Hidden symmetries of Scattering Amplitudes
(and of Hydrogen)

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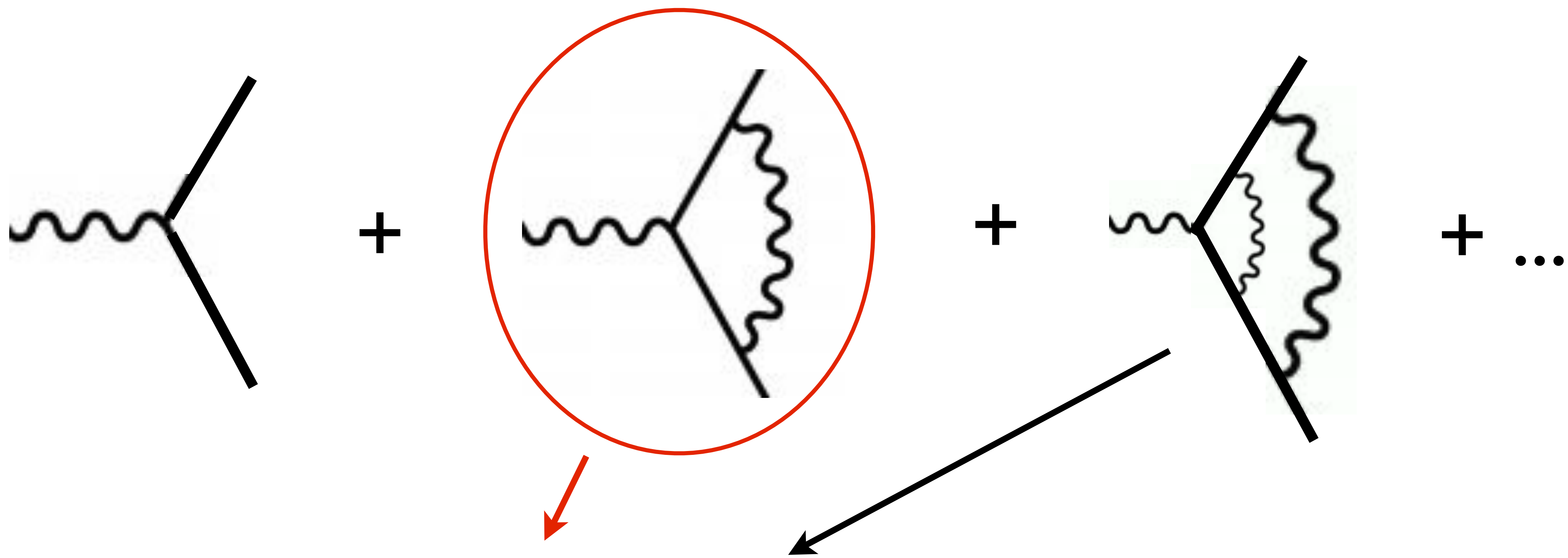
Outline

- Perturbative calculations of amplitudes: why a toy model
- Why a special model, 'N=4 SYM', is solvable: *Laplace-Runge-Lenz* symmetry

- Quantum Field Theory is a successful and powerful tool used in many fields
- Typically calculations are performed in a perturbative expansion
- ‘Scattering amplitudes’ -> probabilities. Needed for precision studies at LHC
- Calculations notoriously difficult. Why?

- Ex.: magnetic moment of the electron



$$g_e = 2 + \frac{\alpha}{\pi} + \ldots = 2.00232\ldots \qquad (\alpha \approx 1/137)$$

- Huge gap between difficulty of calculation, and simplicity of final results
- Gap widens for more complicated processes

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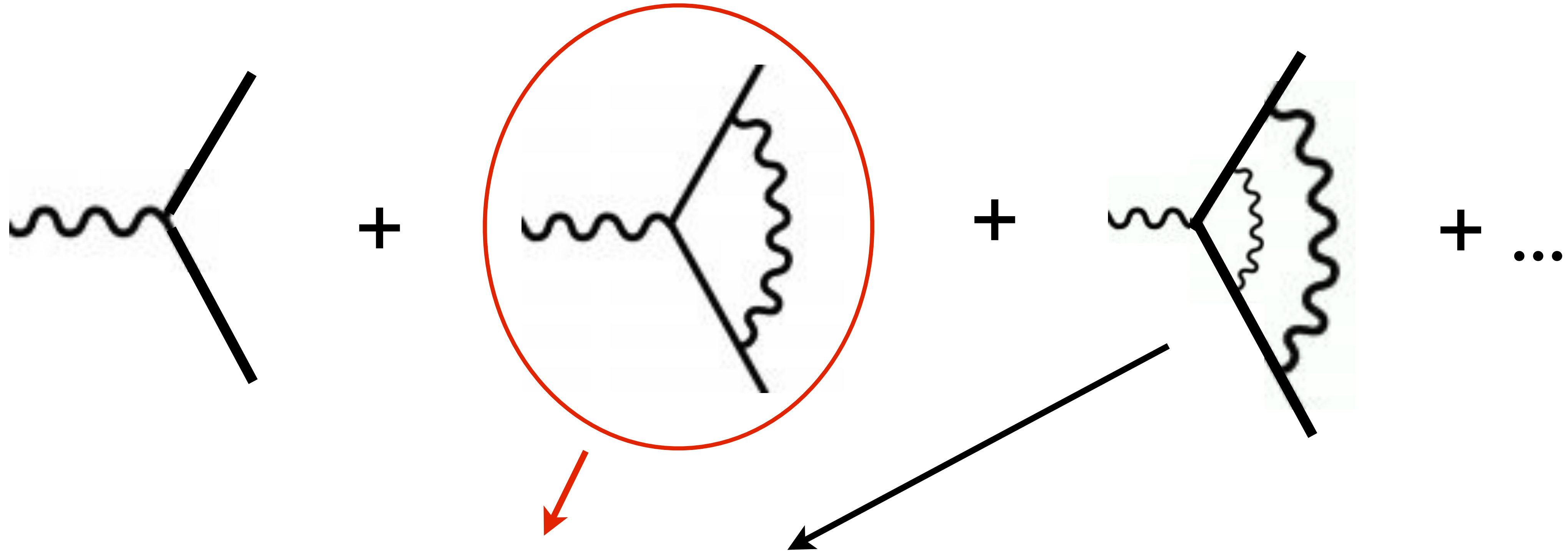

