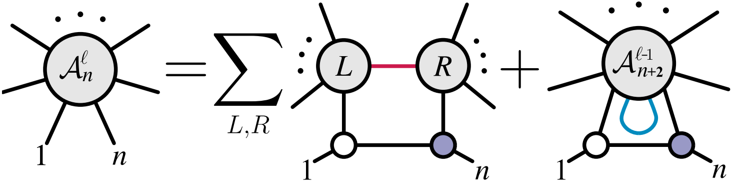

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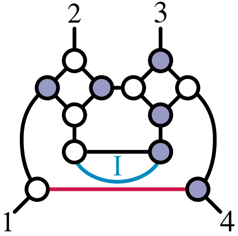

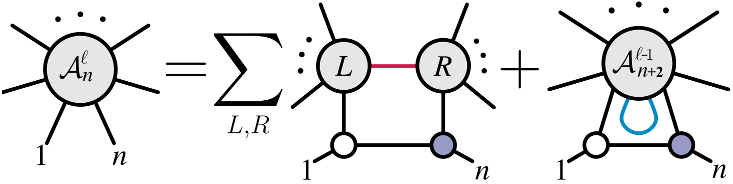

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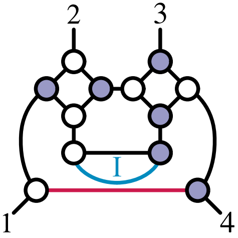
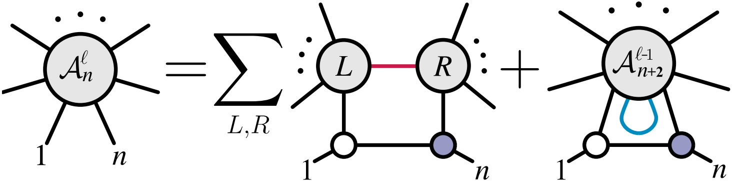


$$\int\limits_{\ell\in\mathbb{R}^{3,1}} d^4\ell$$



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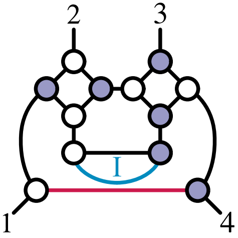

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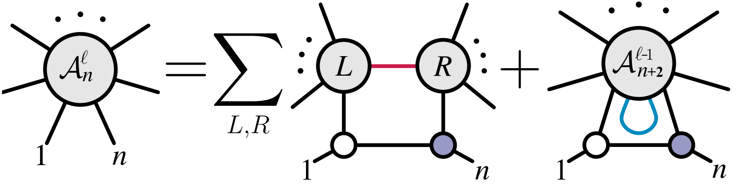

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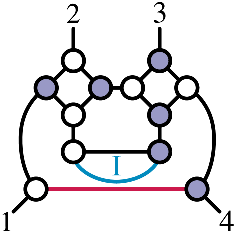

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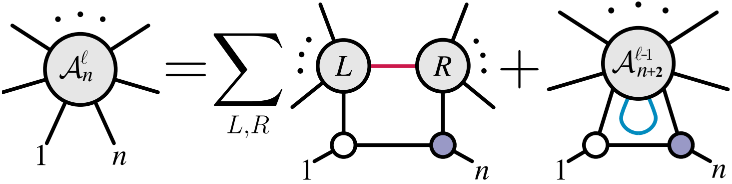

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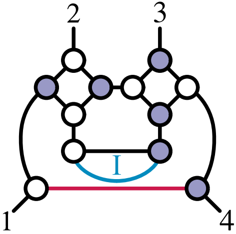

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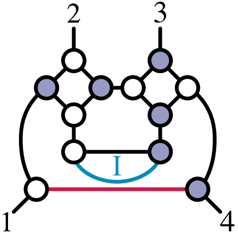

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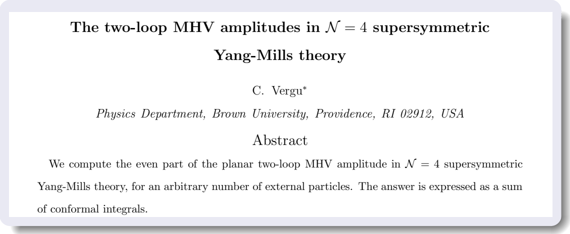

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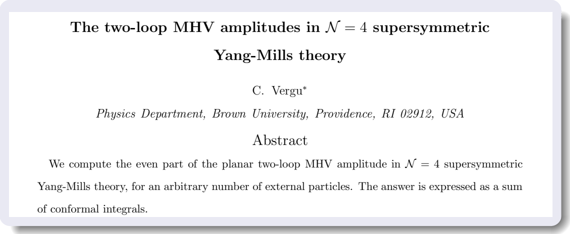

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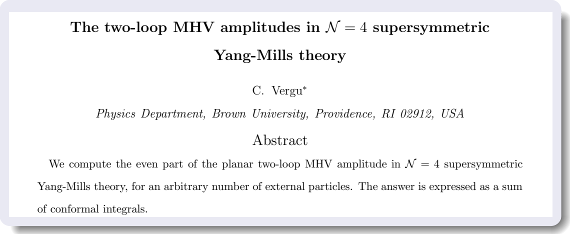

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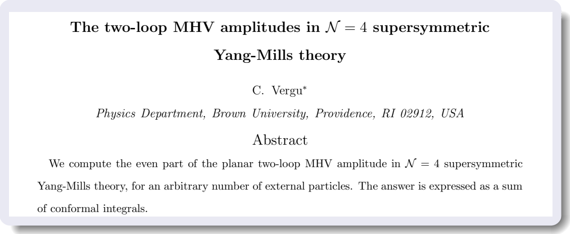

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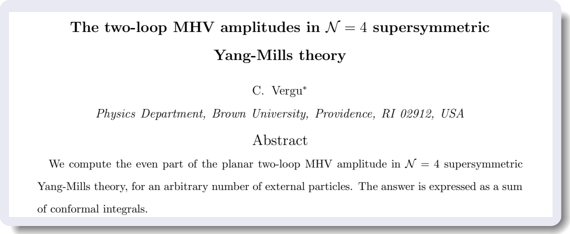

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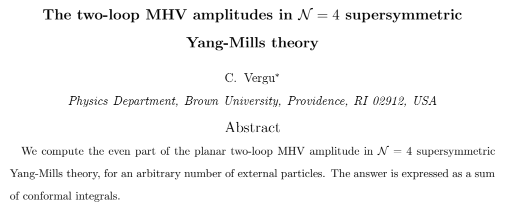

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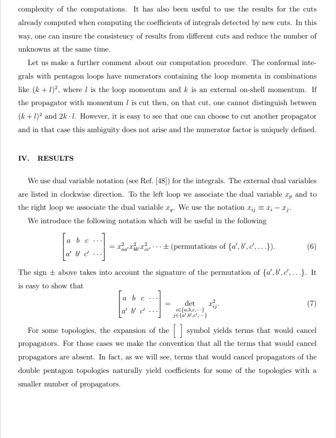

complexity of the computations. It has also been useful to use the results for the cuts already computed when computing the coefficients of integrals detected by new cuts. In this way, one can insure the consistency of results from different cuts and reduce the number of unknowns at the same time.

Let us make a further comment about our computation procedure. The conformal integrals with pentagon loops have numerators containing the loop momenta in combinations like $(k+l)^2$, where l is the loop momentum and k is an external on-shell momentum. If the propagator with momentum l is cut then, on that cut, one cannot distinguish between $(k+l)^2$ and $2k \cdot l$. However, it is easy to see that one can choose to cut another propagator and in that case this ambiguity does not arise and the numerator factor is uniquely defined.

IV. RESULTS

We use dual variable notation (see Ref. [48]) for the integrals. The external dual variables are listed in clockwise direction. To the left loop we associate the dual variable x_p and to the right loop we associate the dual variable x_q. We use the notation $x_{ij} \equiv x_i - x_j$.

We introduce the following notation which will be useful in the following

$$\begin{bmatrix} a & b & c & \cdots \\ a' & b' & c' & \cdots \end{bmatrix} = x^2_{aa'} x^2_{bb'} x^2_{cc'} \cdots \pm (\text{permutations of } \{a', b', c', \ldots\}). \tag{6}$$

The sign ± above takes into account the signature of the permutation of $\{a', b', c', \ldots\}$. It is easy to show that

$$\begin{bmatrix} a & b & c & \cdots \\ a' & b' & c' & \cdots \end{bmatrix} = \det_{\substack{i \in \{a,b,c,\cdots\} \\ j \in \{a',b',c',\cdots\}}} x^2_{ij}. \tag{7}$$

For some topologies, the expansion of the $\left[\;\right]$ symbol yields terms that would cancel propagators. For those cases we make the convention that all the terms that would cancel propagators are absent. In fact, as we will see, terms that would cancel propagators of the double pentagon topologies naturally yield coefficients for some of the topologies with a smaller number of propagators.

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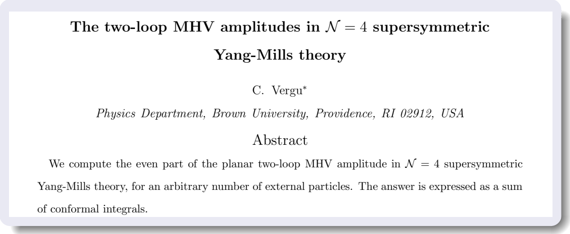

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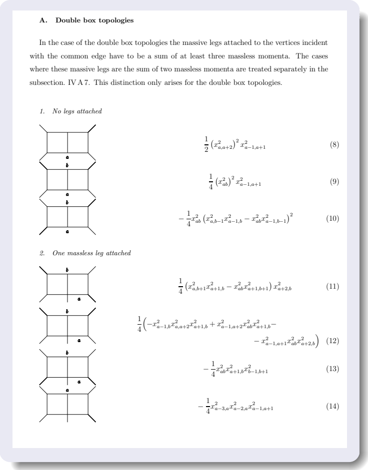

A. Double box topologies

In the case of the double box topologies the massive legs attached to the vertices incident with the common edge have to be a sum of at least three massless momenta. The cases where these massive legs are the sum of two massless momenta are treated separately in the subsection. IV A 7. This distinction only arises for the double box topologies.

1. No legs attached

$$\frac{1}{2}\left(x^2_{a,a+2}\right)^2 x^2_{a-1,a+1} \qquad (8)$$

$$\frac{1}{4}\left(x^2_{ab}\right)^2 x^2_{a-1,a+1} \qquad (9)$$

$$-\frac{1}{4}x^2_{ab}\left(x^2_{a,b-1}x^2_{a-1,b} - x^2_{ab}x^2_{a-1,b-1}\right)^2 \qquad (10)$$

2. One massless leg attached

$$\frac{1}{4}\left(x^2_{a,b+1}x^2_{a+1,b} - x^2_{ab}x^2_{a+1,b+1}\right)x^2_{a+2,b} \qquad (11)$$

$$\frac{1}{4}\Big(-x^2_{a-1,b}x^2_{a,a+2}x^2_{a+1,b} + x^2_{a-1,a+2}x^2_{ab}x^2_{a+1,b}- \\ - x^2_{a-1,a+1}x^2_{ab}x^2_{a+2,b}\Big) \qquad (12)$$

$$-\frac{1}{4}x^2_{ab}x^2_{a+1,b}x^2_{b-1,b+1} \qquad (13)$$

$$-\frac{1}{4}x^2_{a-3,a}x^2_{a-2,a}x^2_{a-1,a+1} \qquad (14)$$

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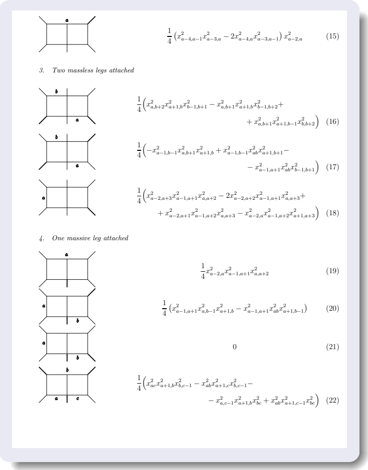


$$\frac{1}{4}\left(x^2_{a-4,a-1}x^2_{a-3,a} - 2x^2_{a-4,a}x^2_{a-3,a-1}\right)x^2_{a-2,a} \qquad (15)$$

3. Two massless legs attached

$$\frac{1}{4}\Big(x^2_{a,b+2}x^2_{a+1,b}x^2_{b-1,b+1} - x^2_{a,b+1}x^2_{a+1,b}x^2_{b-1,b+2}+ \\ + x^2_{a,b+1}x^2_{a+1,b-1}x^2_{b,b+2}\Big) \qquad (16)$$

$$\frac{1}{4}\Big(-x^2_{a-1,b-1}x^2_{a,b+1}x^2_{a+1,b} + x^2_{a-1,b-1}x^2_{ab}x^2_{a+1,b+1}- \\ - x^2_{a-1,a+1}x^2_{ab}x^2_{b-1,b+1}\Big) \qquad (17)$$

$$\frac{1}{4}\Big(x^2_{a-2,a+3}x^2_{a-1,a+1}x^2_{a,a+2} - 2x^2_{a-2,a+2}x^2_{a-1,a+1}x^2_{a,a+3}+ \\ + x^2_{a-2,a+1}x^2_{a-1,a+2}x^2_{a,a+3} - x^2_{a-2,a}x^2_{a-1,a+2}x^2_{a+1,a+3}\Big) \qquad (18)$$

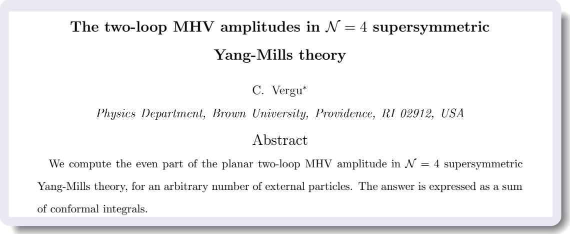
4. One massive leg attached

$$\frac{1}{4}x^2_{a-2,a}x^2_{a-1,a+1}x^2_{a,a+2} \qquad (19)$$

$$\frac{1}{4}\left(x^2_{a-1,a+1}x^2_{a,b-1}x^2_{a+1,b} - x^2_{a-1,a+1}x^2_{ab}x^2_{a+1,b-1}\right) \qquad (20)$$

$$0 \qquad (21)$$

$$\frac{1}{4}\Big(x^2_{ac}x^2_{a+1,b}x^2_{b,c-1} - x^2_{ab}x^2_{a+1,c}x^2_{b,c-1}- \\ - x^2_{a,c-1}x^2_{a+1,b}x^2_{bc} + x^2_{ab}x^2_{a+1,c-1}x^2_{bc}\Big) \qquad (22)$$



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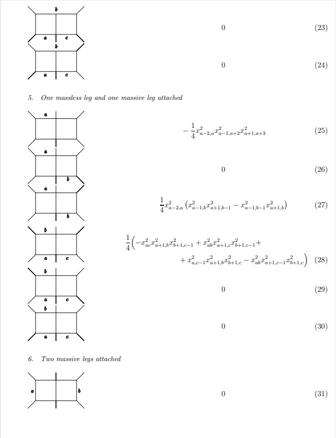


$$0 \qquad (23)$$

$$0 \qquad (24)$$

5. One massless leg and one massive leg attached

$$-\frac{1}{4}x^2_{a-2,a}x^2_{a-1,a+2}x^2_{a+1,a+3} \qquad (25)$$

$$0 \qquad (26)$$

$$\frac{1}{4}x^2_{a-2,a}\left(x^2_{a-1,b}x^2_{a+1,b-1} - x^2_{a-1,b-1}x^2_{a+1,b}\right) \qquad (27)$$