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- More detailed example: at $(\alpha/\pi)^2$, one gets a few transcendental numbers: $\zeta(3)$, $\pi^2/6$, $\log(2)$, $1/6$.
- Analytic result is a list of integers:
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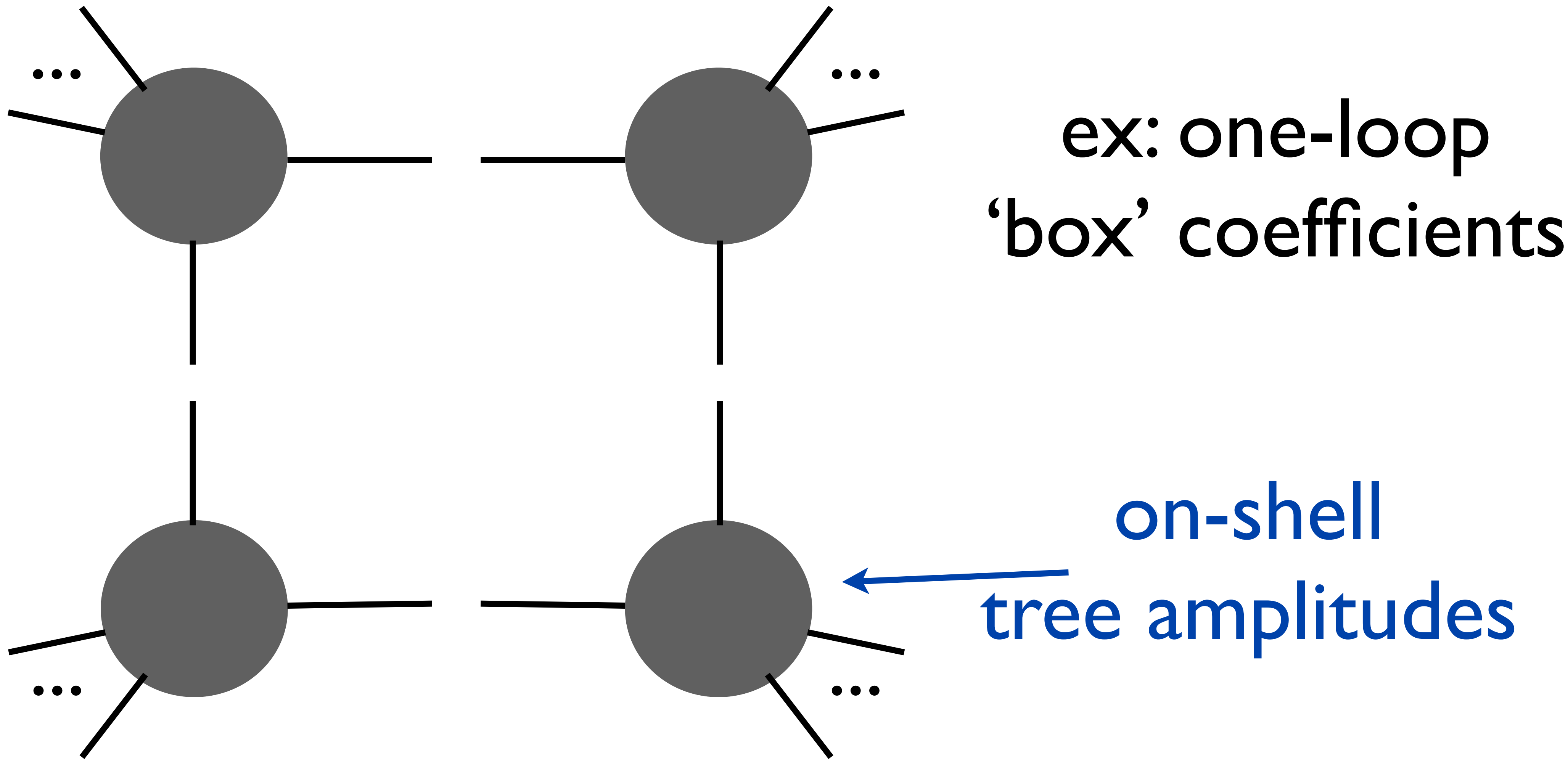

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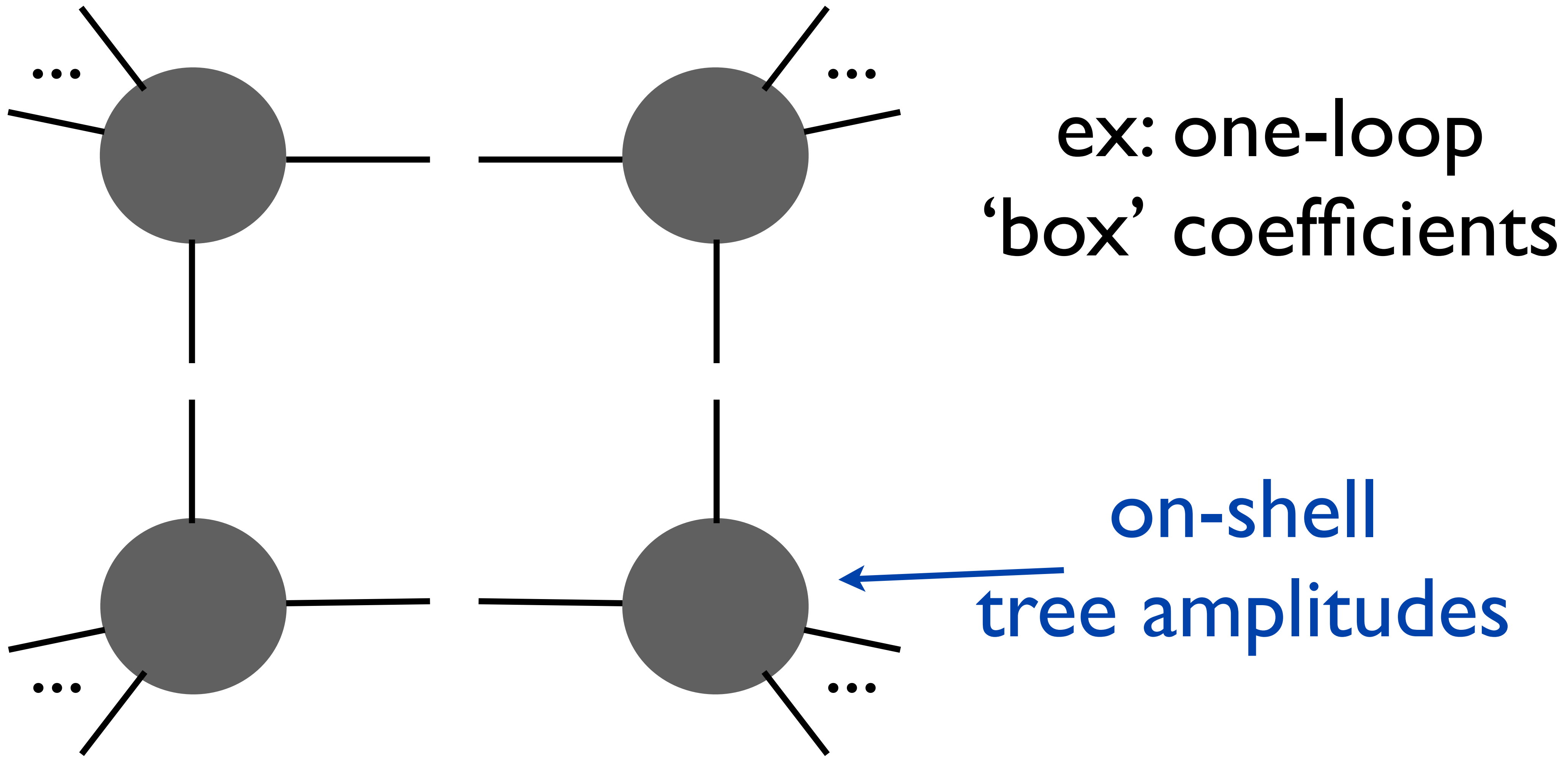


- Predictions constructed out of physically meaningful building blocks
- To generalize and understand how this works: toy model

The Kepler problem, 1.

- Classical two-body problem with 1/r potential:

$$H = \frac{p^2}{2\mu} - \frac{\lambda}{4\pi r}$$

- We can go to a center-of-mass frame;
four conserved quantities are apparent:
angular momentum $\vec{L}$ and energy