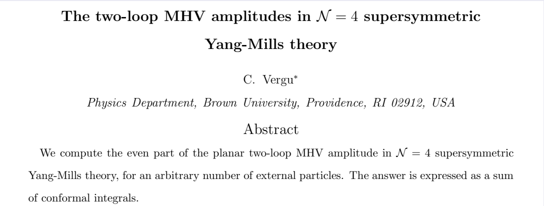
$$\frac{1}{4}\Big(-x^2_{ac}x^2_{a+1,b}x^2_{b+1,c-1} + x^2_{ab}x^2_{a+1,c}x^2_{b+1,c-1}+ \\ + x^2_{a,c-1}x^2_{a+1,b}x^2_{b+1,c} - x^2_{ab}x^2_{a+1,c-1}x^2_{b+1,c}\Big) \qquad (28)$$

$$0 \qquad (29)$$

$$0 \qquad (30)$$

6. Two massive legs attached

$$0 \qquad (31)$$



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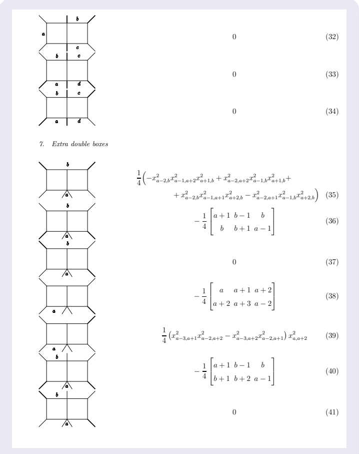


$$0 \qquad (32)$$

$$0 \qquad (33)$$

$$0 \qquad (34)$$

7. Extra double boxes

$$\frac{1}{4}\Big(-x^2_{a-2,b}x^2_{a-1,a+2}x^2_{a+1,b} + x^2_{a-2,a+2}x^2_{a-1,b}x^2_{a+1,b}+ + x^2_{a-2,b}x^2_{a-1,a+1}x^2_{a+2,b} - x^2_{a-2,a+1}x^2_{a-1,b}x^2_{a+2,b}\Big) \qquad (35)$$

$$-\frac{1}{4}\begin{bmatrix} a+1 & b-1 & b \\ b & b+1 & a-1 \end{bmatrix} \qquad (36)$$

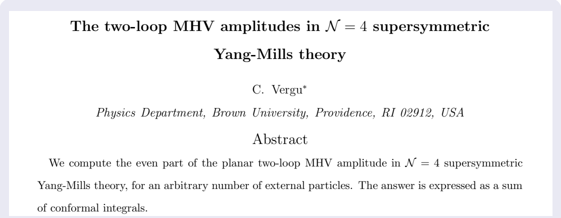
$$0 \qquad (37)$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & a+2 \\ a+2 & a+3 & a-2 \end{bmatrix} \qquad (38)$$

$$\frac{1}{4}\left(x^2_{a-3,a+1}x^2_{a-2,a+2} - x^2_{a-3,a+2}x^2_{a-2,a+1}\right)x^2_{a,a+2} \qquad (39)$$

$$-\frac{1}{4}\begin{bmatrix} a+1 & b-1 & b \\ b+1 & b+2 & a-1 \end{bmatrix} \qquad (40)$$

$$0 \qquad (41)$$



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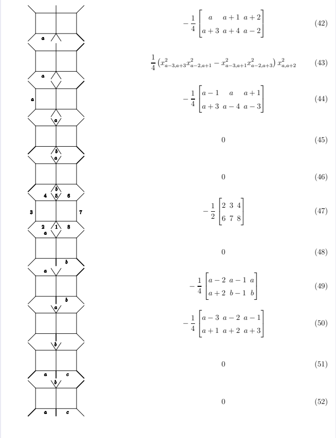

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & a+2 \\ a+3 & a+4 & a-2 \end{bmatrix} \tag{42}$$

$$\frac{1}{4}\left(x^2_{a-3,a+3}x^2_{a-2,a+1} - x^2_{a-3,a+1}x^2_{a-2,a+3}\right) x^2_{a,a+2} \tag{43}$$

$$-\frac{1}{4}\begin{bmatrix} a-1 & a & a+1 \\ a+3 & a-4 & a-3 \end{bmatrix} \tag{44}$$

$$0 \tag{45}$$

$$0 \tag{46}$$

$$-\frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix} \tag{47}$$

$$0 \tag{48}$$

$$-\frac{1}{4}\begin{bmatrix} a-2 & a-1 & a \\ a+2 & b-1 & b \end{bmatrix} \tag{49}$$

$$-\frac{1}{4}\begin{bmatrix} a-3 & a-2 & a-1 \\ a+1 & a+2 & a+3 \end{bmatrix} \tag{50}$$

$$0 \tag{51}$$

$$0 \tag{52}$$

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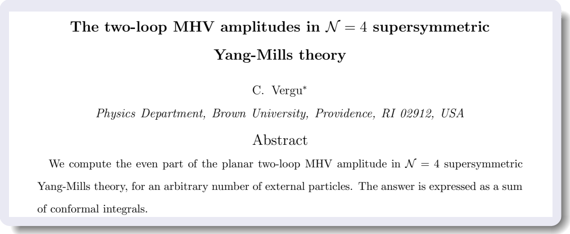

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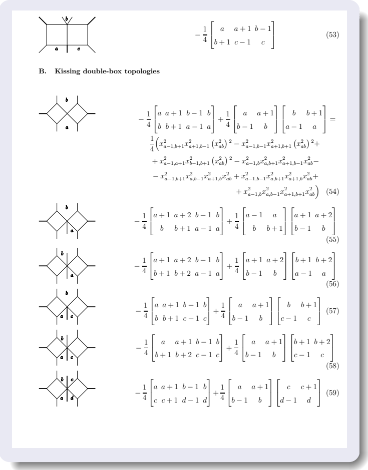

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 \\ b+1 & c-1 & c \end{bmatrix} \tag{53}$$

B. Kissing double-box topologies

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b & b+1 & a-1 & a \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b & b+1 \\ a-1 & a \end{bmatrix}= \\ \frac{1}{4}\Big(x^2_{a-1,b+1}x^2_{a+1,b-1}\left(x^2_{ab}\right)^2-x^2_{a-1,b-1}x^2_{a+1,b+1}\left(x^2_{ab}\right)^2+ \\ +x^2_{a-1,a+1}x^2_{b-1,b+1}\left(x^2_{ab}\right)^2-x^2_{a-1,b}x^2_{a,b+1}x^2_{a+1,b-1}x^2_{ab}- \\ -x^2_{a-1,b+1}x^2_{a,b-1}x^2_{a+1,b}x^2_{ab}+x^2_{a-1,b-1}x^2_{a,b+1}x^2_{a+1,b}x^2_{ab}+ \\ +x^2_{a-1,b}x^2_{a,b-1}x^2_{a+1,b+1}x^2_{ab}\Big) \tag{54}$$

$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a-1 & a \\ b & b+1 \end{bmatrix}\begin{bmatrix} a+1 & a+2 \\ b-1 & b \end{bmatrix} \tag{55}$$

$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a+1 & a+2 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b+1 & b+2 \\ a-1 & a \end{bmatrix} \tag{56}$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b & b+1 & c-1 & c \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b & b+1 \\ c-1 & c \end{bmatrix} \tag{57}$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} b+1 & b+2 \\ c-1 & c \end{bmatrix} \tag{58}$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ c & c+1 & d-1 & d \end{bmatrix}+\frac{1}{4}\begin{bmatrix} a & a+1 \\ b-1 & b \end{bmatrix}\begin{bmatrix} c & c+1 \\ d-1 & d \end{bmatrix} \tag{59}$$

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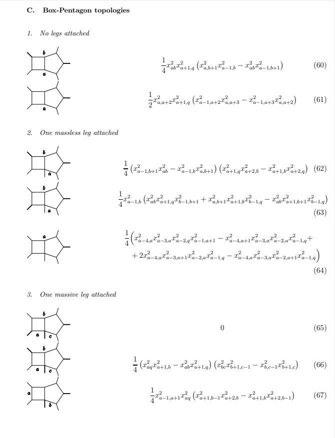


C. Box-Pentagon topologies

1. No legs attached

$$\frac{1}{4}x^2_{ab}x^2_{a+1,q}\left(x^2_{a,b+1}x^2_{a-1,b} - x^2_{ab}x^2_{a-1,b+1}\right) \qquad (60)$$

$$\frac{1}{2}x^2_{a,a+2}x^2_{a+1,q}\left(x^2_{a-1,a+2}x^2_{a,a+3} - x^2_{a-1,a+3}x^2_{a,a+2}\right) \qquad (61)$$

2. One massless leg attached

$$\frac{1}{4}\left(x^2_{a-1,b+1}x^2_{ab} - x^2_{a-1,b}x^2_{a,b+1}\right)\left(x^2_{a+1,q}x^2_{a+2,b} - x^2_{a+1,b}x^2_{a+2,q}\right) \qquad (62)$$

$$\frac{1}{4}x^2_{a-1,b}\left(x^2_{ab}x^2_{a+1,q}x^2_{b-1,b+1} + x^2_{a,b+1}x^2_{a+1,b}x^2_{b-1,q} - x^2_{ab}x^2_{a+1,b+1}x^2_{b-1,q}\right) \qquad (63)$$

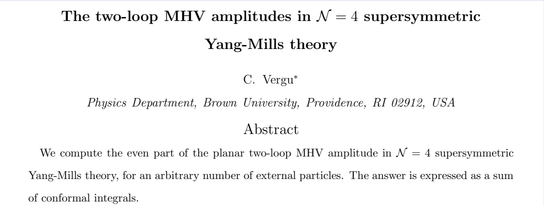
$$\frac{1}{4}\Big(x^2_{a-4,a}x^2_{a-3,a}x^2_{a-2,q}x^2_{a-1,a+1} - x^2_{a-4,a+1}x^2_{a-3,a}x^2_{a-2,a}x^2_{a-1,q} + {} + 2x^2_{a-4,a}x^2_{a-3,a+1}x^2_{a-2,a}x^2_{a-1,q} - x^2_{a-4,a}x^2_{a-3,a}x^2_{a-2,a+1}x^2_{a-1,q}\Big) \qquad (64)$$

3. One massive leg attached

$$0 \qquad (65)$$

$$\frac{1}{4}\left(x^2_{aq}x^2_{a+1,b} - x^2_{ab}x^2_{a+1,q}\right)\left(x^2_{bc}x^2_{b+1,c-1} - x^2_{b,c-1}x^2_{b+1,c}\right) \qquad (66)$$

$$\frac{1}{4}x^2_{a-1,a+1}x^2_{aq}\left(x^2_{a+1,b-1}x^2_{a+2,b} - x^2_{a+1,b}x^2_{a+2,b-1}\right) \qquad (67)$$



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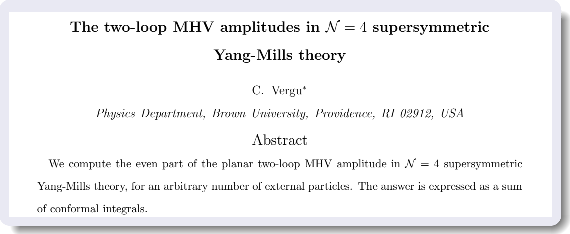

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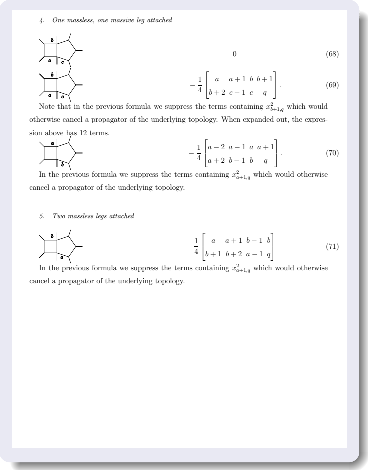

4. One massless, one massive leg attached

$$0 \tag{68}$$

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b & b+1 \\ b+2 & c-1 & c & q \end{bmatrix}. \tag{69}$$

Note that in the previous formula we suppress the terms containing $x^2_{b+1,q}$ which would otherwise cancel a propagator of the underlying topology. When expanded out, the expression above has 12 terms.

$$-\frac{1}{4}\begin{bmatrix} a-2 & a-1 & a & a+1 \\ a+2 & b-1 & b & q \end{bmatrix}. \tag{70}$$

In the previous formula we suppress the terms containing $x^2_{a+1,q}$ which would otherwise cancel a propagator of the underlying topology.

5. Two massless legs attached

$$\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b \\ b+1 & b+2 & a-1 & q \end{bmatrix} \tag{71}$$

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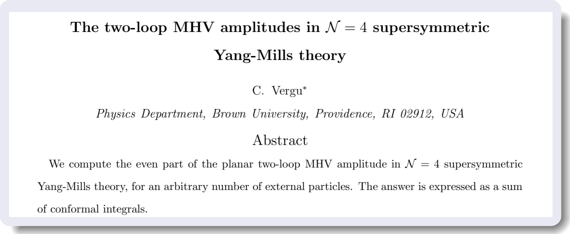

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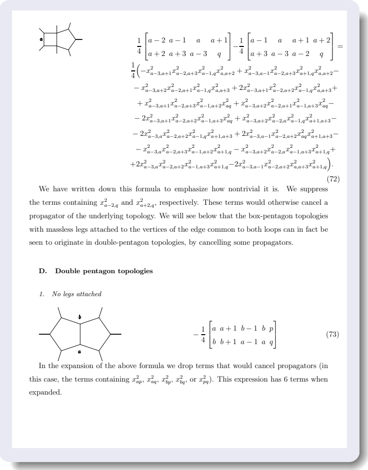

$$\frac{1}{4}\begin{bmatrix} a-2 & a-1 & a & a+1 \\ a+2 & a+3 & a-3 & q \end{bmatrix} - \frac{1}{4}\begin{bmatrix} a-1 & a & a+1 & a+2 \\ a+3 & a-3 & a-2 & q \end{bmatrix} =$$
$$\frac{1}{4}\Big(-x^2_{a-3,a+1}x^2_{a-2,a+3}x^2_{a-1,q}x^2_{a,a+2} + x^2_{a-3,a-1}x^2_{a-2,a+3}x^2_{a+1,q}x^2_{a,a+2} -$$
$$- x^2_{a-3,a+2}x^2_{a-2,a+1}x^2_{a-1,q}x^2_{a,a+3} + 2x^2_{a-3,a+1}x^2_{a-2,a+2}x^2_{a-1,q}x^2_{a,a+3} +$$
$$+ x^2_{a-3,a+1}x^2_{a-2,a+3}x^2_{a-1,a+2}x^2_{aq} + x^2_{a-3,a+2}x^2_{a-2,a+1}x^2_{a-1,a+3}x^2_{aq} -$$
$$- 2x^2_{a-3,a+1}x^2_{a-2,a+2}x^2_{a-1,a+3}x^2_{aq} + x^2_{a-3,a+2}x^2_{a-2,a}x^2_{a-1,q}x^2_{a+1,a+3} -$$
$$- 2x^2_{a-3,a}x^2_{a-2,a+2}x^2_{a-1,q}x^2_{a+1,a+3} + 2x^2_{a-3,a-1}x^2_{a-2,a+2}x^2_{aq}x^2_{a+1,a+3} -$$
$$- x^2_{a-3,a}x^2_{a-2,a+3}x^2_{a-1,a+2}x^2_{a+1,q} - x^2_{a-3,a+2}x^2_{a-2,a}x^2_{a-1,a+3}x^2_{a+1,q} +$$
$$+2x^2_{a-3,a}x^2_{a-2,a+2}x^2_{a-1,a+3}x^2_{a+1,q} - 2x^2_{a-3,a-1}x^2_{a-2,a+2}x^2_{a,a+3}x^2_{a+1,q}\Big). \tag{72}$$

We have written down this formula to emphasize how nontrivial it is. We suppress the terms containing $x^2_{a-2,q}$ and $x^2_{a+2,q}$, respectively. These terms would otherwise cancel a propagator of the underlying topology. We will see below that the box-pentagon topologies with massless legs attached to the vertices of the edge common to both loops can in fact be seen to originate in double-pentagon topologies, by cancelling some propagators.

D. Double pentagon topologies

1. *No legs attached*

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ b & b+1 & a-1 & a & q \end{bmatrix} \tag{73}$$

In the expansion of the above formula we drop terms that would cancel propagators (in this case, the terms containing x^2_{ap}, x^2_{aq}, x^2_{bp}, x^2_{bq}, or x^2_{pq}). This expression has 6 terms when expanded.