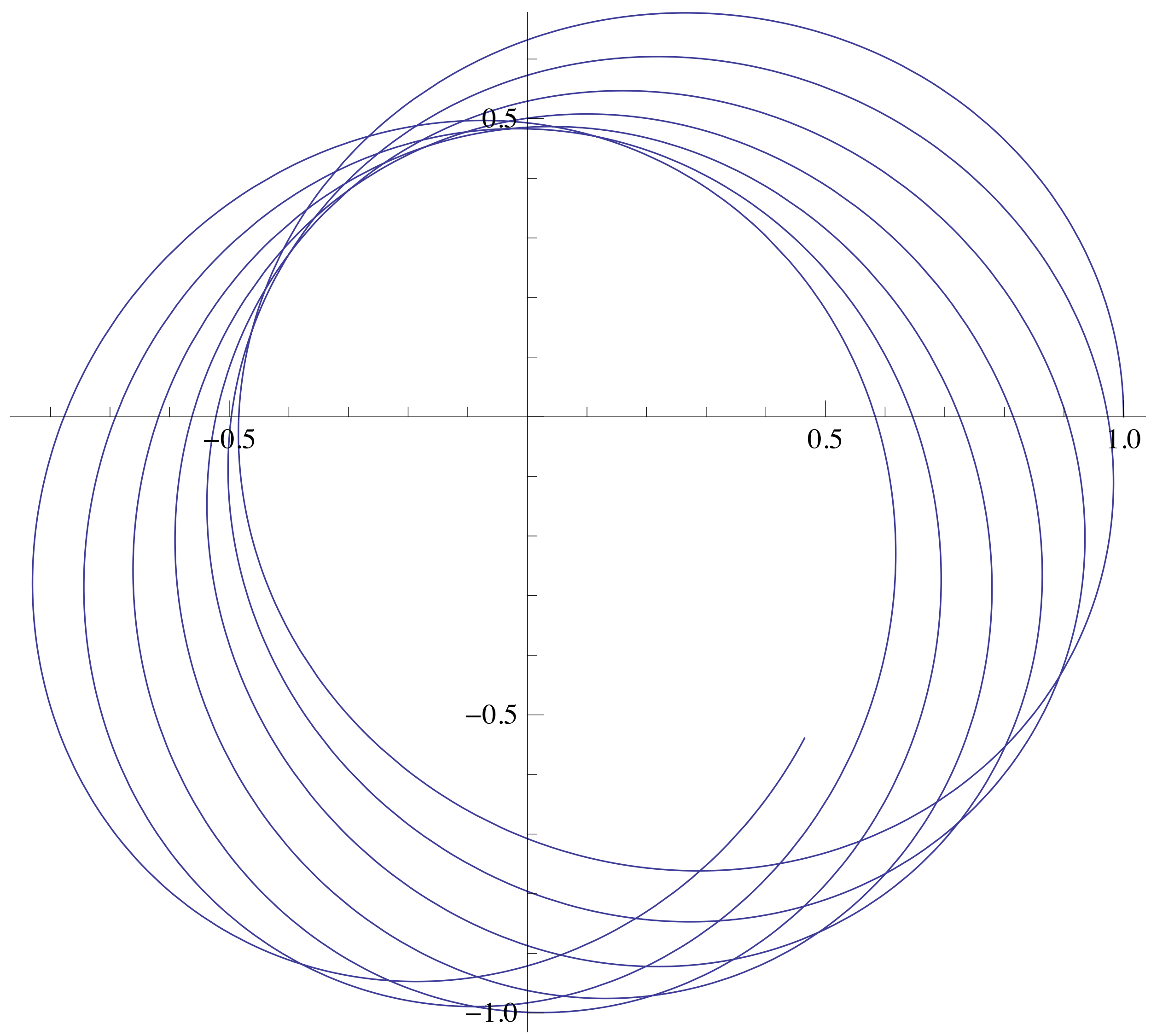
- Just from these, one deduces that motion takes place in a plane, Kepler's area law, etc.

The Kepler problem, 2.

- Something special happens when the force is $1/r^2$: the orbits do not precess

Example:

$$F = -1/r^{1.9}$$



The Kepler problem, 3.

- For $V \propto -1/r$ the system possess an additional, *non-obvious* conserved vector:

$$\vec{A} = \vec{p} \times \vec{L} - \mu \frac{\lambda}{4\pi} \frac{\vec{x}}{|x|}$$

(« Laplace-Runge-Lenz » vector)

- It points in the direction of the eccentricity, preventing it from precessing

- Quantum mechanically, the Laplace-Runge-Lenz vector is *still* conserved
- It explains the well-known degeneracy [SO(4) symmetry] of the hydrogen atom spectrum (this was quickly pointed out by Pauli in the early days of the subject)

  1s

  2s 2p

  3s 3p 3d ...

- In the real world, its conservation is broken by relativistic effects (spin-orbit, …)

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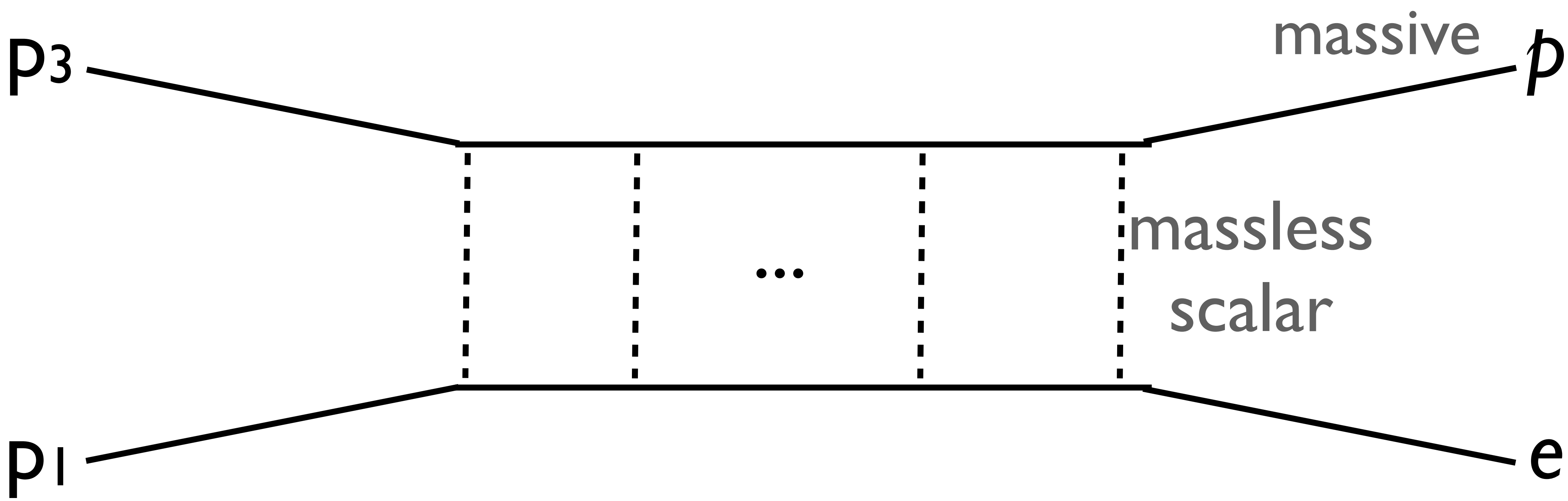
Does there exist a relativistic quantum field theory in which the Runge-Lenz vector is conserved?

- In the early days of relativistic QFT, Wick and Cutkowski considered the following model:



- This is the ladder approximation to $ep \to ep$, ignoring the spin of the photon.
- In the nonrelativistic limit this reduces to the hydrogen atom Hamiltonian

- They found that this model possesses an exact SO(4) symmetry, even *away* from the NR limit
- Consider just one rung

$$\cdots \int \frac{d^4\ell_2}{(\ell_2-\ell_1)^2\left[(\ell_2-p_1)^2+m_1^2\right]\left[(\ell_2+p_2)^2+m_3^2\right](\ell_2-\ell_3)^2}\cdots$$

- The symmetry is non-obvious in this form, but there is a conformal symmetry in momentum space

- Can be revealed easily using Dirac's embedding formalism (much used in CFT bootstrap)
- Rewrite each vector as a 6-vector, with $L^2=0$:

$$L_i^a \equiv \begin{pmatrix} \ell_i^\mu \\ L_i^+ \\ L_i^- \end{pmatrix} = \begin{pmatrix} \ell_i^\mu \\ \ell_i^2 \\ 1 \end{pmatrix}$$

and similarly for the external momenta:

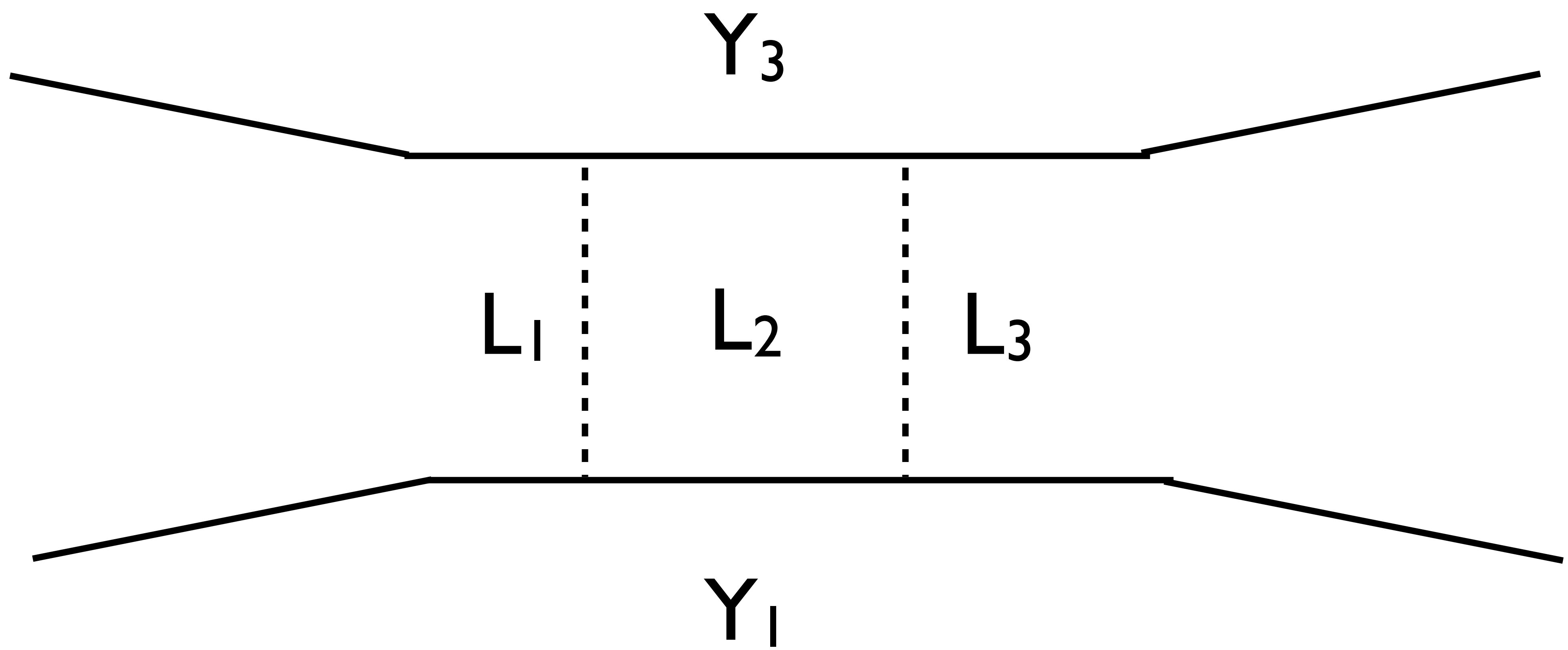
$$Y_1^a = \begin{pmatrix} p_1^\mu \\ p_1^2 + m_1^2 \\ 1 \end{pmatrix}, \qquad Y_3^a = \begin{pmatrix} -p_2^\mu \\ p_2^2 + m_3^2 \\ 1 \end{pmatrix}$$