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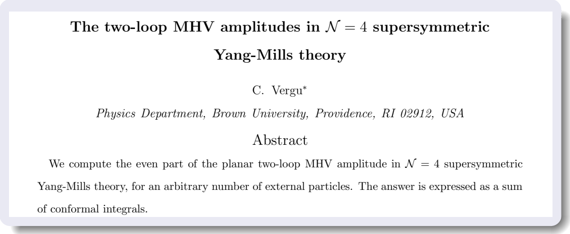

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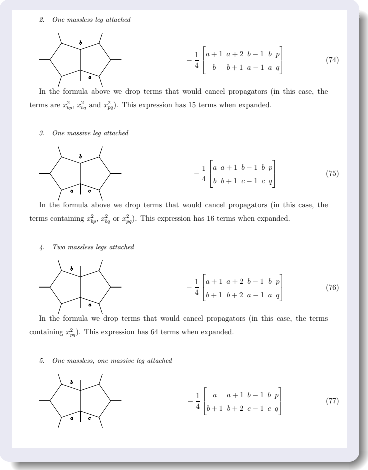

2. *One massless leg attached*

$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b & p \\ b & b+1 & a-1 & a & q \end{bmatrix} \tag{74}$$

In the formula above we drop terms that would cancel propagators (in this case, the terms are x_{bp}^2, x_{bq}^2 and x_{pq}^2). This expression has 15 terms when expanded.

3. *One massive leg attached*

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ b & b+1 & c-1 & c & q \end{bmatrix} \tag{75}$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{bp}^2, x_{bq}^2 or x_{pq}^2). This expression has 16 terms when expanded.

4. *Two massless legs attached*

$$-\frac{1}{4}\begin{bmatrix} a+1 & a+2 & b-1 & b & p \\ b+1 & b+2 & a-1 & a & q \end{bmatrix} \tag{76}$$

In the formula we drop terms that would cancel propagators (in this case, the terms containing x_{pq}^2). This expression has 64 terms when expanded.

5. *One massless, one massive leg attached*

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ b+1 & b+2 & c-1 & c & q \end{bmatrix} \tag{77}$$

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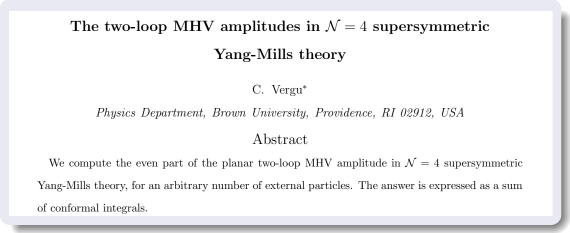

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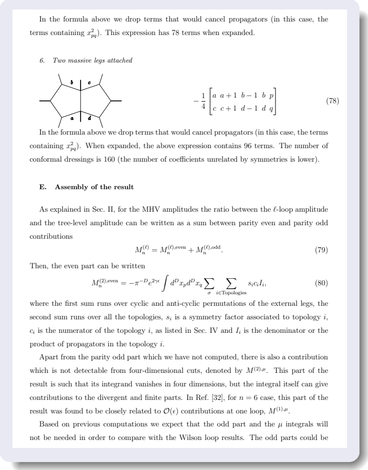

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{pq}^2). This expression has 78 terms when expanded.

6. Two massive legs attached

$$-\frac{1}{4}\begin{bmatrix} a & a+1 & b-1 & b & p \\ c & c+1 & d-1 & d & q \end{bmatrix} \tag{78}$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{pq}^2). When expanded, the above expression contains 96 terms. The number of conformal dressings is 160 (the number of coefficients unrelated by symmetries is lower).

E. Assembly of the result

As explained in Sec. II, for the MHV amplitudes the ratio between the ℓ-loop amplitude and the tree-level amplitude can be written as a sum between parity even and parity odd contributions

$$M_n^{(\ell)} = M_n^{(\ell),\text{even}} + M_n^{(\ell),\text{odd}}. \tag{79}$$

Then, the even part can be written

$$M_n^{(2),\text{even}} = -\pi^{-D} e^{2\gamma\epsilon} \int d^D x_p d^D x_q \sum_{\sigma} \sum_{i \in \text{Topologies}} s_i c_i I_i, \tag{80}$$

where the first sum runs over cyclic and anti-cyclic permutations of the external legs, the second sum runs over all the topologies, s_i is a symmetry factor associated to topology i, c_i is the numerator of the topology i, as listed in Sec. IV and I_i is the denominator or the product of propagators in the topology i.

Apart from the parity odd part which we have not computed, there is also a contribution which is not detectable from four-dimensional cuts, denoted by $M^{(2),\mu}$. This part of the result is such that its integrand vanishes in four dimensions, but the integral itself can give contributions to the divergent and finite parts. In Ref. [32], for $n = 6$ case, this part of the result was found to be closely related to $\mathcal{O}(\epsilon)$ contributions at one loop, $M^{(1),\mu}$.

Based on previous computations we expect that the odd part and the μ integrals will not be needed in order to compare with the Wilson loop results. The odd parts could be

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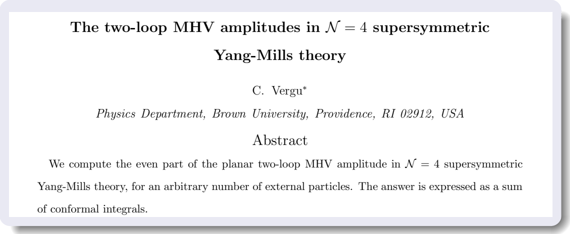

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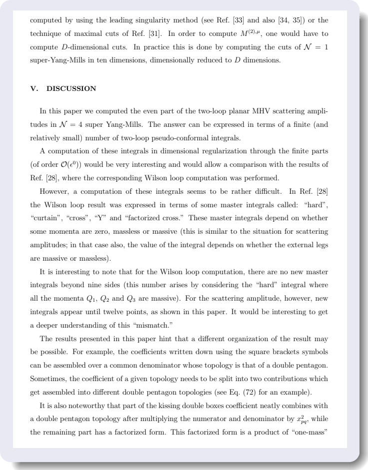

computed by using the leading singularity method (see Ref. [33] and also [34, 35]) or the technique of maximal cuts of Ref. [31]. In order to compute $M^{(2),\mu}$, one would have to compute D-dimensional cuts. In practice this is done by computing the cuts of $\mathcal{N} = 1$ super-Yang-Mills in ten dimensions, dimensionally reduced to D dimensions.

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In this paper we computed the even part of the two-loop planar MHV scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills. The answer can be expressed in terms of a finite (and relatively small) number of two-loop pseudo-conformal integrals.

A computation of these integrals in dimensional regularization through the finite parts (of order $\mathcal{O}(\epsilon^0)$) would be very interesting and would allow a comparison with the results of Ref. [28], where the corresponding Wilson loop computation was performed.

However, a computation of these integrals seems to be rather difficult. In Ref. [28] the Wilson loop result was expressed in terms of some master integrals called: "hard", "curtain", "cross", "Y" and "factorized cross." These master integrals depend on whether some momenta are zero, massless or massive (this is similar to the situation for scattering amplitudes; in that case also, the value of the integral depends on whether the external legs are massive or massless).

It is interesting to note that for the Wilson loop computation, there are no new master integrals beyond nine sides (this number arises by considering the "hard" integral where all the momenta Q_1, Q_2 and Q_3 are massive). For the scattering amplitude, however, new integrals appear until twelve points, as shown in this paper. It would be interesting to get a deeper understanding of this "mismatch."

The results presented in this paper hint that a different organization of the result may be possible. For example, the coefficients written down using the square brackets symbols can be assembled over a common denominator whose topology is that of a double pentagon. Sometimes, the coefficient of a given topology needs to be split into two contributions which get assembled into different double pentagon topologies (see Eq. (72) for an example).

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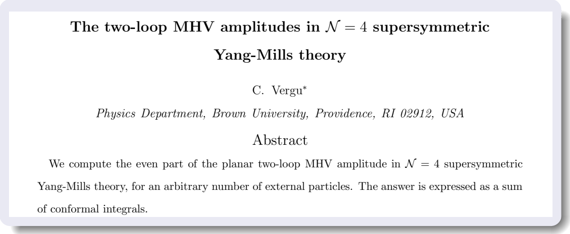

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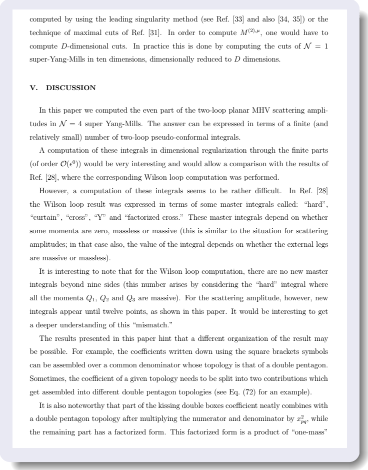

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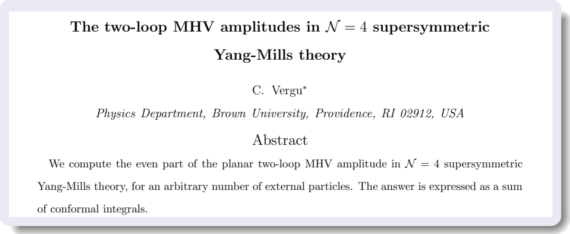

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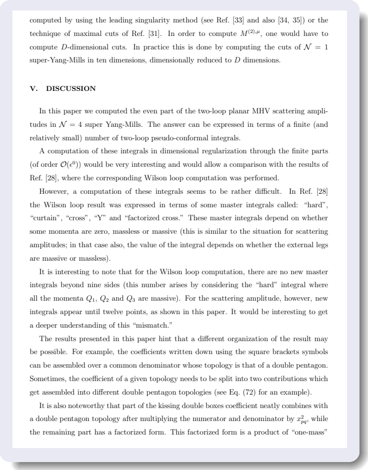

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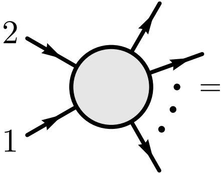

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n1\rangle}\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

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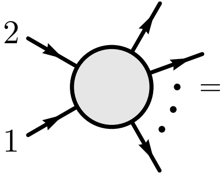

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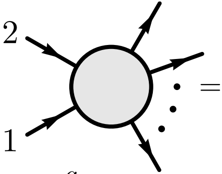

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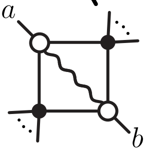

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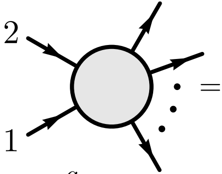

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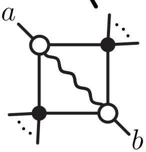
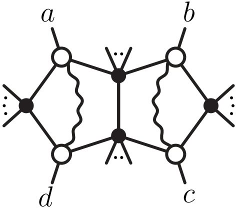


$$+\sum_{a<b<c<d}$$



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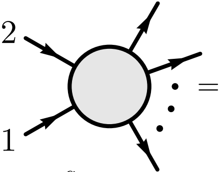

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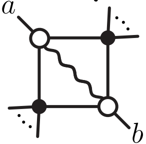

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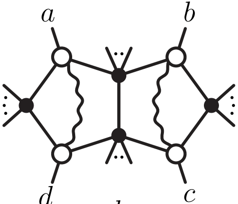
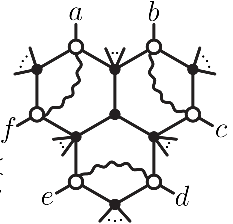
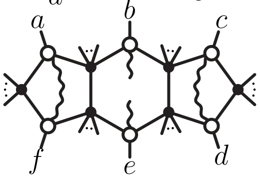


$$+\sum_{\substack{a<b\leq c<\\<d\leq e<f}}$$



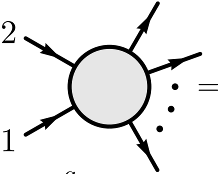
$$+\sum_{\substack{a\leq b<c<\\<d\leq e<f}}$$



$$+\ \ldots$$

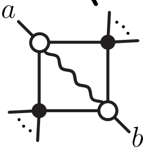
Pushing Parke and Taylor's Amplitude Beyond Tree-Level

Using the recursion relations, (dramatically) simplified formulae were found:

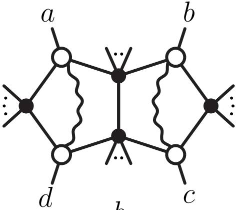


$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n1\rangle}\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

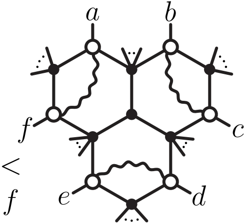
$$\times\Big\{1+\sum_{a<b}$$



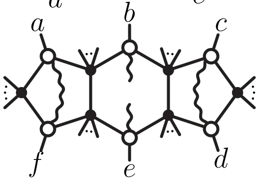
$$+\sum_{a<b<c<d}$$



$$+\sum_{\substack{a<b\leq c<\\ <d\leq e<f}}$$



$$+\sum_{\substack{a\leq b<c<\\ <d\leq e<f}}$$



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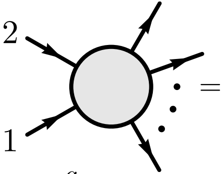

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n1\rangle}\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

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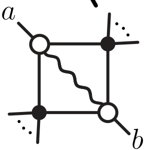
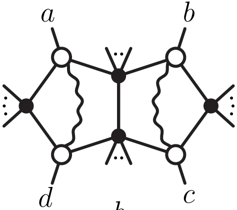
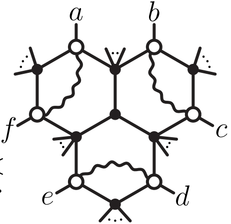


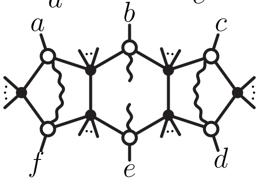
$$+\sum_{a<b<c<d}$$



$$+\sum_{\substack{a<b\leq c<\\<d\leq e<f}}$$



$$+\sum_{\substack{a\leq b<c<\\<d\leq e<f}}$$



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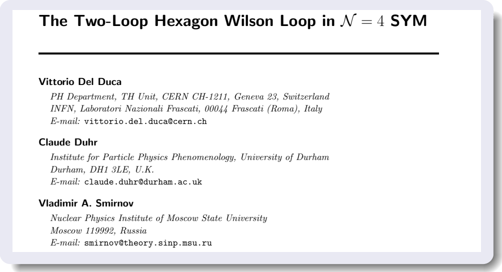

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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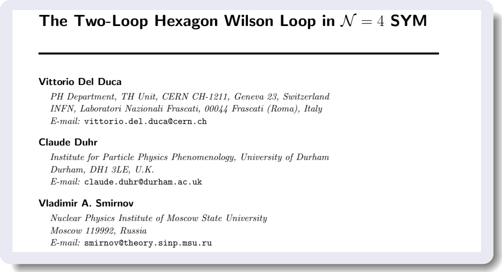

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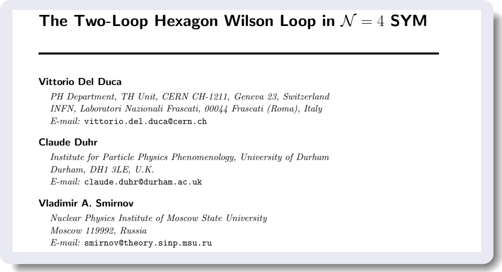

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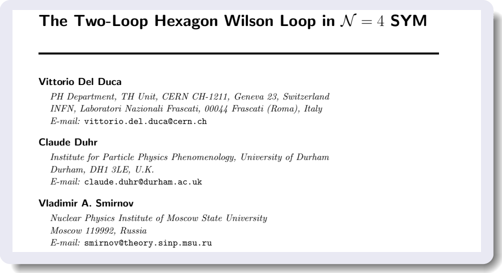

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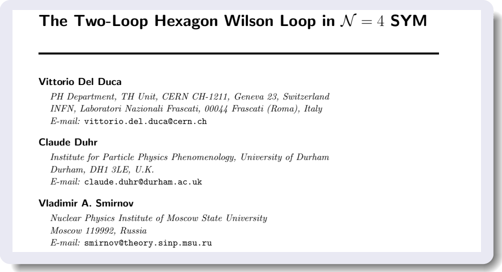