- Propagators become simple 6D vector products:

$$L_i \cdot L_j = (\ell_i - \ell_j)^2 \qquad L_i \cdot Y_1 = (\ell_i - p_1)^2 + m^2$$

$$L_i \cdot Y_3 = (\ell_i + p_2)^2 + m^2$$

- The L's and Y's 'live' in regions of the planar graph



$$\cdots \int \text{“}d^4 L_2\text{”} \frac{1}{(L_1{\cdot}L_2)(L_2{\cdot}Y_1)(L_2{\cdot}Y_3)(L_2{\cdot}L_3)} \cdots$$

- Ladder has SO(6) rotation symmetry! [SO(4,2)]

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- SO(4,2)=Conformal symmetry in momentum space
- Seems to big!
- The masses are not invariant; the external vectors Y_1, Y_3 break the symmetry of bound states:
- This is precisely Pauli's SO(4): generated by SO(3) rotations, plus the Laplace-Runge-Lenz vector

[Fock; Itsykson& Bander; Itsykson& Zuber]

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- The broken part of the SO(4,2) is also useful
- It implies that the physics depends on only the 'angle' between Y_1 and Y_3:

$$u = \frac{2\sqrt{Y_1^2 Y_3^2}}{Y_1 \cdot Y_3} = \frac{4m_1 m_3}{-s + (m_1 - m_3)^2}$$
$$= \frac{-1}{1 - \frac{(m_1+m_3)E_{\rm kin}}{2m_1 m_3} + \frac{\delta E_{\rm kin}^2}{4m_1 m_3}}$$

[Cutkowski, `54]

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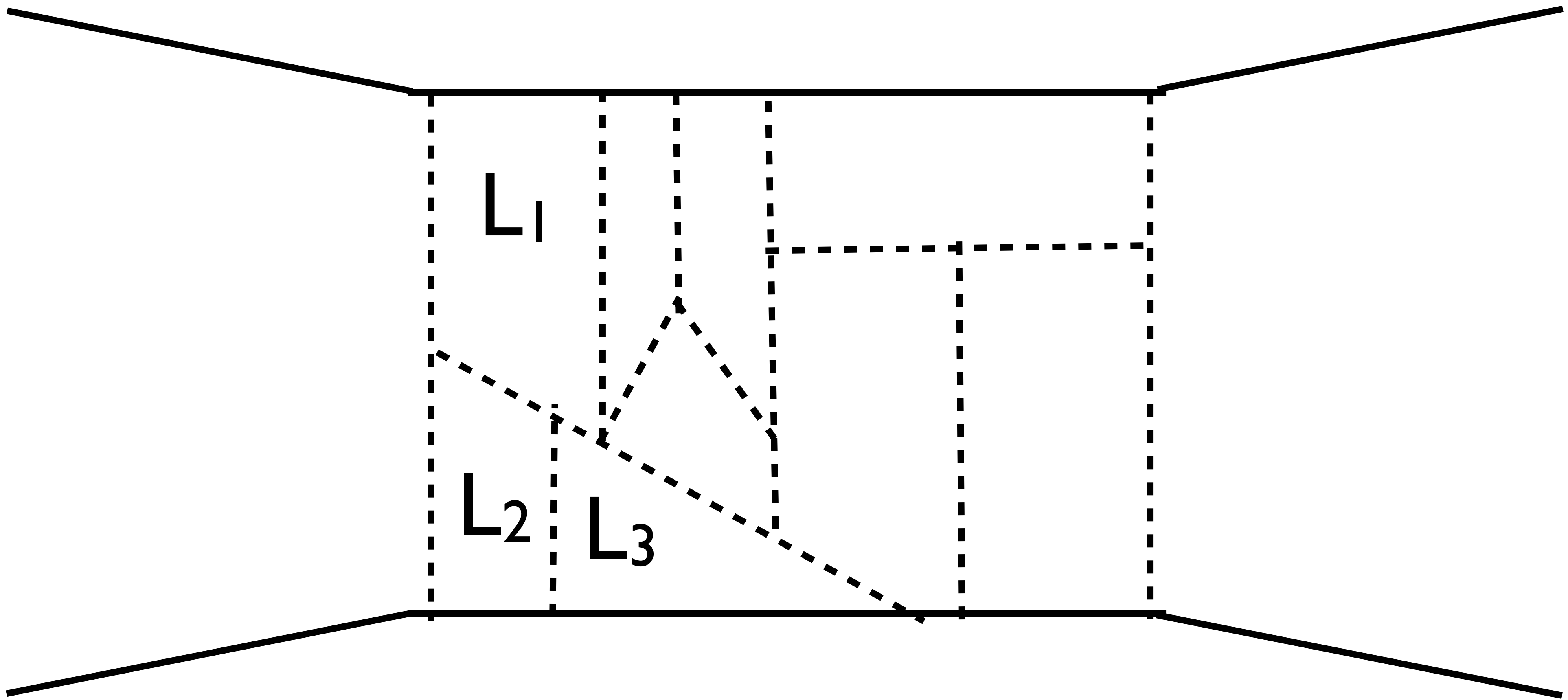
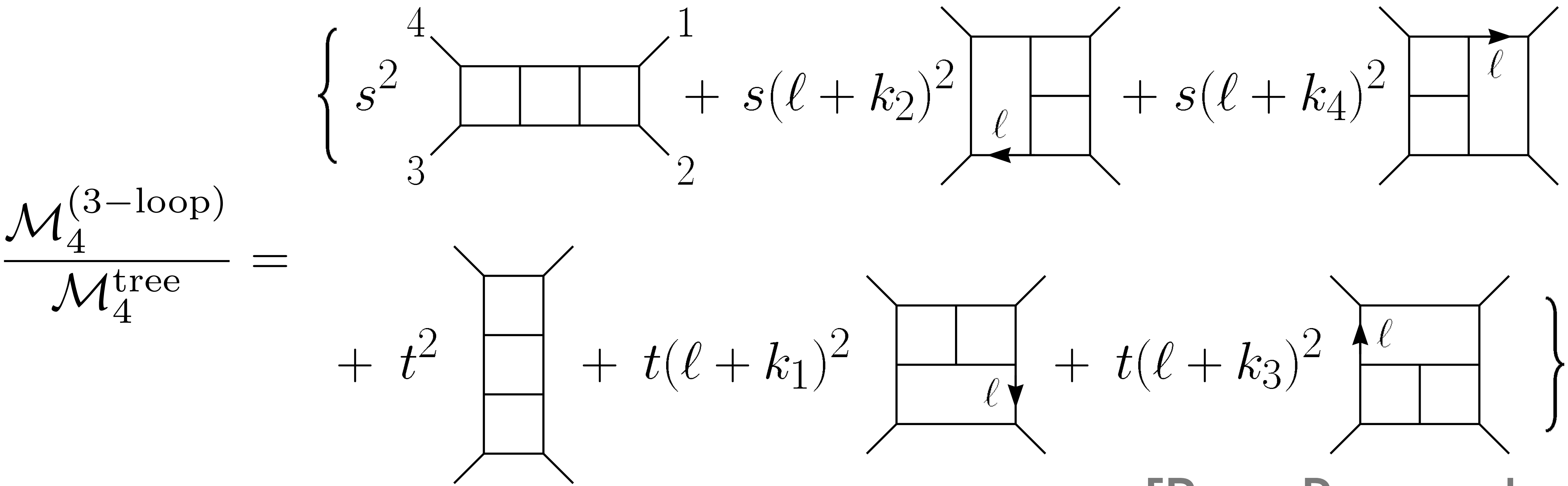
- Unfortunately, the ladder approximation is not consistent quantum-mechanically
- For this reason, this symmetry appears to have been mostly forgotten, like a curiosity
- Wick and Cutkowski's investigations left us the ``Wick rotation"