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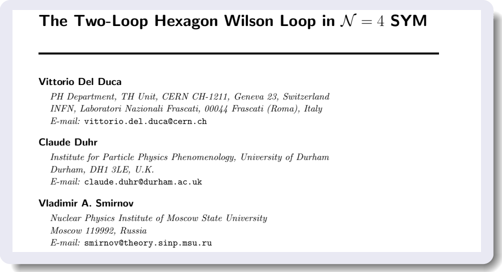

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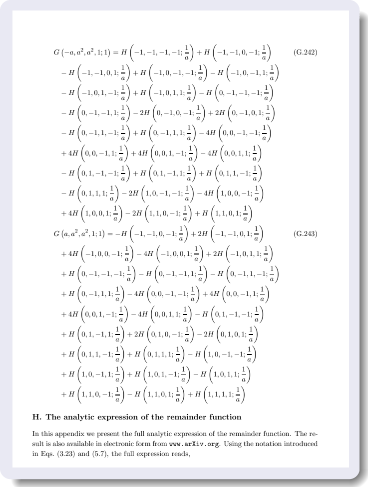

$$\begin{aligned} G\left(-a,a^2,a^2,1;1\right) = {} & H\left(-1,-1,-1,-1;\frac{1}{a}\right) + H\left(-1,-1,0,-1;\frac{1}{a}\right) \\ & - H\left(-1,-1,0,1;\frac{1}{a}\right) + H\left(-1,0,-1,-1;\frac{1}{a}\right) - H\left(-1,0,-1,1;\frac{1}{a}\right) \\ & - H\left(-1,0,1,-1;\frac{1}{a}\right) + H\left(-1,0,1,1;\frac{1}{a}\right) - H\left(0,-1,-1,-1;\frac{1}{a}\right) \\ & - H\left(0,-1,-1,1;\frac{1}{a}\right) - 2H\left(0,-1,0,-1;\frac{1}{a}\right) + 2H\left(0,-1,0,1;\frac{1}{a}\right) \\ & - H\left(0,-1,1,-1;\frac{1}{a}\right) + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) \\ & + 4H\left(0,0,-1,1;\frac{1}{a}\right) + 4H\left(0,0,1,-1;\frac{1}{a}\right) - 4H\left(0,0,1,1;\frac{1}{a}\right) \\ & - H\left(0,1,-1,-1;\frac{1}{a}\right) + H\left(0,1,-1,1;\frac{1}{a}\right) + H\left(0,1,1,-1;\frac{1}{a}\right) \\ & - H\left(0,1,1,1;\frac{1}{a}\right) - 2H\left(1,0,-1,-1;\frac{1}{a}\right) - 4H\left(1,0,0,-1;\frac{1}{a}\right) \\ & + 4H\left(1,0,0,1;\frac{1}{a}\right) - 2H\left(1,1,0,-1;\frac{1}{a}\right) + H\left(1,1,0,1;\frac{1}{a}\right) \end{aligned} \tag{G.242}$$

$$\begin{aligned} G\left(a,a^2,a^2,1;1\right) = {} & -H\left(-1,-1,0,-1;\frac{1}{a}\right) + 2H\left(-1,-1,0,1;\frac{1}{a}\right) \\ & + 4H\left(-1,0,0,-1;\frac{1}{a}\right) - 4H\left(-1,0,0,1;\frac{1}{a}\right) + 2H\left(-1,0,1,1;\frac{1}{a}\right) \\ & + H\left(0,-1,-1,-1;\frac{1}{a}\right) - H\left(0,-1,-1,1;\frac{1}{a}\right) - H\left(0,-1,1,-1;\frac{1}{a}\right) \\ & + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) + 4H\left(0,0,-1,1;\frac{1}{a}\right) \\ & + 4H\left(0,0,1,-1;\frac{1}{a}\right) - 4H\left(0,0,1,1;\frac{1}{a}\right) - H\left(0,1,-1,-1;\frac{1}{a}\right) \\ & + H\left(0,1,-1,1;\frac{1}{a}\right) + 2H\left(0,1,0,-1;\frac{1}{a}\right) - 2H\left(0,1,0,1;\frac{1}{a}\right) \\ & + H\left(0,1,1,-1;\frac{1}{a}\right) + H\left(0,1,1,1;\frac{1}{a}\right) - H\left(1,0,-1,-1;\frac{1}{a}\right) \\ & + H\left(1,0,-1,1;\frac{1}{a}\right) + H\left(1,0,1,-1;\frac{1}{a}\right) - H\left(1,0,1,1;\frac{1}{a}\right) \\ & + H\left(1,1,0,-1;\frac{1}{a}\right) - H\left(1,1,0,1;\frac{1}{a}\right) + H\left(1,1,1,1;\frac{1}{a}\right) \end{aligned} \tag{G.243}$$

H. The analytic expression of the remainder function

In this appendix we present the full analytic expression of the remainder function. The result is also available in electronic form from `www.arXiv.org`. Using the notation introduced in Eqs. (3.23) and (5.7), the full expression reads,

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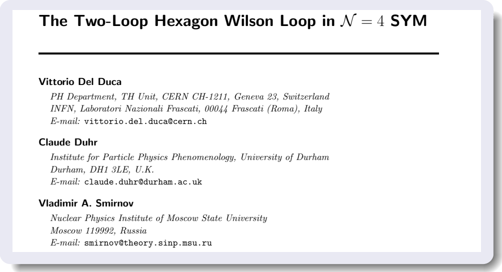

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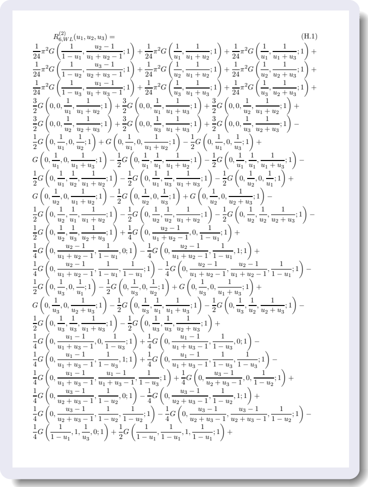

$$
\begin{gathered}
R_{6,WL}^{(2)}(u_1,u_2,u_3) = \\
\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+ \\
\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+ \\
\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+ \\
\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+ \\
\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+ \\
G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+ \\
G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+ \\
\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+ \\
\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+ \\
G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+ \\
\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)- \\
\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)- \\
\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+ \\
\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+ \\
\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)- \\
\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+
\end{gathered}
\tag{H.1}
$$

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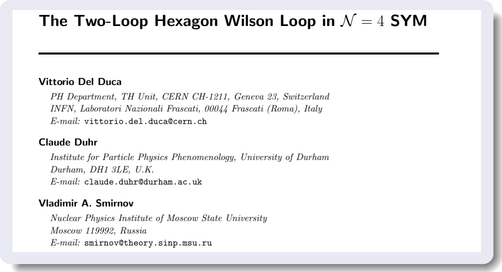

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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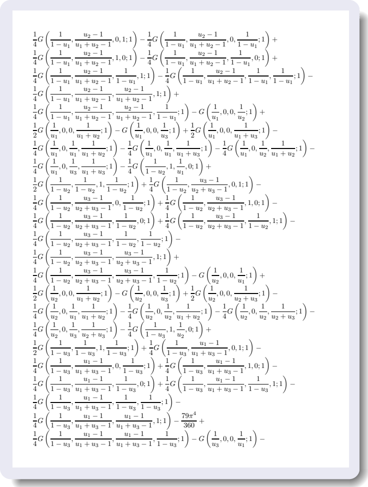

$$
\begin{aligned}
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+\\
&\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+\\
&\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+\\
&\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+\\
&\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-
\end{aligned}
$$

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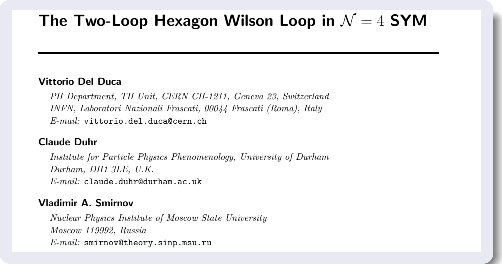

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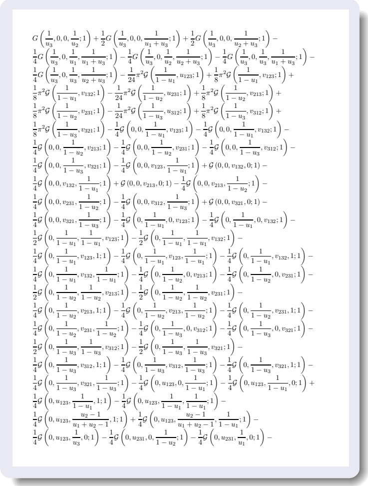

$$G\left(\frac{1}{u_3},0,0,\frac{1}{u_2};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_1+u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{132},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{213},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{213},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{312},\frac{1}{1-u_3};1\right)+\mathcal{G}\left(0,0,v_{321},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{132};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{231};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{321};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1},0;1\right)-$$

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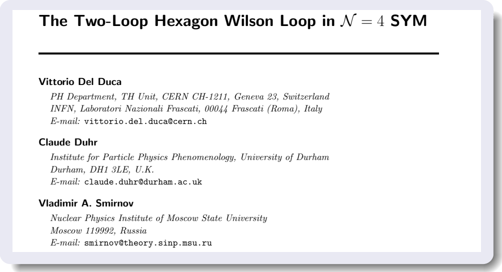

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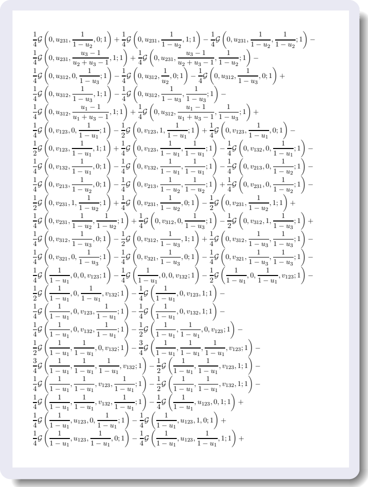

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},0;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{123},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},0;1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},0,\frac{1}{1-u_1};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},0,\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},0,\frac{1}{1-u_2};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},1;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},1,\frac{1}{1-u_3};1\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{123};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{123};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{132};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,1;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,0;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},1;1\right)+
\end{aligned}
$$

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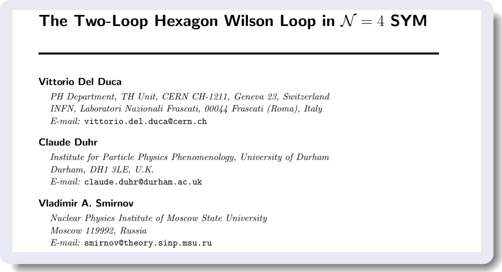

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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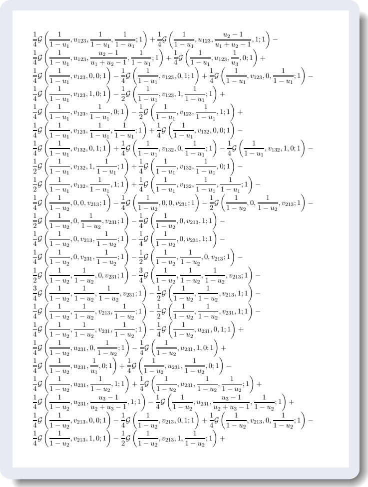

$$\begin{gathered}
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3},0;1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,\frac{1}{1-u_1};1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,\frac{1}{1-u_1};1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},1;1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,0;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,0;1\right)- \\
\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},0;1\right)- \\
\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{231};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{213};1\right)- \\
\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},1;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},1;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{213};1\right)- \\
\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{231};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)- \\
\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},1;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},1;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,1;1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1,0;1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1},0;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},0;1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)+ \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,\frac{1}{1-u_2};1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,\frac{1}{1-u_2};1\right)+
\end{gathered}$$

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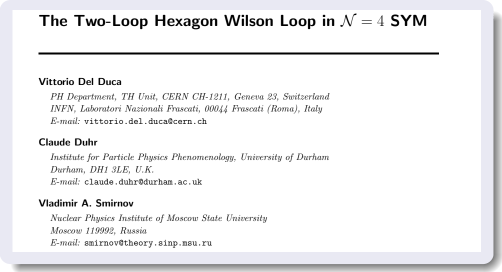

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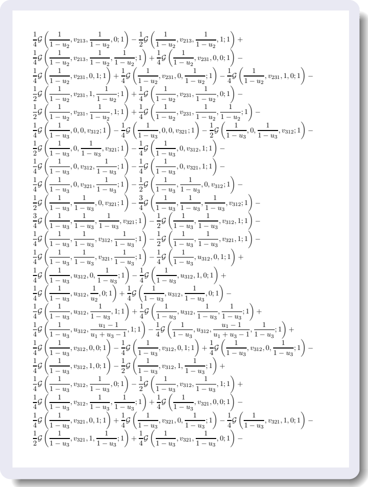

$$\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},1;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,0;1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},0;1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{321};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{312};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{312};1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{321};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\\
&\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,1;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1,0;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,\frac{1}{1-u_3};1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,\frac{1}{1-u_3};1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},1;1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,0;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,0;1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},0;1\right)-
\end{aligned}$$

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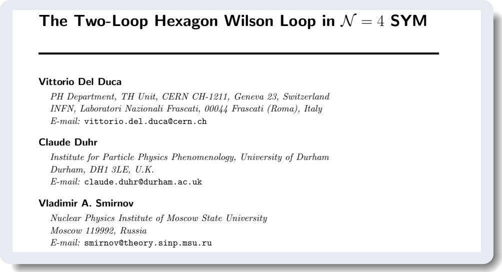

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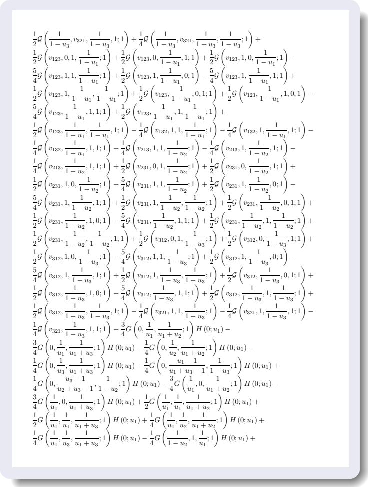

$$\begin{aligned}
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-\\
&\frac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\\
&\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-\\
&\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\\
&\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H\left(0;u_1\right)+
\end{aligned}$$

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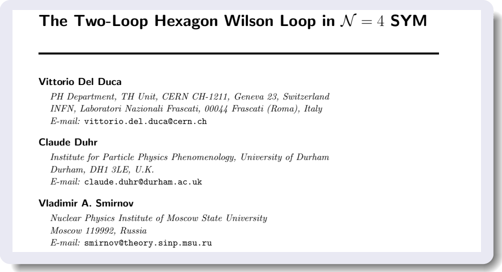

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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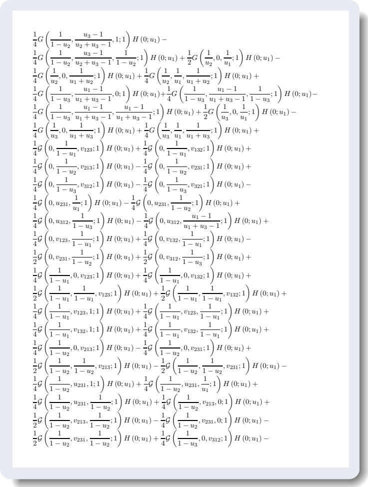

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1;1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0;1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H\left(0;u_1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_1\right)-$$

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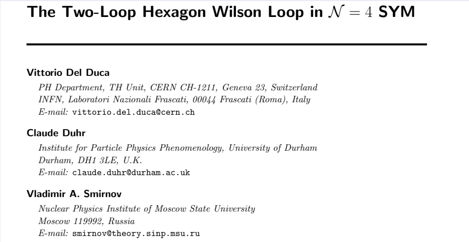