- The simplest way to imagine a consistent QFT with this symmetry is to take a planar limit:



- The Feynman rules would then 'just' need to respect the SO(6) symmetry, which acts in *momentum space*
- Can such a thing exist?

Fast forward to the 2000's



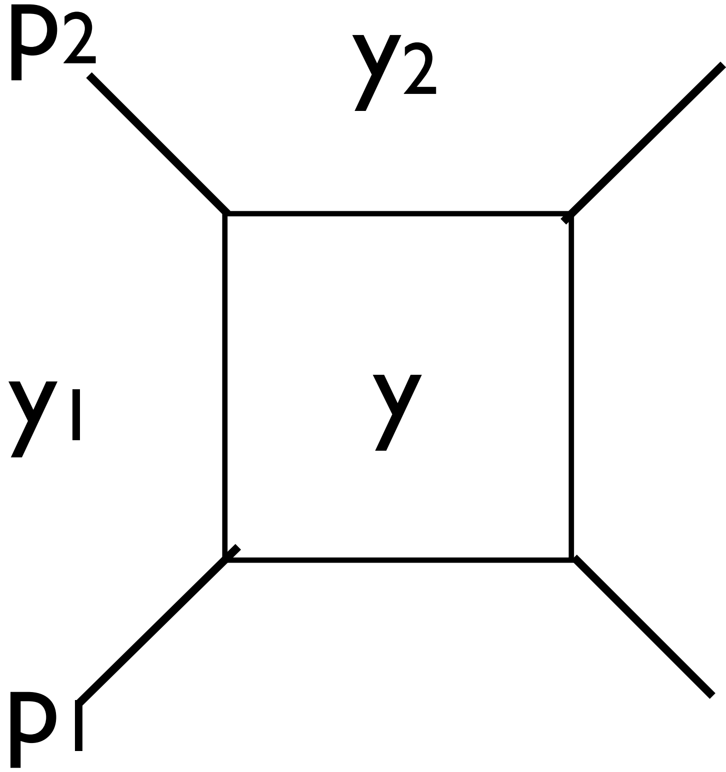
[Bern,Rozowsky&Yan, `97]

- Bern-Dixon-Smirnov-(Kosower-Anastasiou), and Drummond-Henn-Smirnov-Sokatchev observed:

  All integrals are SO(6)-invariant!

- Dual conformal symmetry in massless case

(Drummond, Henn, Smirnov& Sokatchev,
Bern, Dixon& Smirnov,
Alday&Maldacena,
Berkovitz&Maldacena
Beisert,Ricci,Tseytlin&Wolf,
...)



$$= \int d^4\ell \frac{st}{\ell^2(\ell - p_1)^2(\ell - p_1 - p_2^2)(\ell + p_4)^2}$$

$$= \int d^4y \frac{(y_1 - y_3)^2(y_2 - y_4)^2}{(y - y_1)^2(y - y_2)^2(y - y_3)^2(y - y_4)^2}$$

if $p_i = y_i - y_{i-1}$.