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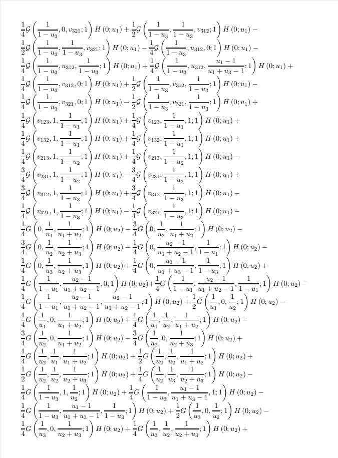

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_1\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0;1\right)H\left(0;u_1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H\left(0;u_1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H\left(0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H\left(0;u_1\right)-\\
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_1\right)+\\
&\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_1\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_1\right)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)-\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0;1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)-\\
&\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)-\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)+\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_2\right)+\\
&\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1;1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_2};1\right)H\left(0;u_2\right)-\\
&\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)+
\end{aligned}
$$

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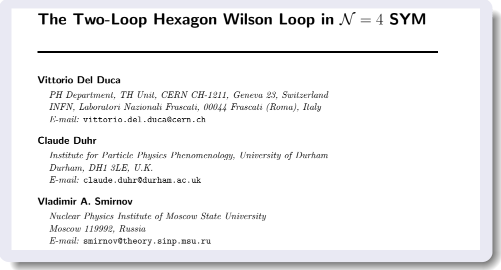

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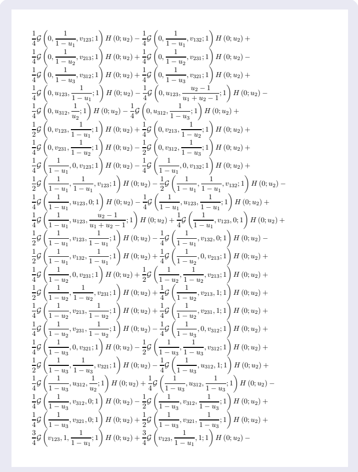

$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_2\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_2\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0;1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H\left(0;u_2\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_2\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H\left(0;u_2\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H\left(0;u_2\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H\left(0;u_2\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H\left(0;u_2\right)+\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H\left(0;u_2\right)-$$

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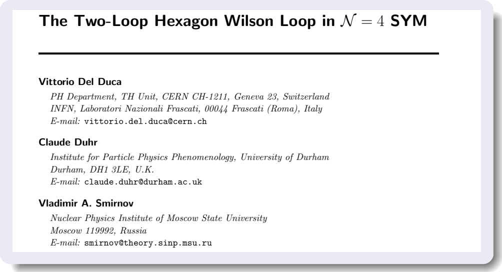

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

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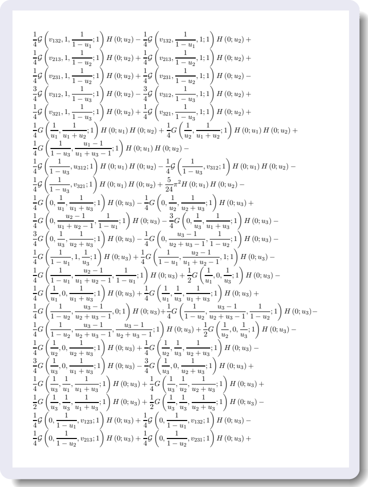

$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_2\right)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_2\right)-\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_2\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)H\left(0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)H\left(0;u_2\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H\left(0;u_1\right)H\left(0;u_2\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_1\right)H\left(0;u_2\right)+\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_2\right)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1;1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0;1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)-\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_3\right)+\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_3\right)+$$

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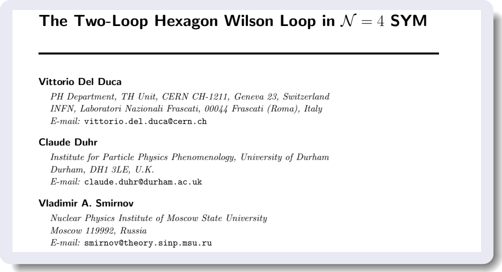

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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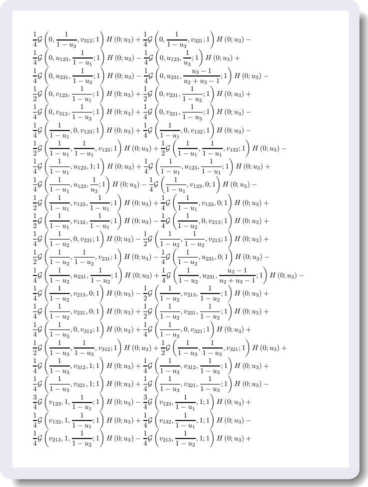

$$\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)-\\
&\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_3\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H\left(0;u_3\right)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_3\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0;1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H\left(0;u_3\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H\left(0;u_3\right)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_3\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1;1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H\left(0;u_3\right)-\\
&\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H\left(0;u_3\right)-\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+
\end{aligned}$$

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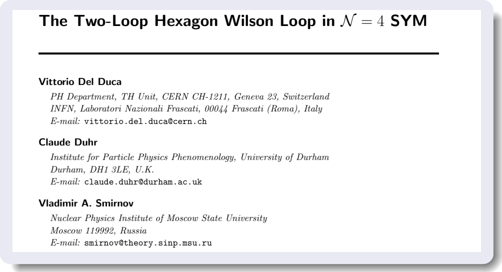

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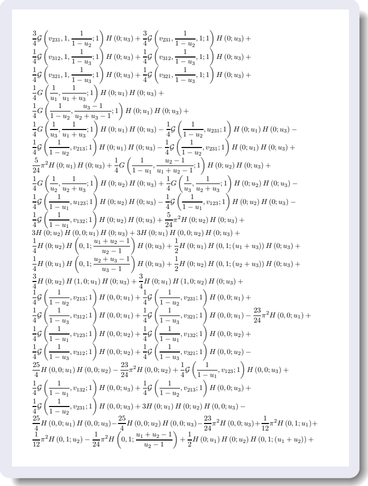

$$
\begin{aligned}
&\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+\\
&\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\\
&\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{5}{24}\pi^2H\left(0;u_2\right)H\left(0;u_3\right)+\\
&3H\left(0;u_2\right)H\left(0,0;u_1\right)H\left(0;u_3\right)+3H\left(0;u_1\right)H\left(0,0;u_2\right)H\left(0;u_3\right)+\\
&\frac{1}{4}H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,1;\left(u_1+u_3\right)\right)H\left(0;u_3\right)+\\
&\frac{1}{4}H\left(0;u_1\right)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,1;\left(u_2+u_3\right)\right)H\left(0;u_3\right)+\\
&\frac{3}{4}H\left(0;u_2\right)H\left(1,0;u_1\right)H\left(0;u_3\right)+\frac{3}{4}H\left(0;u_1\right)H\left(1,0;u_2\right)H\left(0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_1\right)-\frac{23}{24}\pi^2H\left(0,0;u_1\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_2\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_2\right)-\\
&\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_2\right)-\frac{23}{24}\pi^2H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_3\right)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_3\right)+3H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,0;u_3\right)-\\
&\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_3\right)-\frac{25}{4}H\left(0,0;u_2\right)H\left(0,0;u_3\right)-\frac{23}{24}\pi^2H\left(0,0;u_3\right)+\frac{1}{12}\pi^2H\left(0,1;u_1\right)+\\
&\frac{1}{12}\pi^2H\left(0,1;u_2\right)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;\left(u_1+u_2\right)\right)+
\end{aligned}
$$

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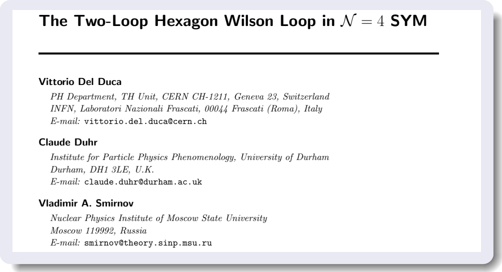

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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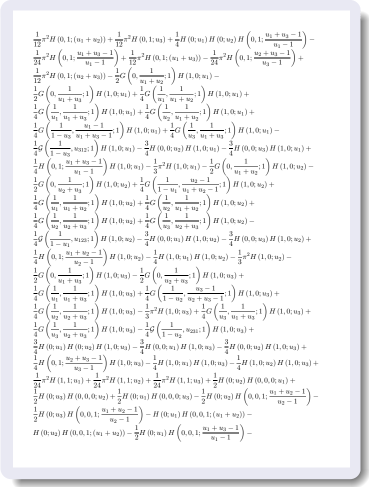

$$\begin{gathered}
\frac{1}{12}\pi^2 H\left(0,1;(u_1+u_2)\right)+\frac{1}{12}\pi^2 H\left(0,1;u_3\right)+\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)- \\
\frac{1}{24}\pi^2 H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\frac{1}{12}\pi^2 H\left(0,1;(u_1+u_3)\right)-\frac{1}{24}\pi^2 H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)+ \\
\frac{1}{12}\pi^2 H\left(0,1;(u_2+u_3)\right)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)+ \\
\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)+ \\
\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H\left(1,0;u_1\right)-\frac{3}{4}H\left(0,0;u_2\right)H\left(1,0;u_1\right)-\frac{3}{4}H\left(0,0;u_3\right)H\left(1,0;u_1\right)+ \\
\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)H\left(1,0;u_1\right)-\frac{1}{3}\pi^2 H\left(1,0;u_1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(1,0;u_2\right)+ \\
\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)+ \\
\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)- \\
\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(1,0;u_2\right)-\frac{3}{4}H\left(0,0;u_1\right)H\left(1,0;u_2\right)-\frac{3}{4}H\left(0,0;u_3\right)H\left(1,0;u_2\right)+ \\
\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(1,0;u_2\right)-\frac{1}{4}H\left(1,0;u_1\right)H\left(1,0;u_2\right)-\frac{1}{3}\pi^2 H\left(1,0;u_2\right)- \\
\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)+ \\
\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(1,0;u_3\right)+ \\
\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{3}\pi^2 H\left(1,0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)+ \\
\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(1,0;u_3\right)+ \\
\frac{3}{4}H\left(0;u_1\right)H\left(0;u_2\right)H\left(1,0;u_3\right)-\frac{3}{4}H\left(0,0;u_1\right)H\left(1,0;u_3\right)-\frac{3}{4}H\left(0,0;u_2\right)H\left(1,0;u_3\right)+ \\
\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(1,0;u_3\right)-\frac{1}{4}H\left(1,0;u_1\right)H\left(1,0;u_3\right)-\frac{1}{4}H\left(1,0;u_2\right)H\left(1,0;u_3\right)+ \\
\frac{1}{24}\pi^2 H\left(1,1;u_1\right)+\frac{1}{24}\pi^2 H\left(1,1;u_2\right)+\frac{1}{24}\pi^2 H\left(1,1;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,0,0;u_1\right)+ \\
\frac{1}{2}H\left(0;u_3\right)H\left(0,0,0;u_2\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,0,0;u_3\right)-\frac{1}{2}H\left(0;u_2\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)- \\
\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-H\left(0;u_1\right)H\left(0,0,1;(u_1+u_2)\right)- \\
H\left(0;u_2\right)H\left(0,0,1;(u_1+u_2)\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-
\end{gathered}$$

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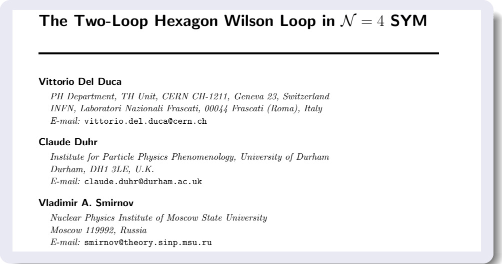

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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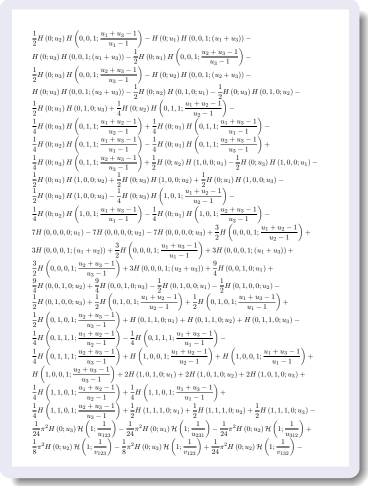

$$\begin{aligned}
&\frac{1}{2}H(0;u_2)\,H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H(0;u_1)\,H(0,0,1;(u_1+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_1+u_3))-\frac{1}{2}H(0;u_1)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&\frac{1}{2}H(0;u_3)\,H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H(0;u_2)\,H(0,0,1;(u_2+u_3))-\\
&H(0;u_3)\,H(0,0,1;(u_2+u_3))-\frac{1}{2}H(0;u_2)\,H(0,1,0;u_1)-\frac{1}{2}H(0;u_3)\,H(0,1,0;u_2)-\\
&\frac{1}{2}H(0;u_1)\,H(0,1,0;u_3)+\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\\
&\frac{1}{4}H(0;u_3)\,H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(0;u_2)\,H(1,0,0;u_1)-\frac{1}{2}H(0;u_3)\,H(1,0,0;u_1)-\\
&\frac{1}{2}H(0;u_1)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_3)\,H(1,0,0;u_2)+\frac{1}{2}H(0;u_1)\,H(1,0,0;u_3)-\\
&\frac{1}{2}H(0;u_2)\,H(1,0,0;u_3)-\frac{1}{4}H(0;u_3)\,H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-\\
&\frac{1}{4}H(0;u_2)\,H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H(0;u_1)\,H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\\
&7H(0,0,0,0;u_1)-7H(0,0,0,0;u_2)-7H(0,0,0,0;u_3)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\\
&3H(0,0,0,1;(u_1+u_2))+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H(0,0,0,1;(u_1+u_3))+\\
&\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H(0,0,0,1;(u_2+u_3))+\frac{9}{4}H(0,0,1,0;u_1)+\\
&\frac{9}{4}H(0,0,1,0;u_2)+\frac{9}{4}H(0,0,1,0;u_3)-\frac{1}{2}H(0,1,0,0;u_1)-\frac{1}{2}H(0,1,0,0;u_2)-\\
&\frac{1}{2}H(0,1,0,0;u_3)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H(0,1,1,0;u_1)+H(0,1,1,0;u_2)+H(0,1,1,0;u_3)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\\
&\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H(1,0,1,0;u_1)+2H(1,0,1,0;u_2)+2H(1,0,1,0;u_3)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\\
&\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H(1,1,1,0;u_1)+\frac{1}{2}H(1,1,1,0;u_2)+\frac{1}{2}H(1,1,1,0;u_3)-\\
&\frac{1}{24}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H(0;u_1)\,\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{8}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H(0;u_3)\,\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H(0;u_2)\,\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-
\end{aligned}$$

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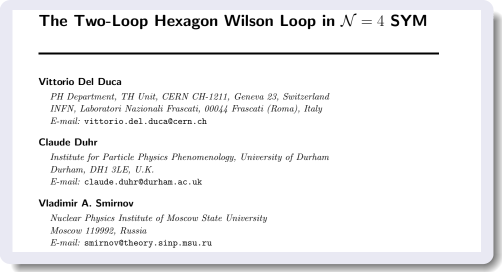

The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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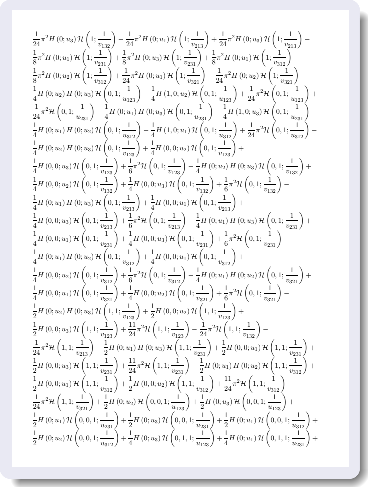

$$\begin{aligned}
&\frac{1}{24}\pi^2 H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-\frac{1}{24}\pi^2 H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{24}\pi^2 H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)-\\
&\frac{1}{8}\pi^2 H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2 H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2 H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)-\\
&\frac{1}{8}\pi^2 H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\frac{1}{24}\pi^2 H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\frac{1}{24}\pi^2 H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\\
&\frac{1}{4}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)-\frac{1}{4}H\left(1,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\\
&\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H\left(1,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\\
&\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\frac{1}{4}H\left(1,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\\
&\frac{1}{4}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\\
&\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\\
&\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)-\\
&\frac{1}{4}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\\
&\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\\
&\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)-\\
&\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\\
&\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\\
&\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)-\\
&\frac{1}{2}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{1}{2}H\left(0,0;u_2\right)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\\
&\frac{1}{2}H\left(0,0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)-\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{132}}\right)-\\
&\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{213}}\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{1}{2}H\left(0,0;u_1\right)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\\
&\frac{1}{2}H\left(0,0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\\
&\frac{1}{2}H\left(0,0;u_1\right)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{1}{2}H\left(0,0;u_2\right)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)-\\
&\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{321}}\right)+\frac{1}{2}H\left(0;u_2\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\frac{1}{2}H\left(0;u_3\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\\
&\frac{1}{2}H\left(0;u_1\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H\left(0;u_3\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H\left(0;u_1\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\\
&\frac{1}{2}H\left(0;u_2\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{231}}\right)+
\end{aligned}$$

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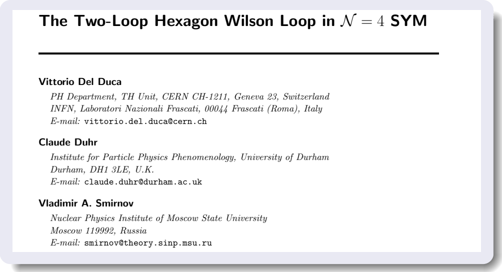

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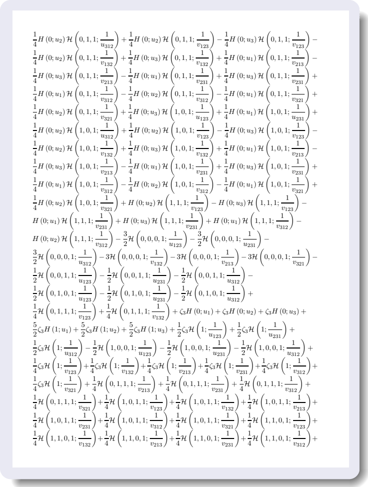

$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+H\left(0;u_2\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-H\left(0;u_3\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-$$
$$H\left(0;u_1\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H\left(0;u_3\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H\left(0;u_1\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-$$
$$H\left(0;u_2\right)\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{132}}\right)+\zeta_3H\left(0;u_1\right)+\zeta_3H\left(0;u_2\right)+\zeta_3H\left(0;u_3\right)+$$
$$\frac{5}{2}\zeta_3H\left(1;u_1\right)+\frac{5}{2}\zeta_3H\left(1;u_2\right)+\frac{5}{2}\zeta_3H\left(1;u_3\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{312}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{132}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}\zeta_3\mathcal{H}\left(1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{312}}\right)+\frac{1}{4}\mathcal{H}\left(1,0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{312}}\right)+$$

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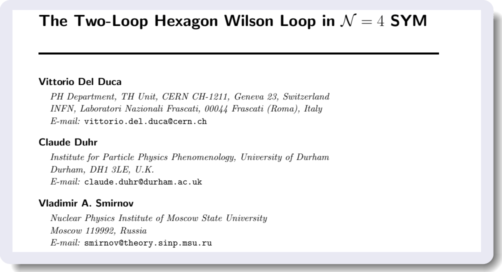

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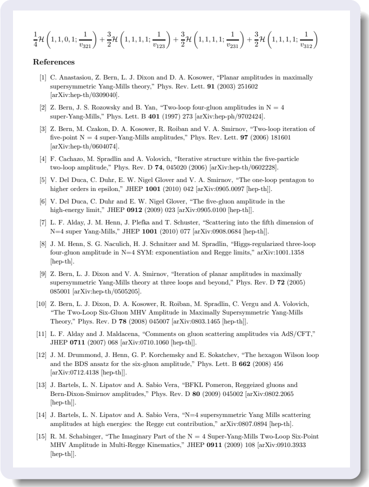

$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$$

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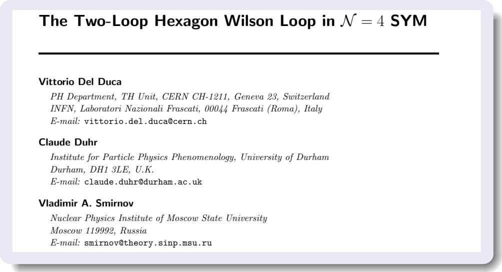
- dimensionally regulating thousands of separately divergent integrals
- final formula: 18 pages of so-called "**Goncharov**" polylogarithms



The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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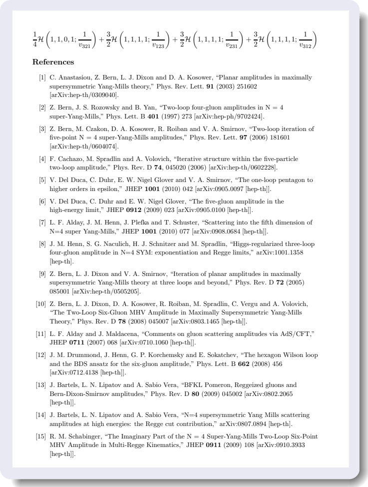

$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$$

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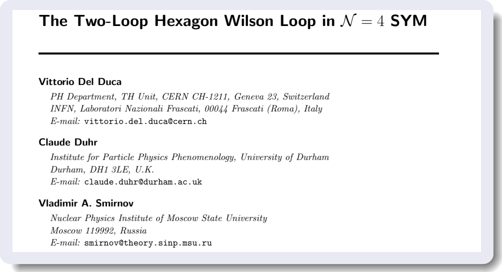

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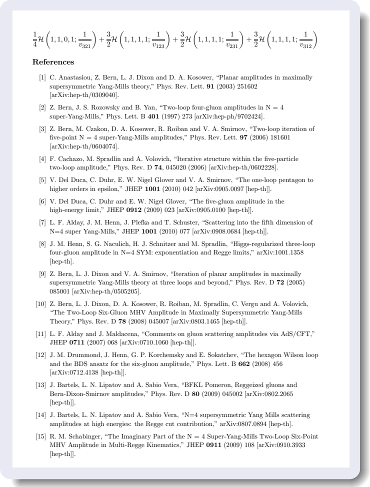

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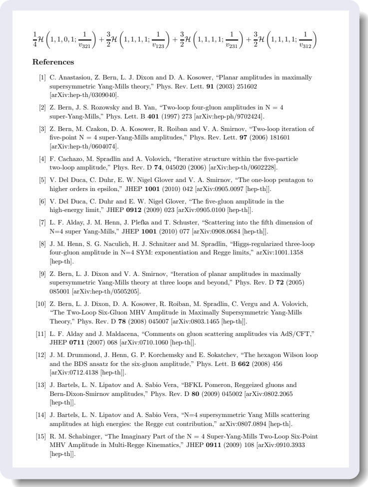

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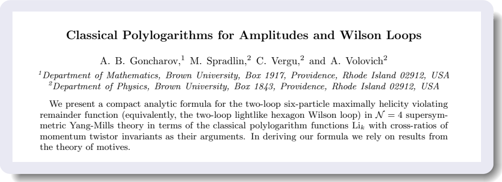

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]
[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

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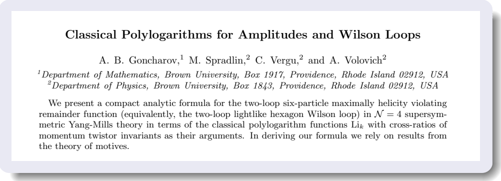
$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}.$$



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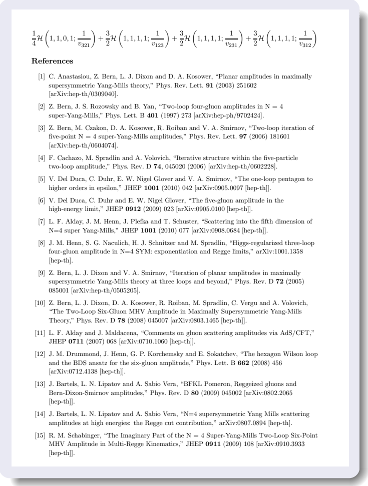

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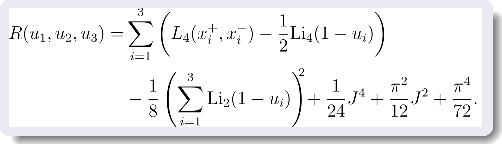

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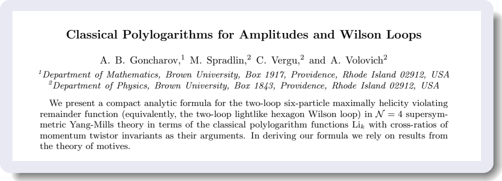

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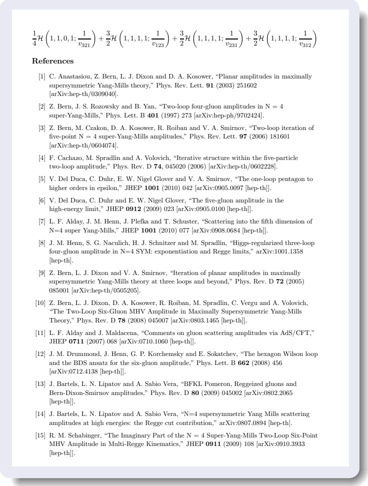

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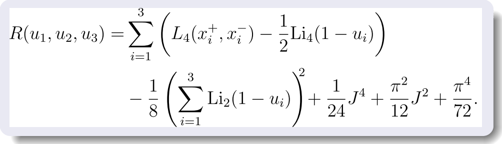
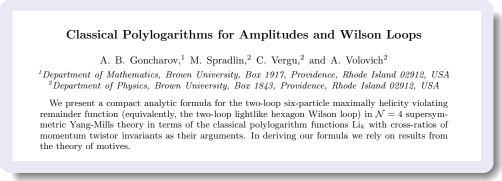


$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)\right)^2+\frac{1}{24}J^4+\frac{\pi^2}{12}J^2+\frac{\pi^4}{72}.$$

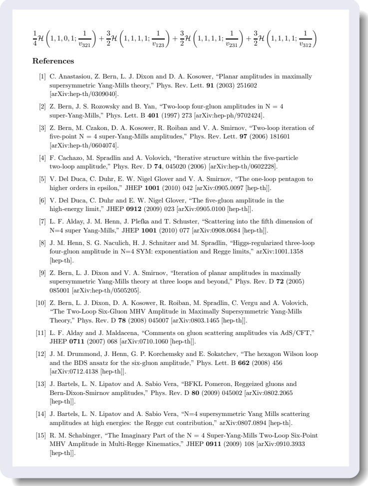


Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]

[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.



$$\frac{1}{4}\mathcal{H}\left(1,1,0,1;\frac{1}{v_{321}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{123}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{231}}\right)+\frac{3}{2}\mathcal{H}\left(1,1,1,1;\frac{1}{v_{312}}\right)$$

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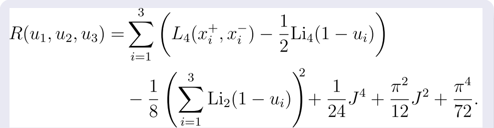

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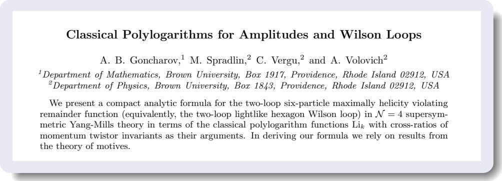

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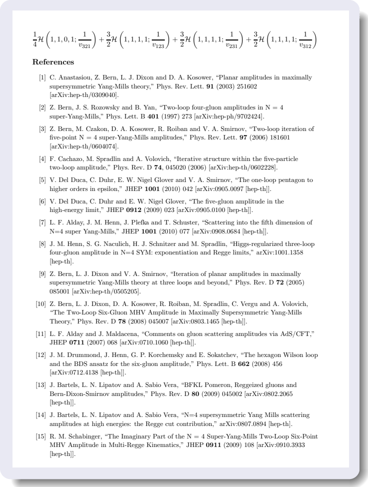

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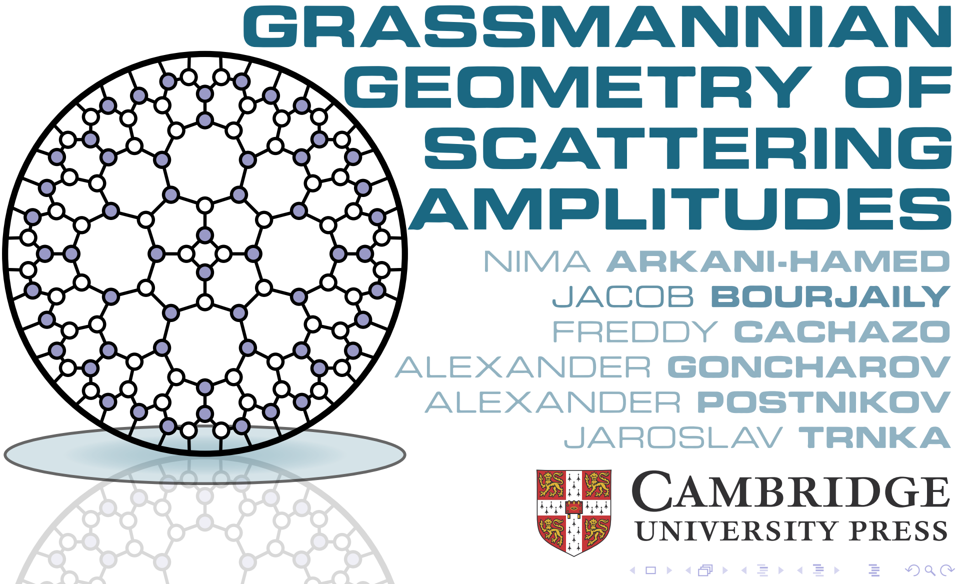

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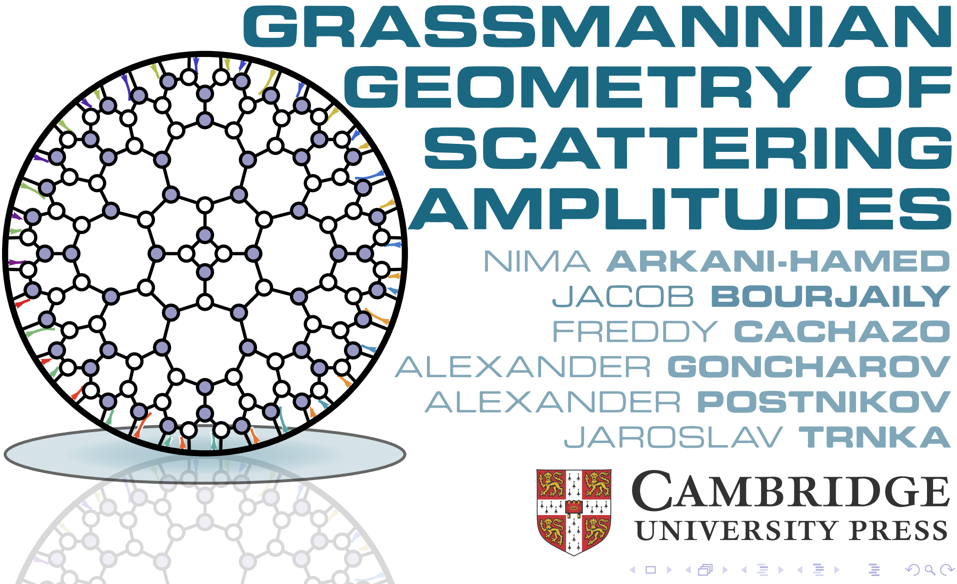

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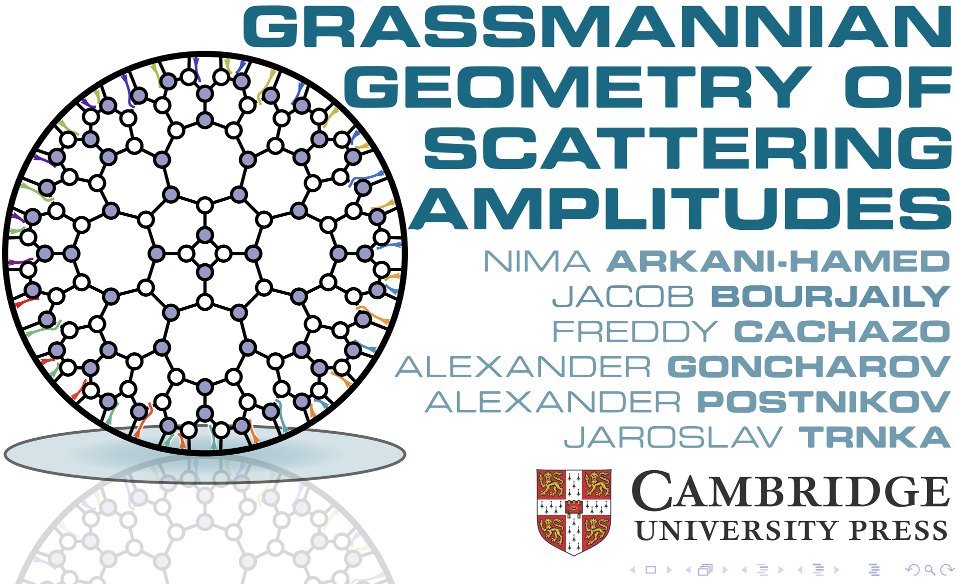

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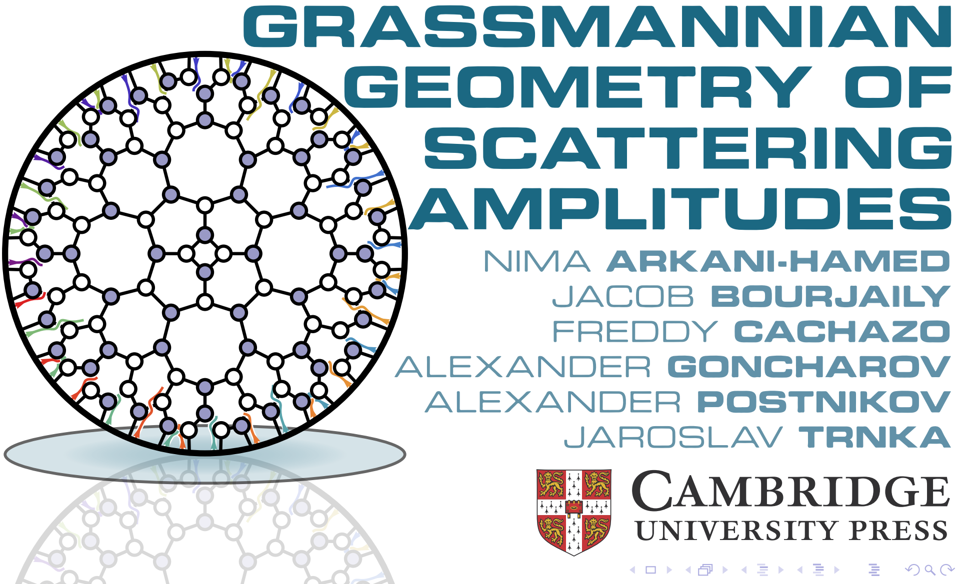

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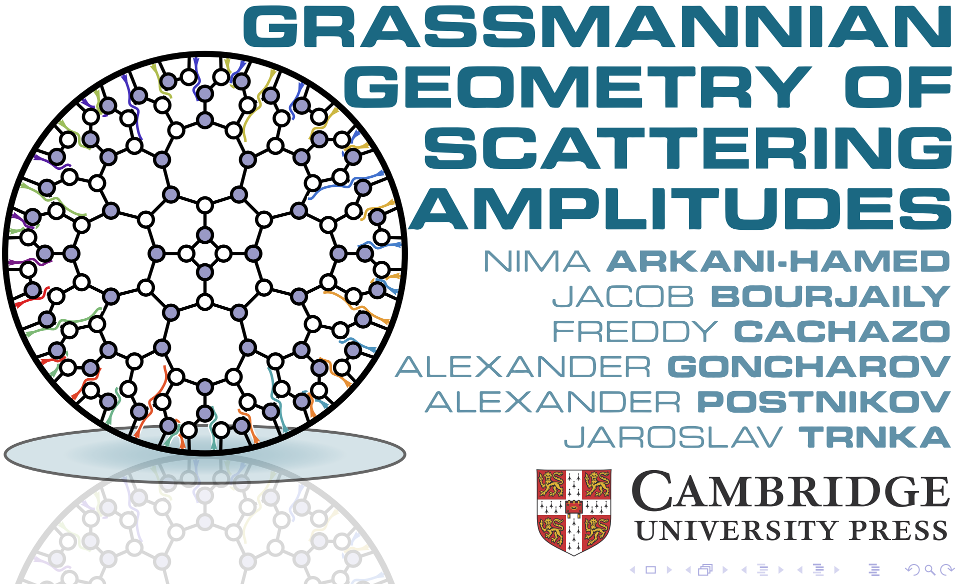

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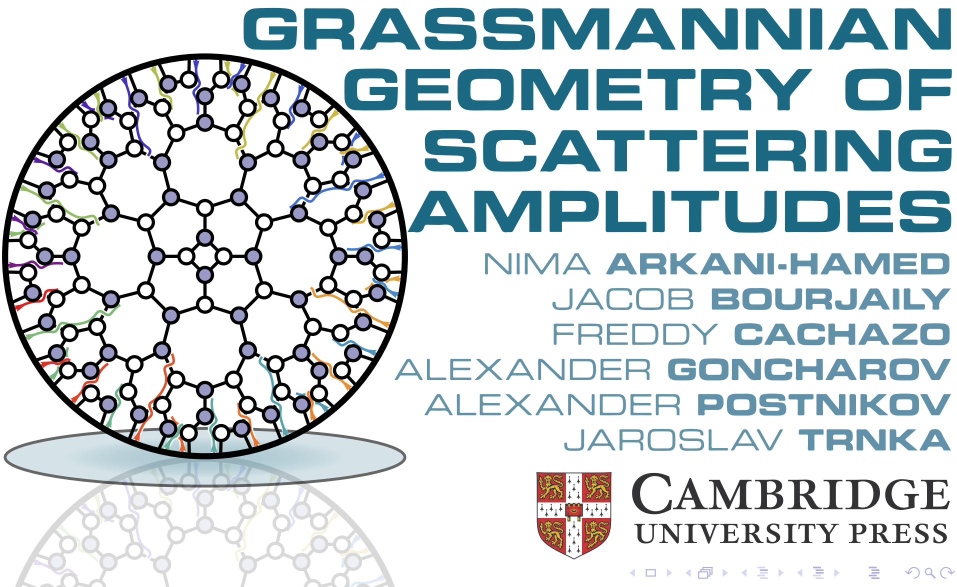

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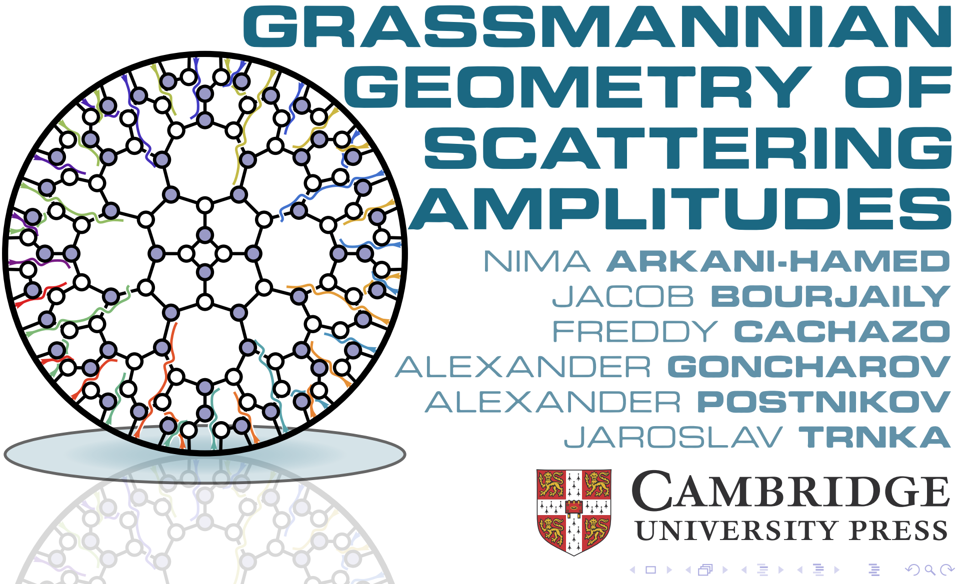

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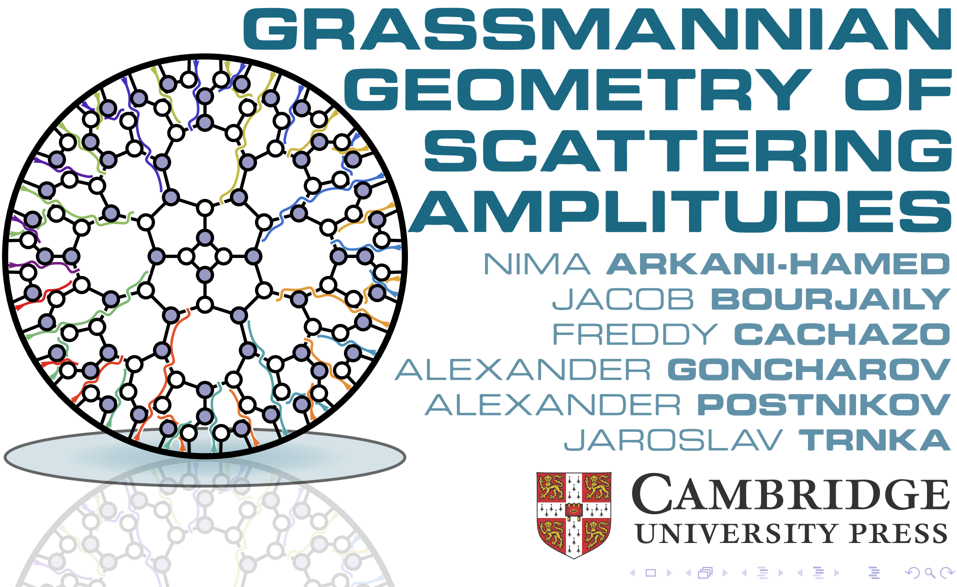

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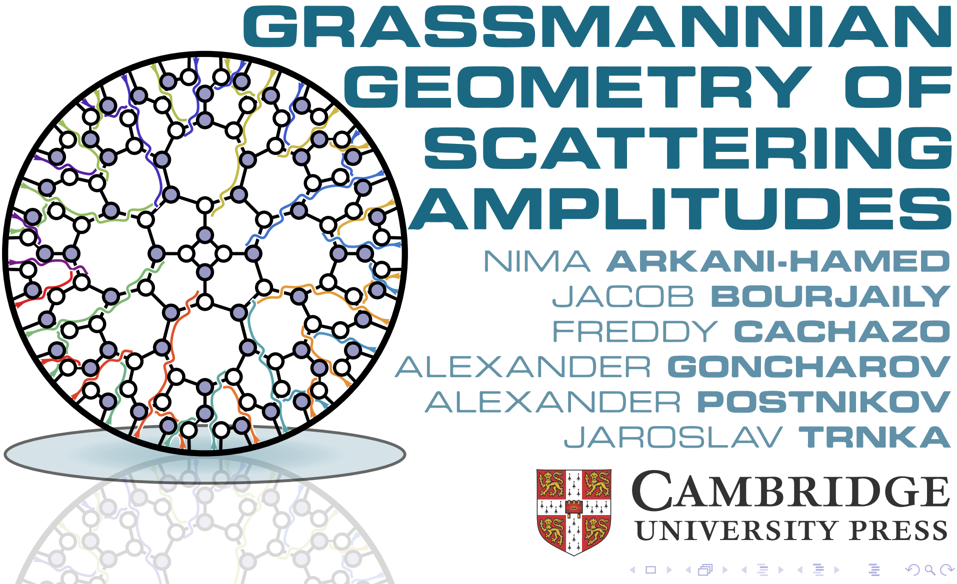

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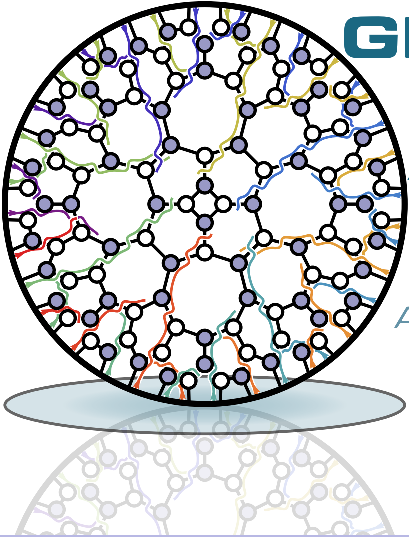

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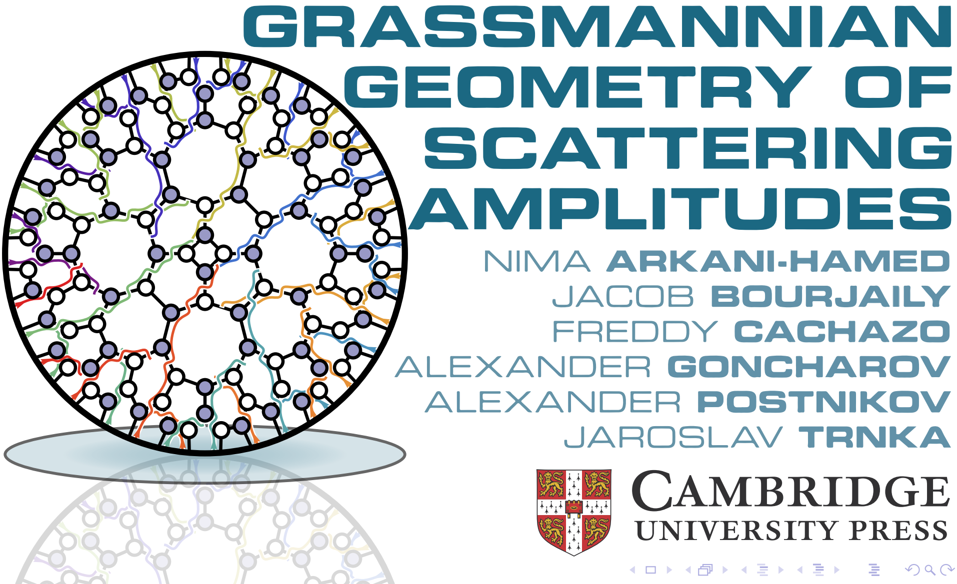

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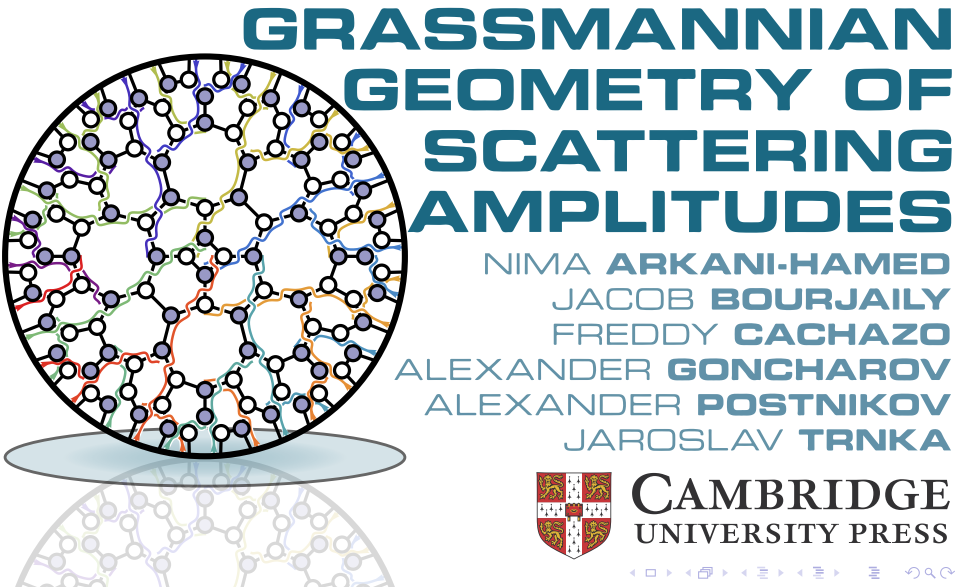

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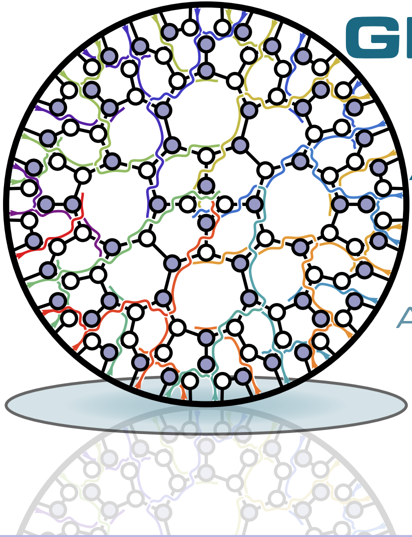

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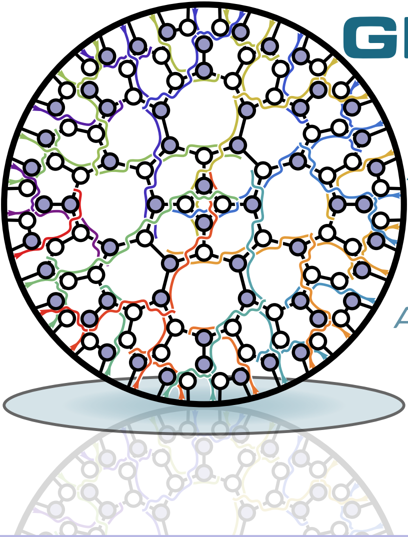

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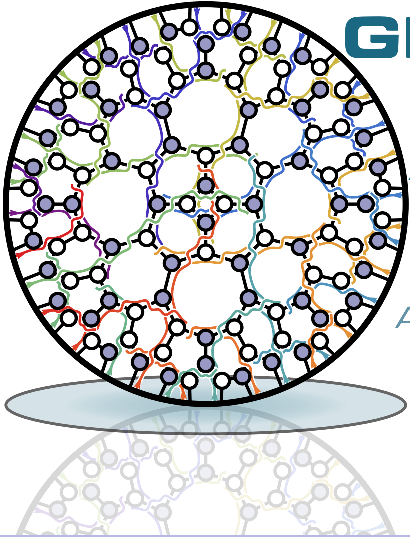

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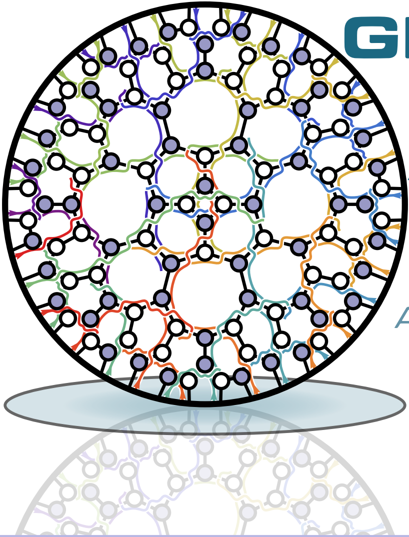

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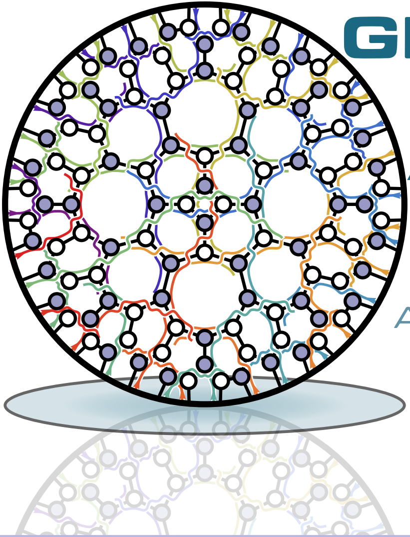

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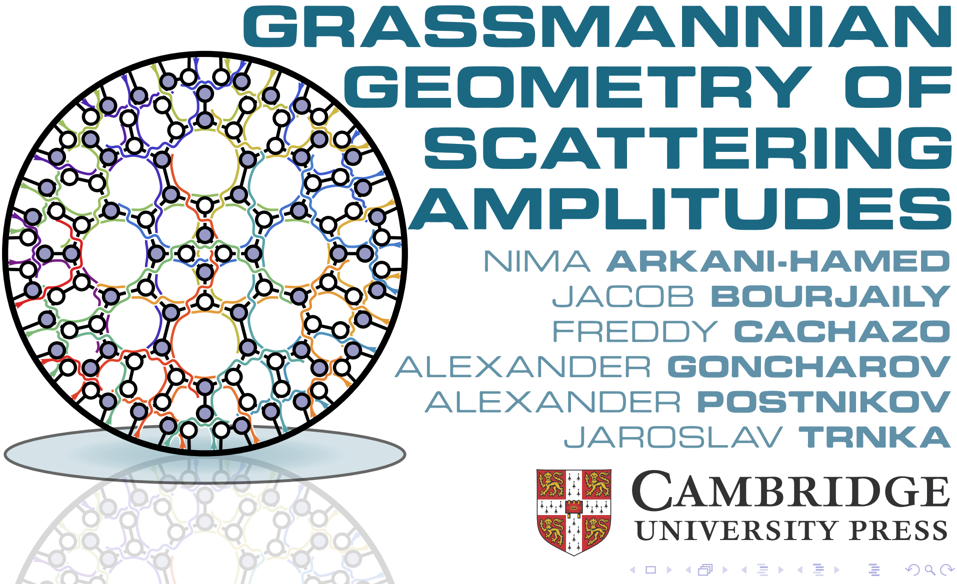

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ALEXANDER **POSTNIKOV**
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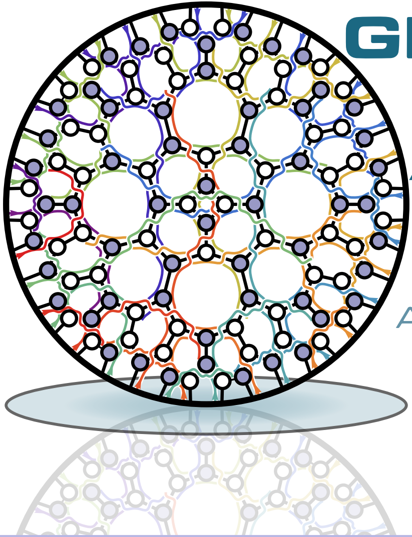

GRASSMANNIAN GEOMETRY OF SCATTERING AMPLITUDES

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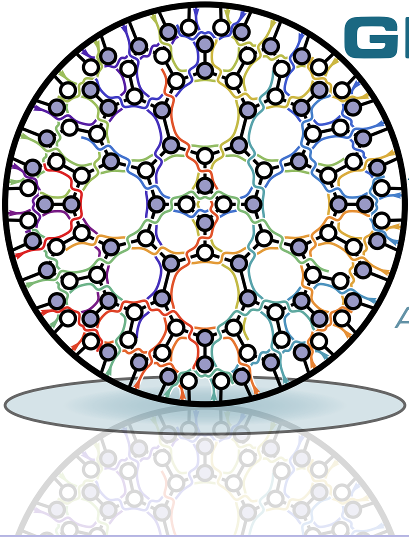

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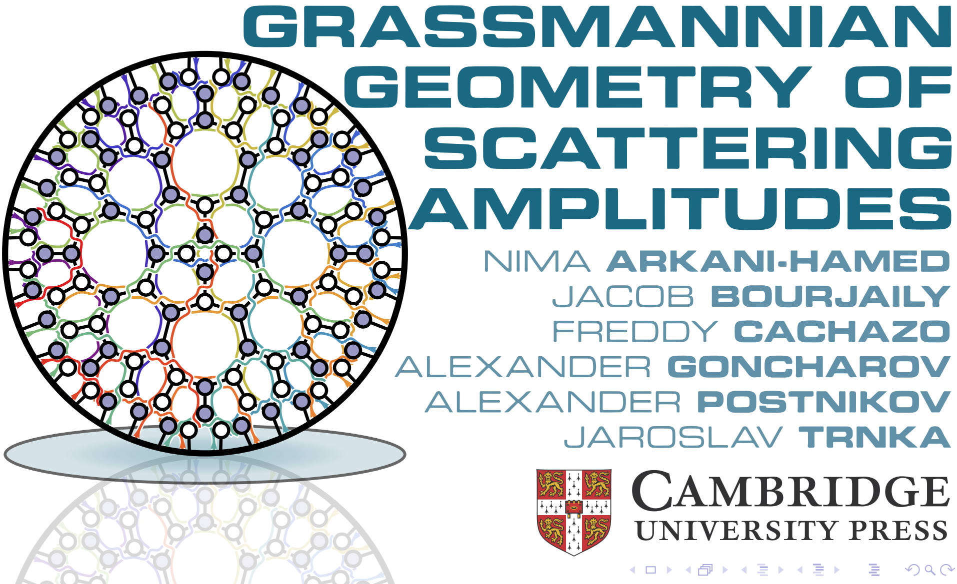

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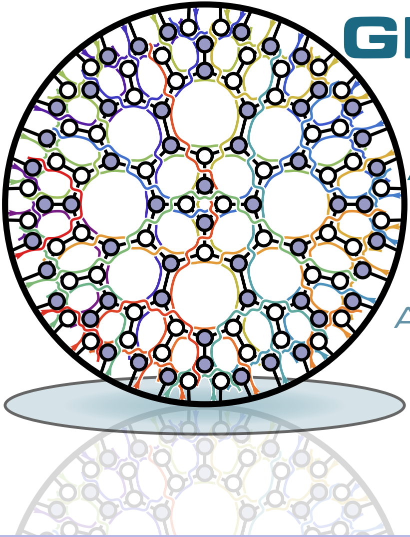

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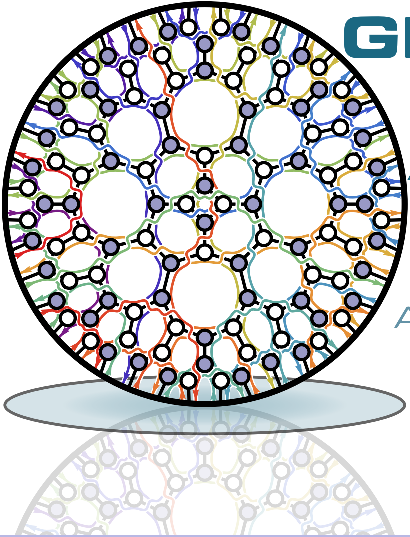

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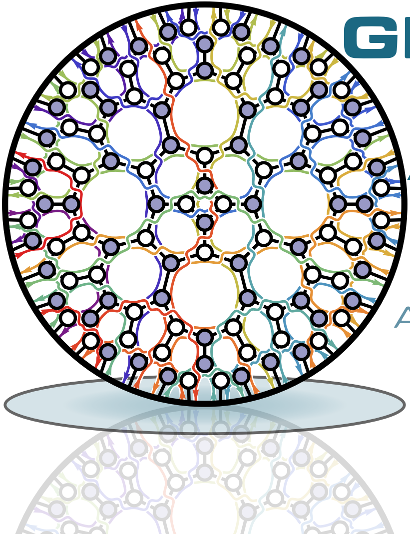

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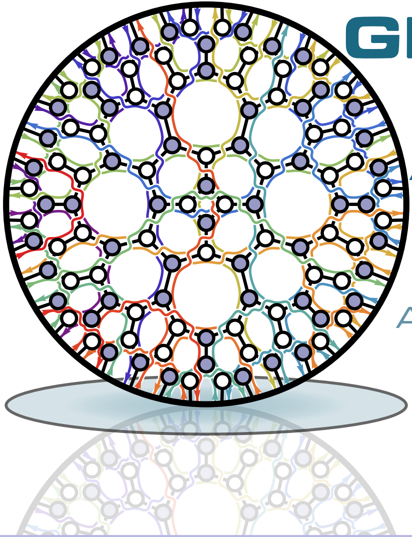

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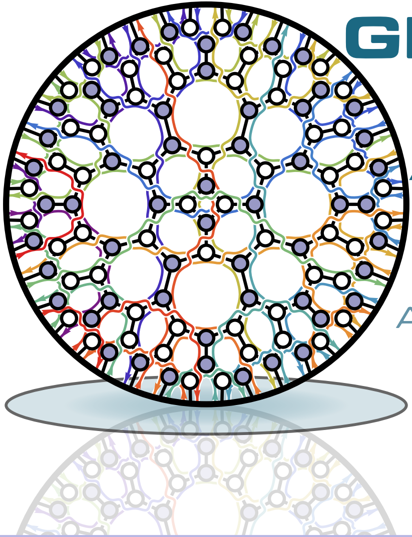

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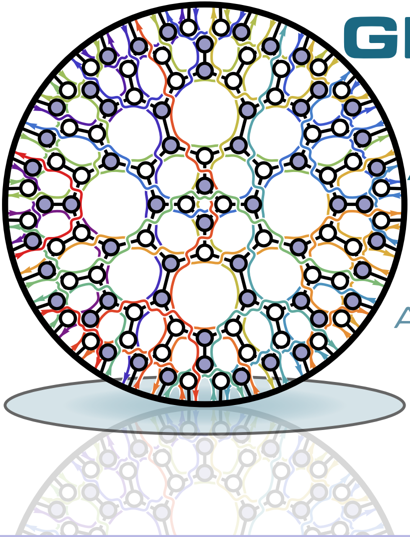

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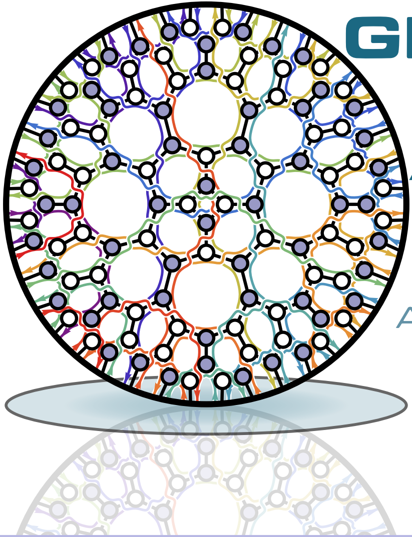

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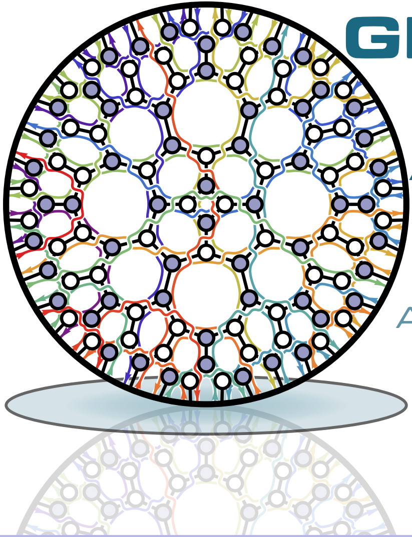

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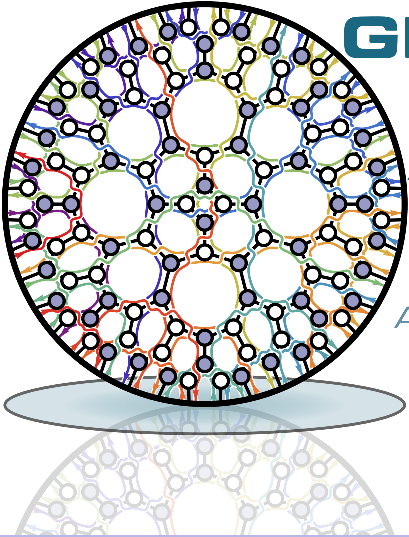

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