Symmetry seen as invariance under inversion: $y_i^\mu \to \frac{y_i^\mu}{y_i^2}$

All integrals in previous slide have this invariance!

Why the N=4 SYM theory?

- The symmetry implies conformal invariance:
  [SO(6) symmetry implies nothing special can happen at infinite momentum -> theory UV finite]

$$\beta(g^2) \propto -g^4 N_c \left(\frac{11}{3} - \frac{2n_{\text{Weyl}}}{3} - \frac{n_s}{6} \right) = 0$$

($n_{\text{Weyl}} \to 4$, $n_s \to 6$)

- The N=4 theory is however much more special: conformal in both x- and p-space!
- Unique known example

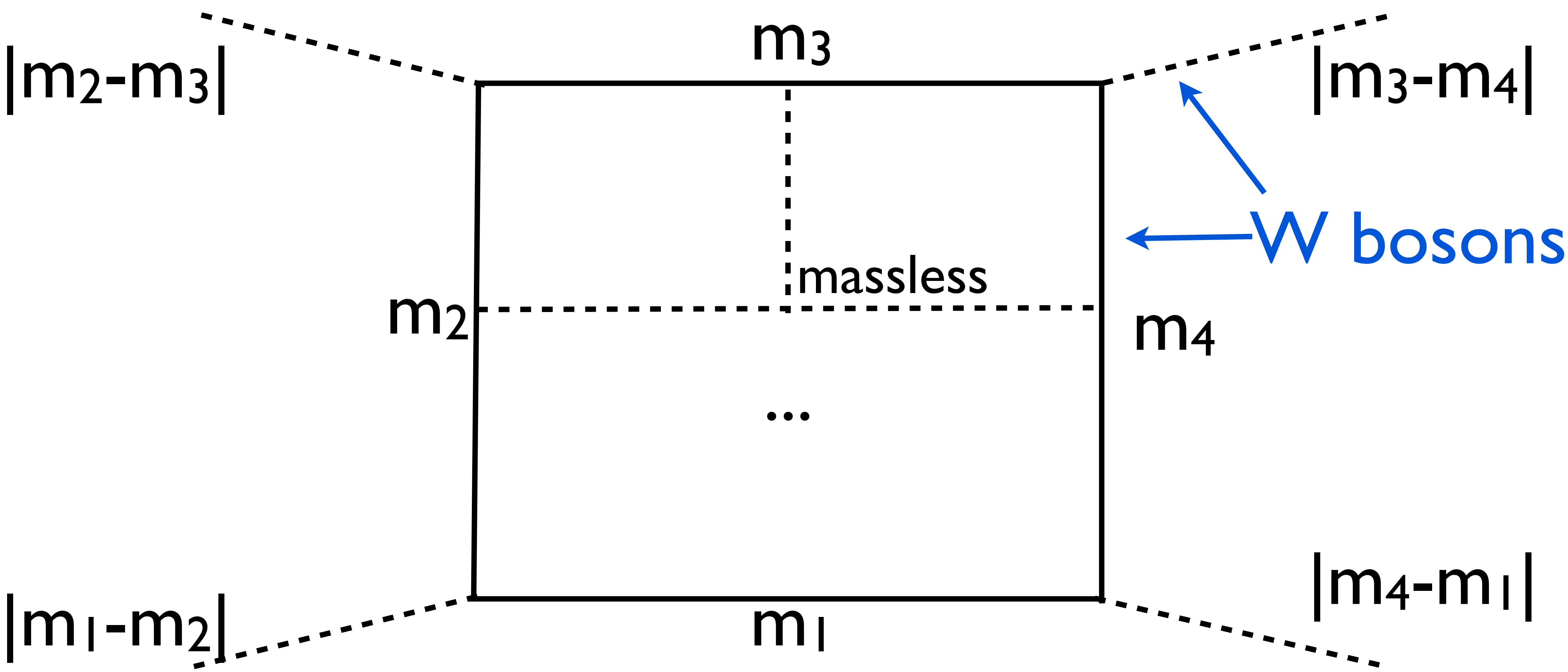
A fresh look at Hydrogen

- The theory is conformal. How can it have masses? --> Higgs mechanism!
- Scalars can be given the vev's we want.
- For planar 2→2 scattering it is interesting to break $U(N_c) \to U(N_c\text{-}4) \times U(1)^4$:

$$\langle \phi_1^a \rangle = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & 0 & & \\ 0 & 0 & m_3 & 0 & & \\ 0 & 0 & 0 & m_4 & & \\ 0 & & \cdots & & 0 & \cdots \end{pmatrix}$$

(Alday,Henn,Plefka&Schuster)

- The four-point color-ordered amplitude from the top U(4) looks like this:



[Alday,Henn,Plefka&Schuste
Dennen& Huang: 6D,
O'Connell&SCH: 10D]

- Contains analogs of both QED light-by-light scattering and e-p scattering

- The W-bosons attract (through gauge and scalar exchange in the unbroken group)

- At weak coupling, bound states are non-relativistic and hydrogen-like

- O(4) unbroken at *all* couplings: degeneracy is exact

- The Laplace-Runge-Lenz symmetry can help us compute the spectrum, *even at finite coupling*

(J.Henn & SCH, PRL 113.161601 (2014))

- Tool: semi-classical quantization -> Regge theory
- Look at the maximum angular momentum for a given energy
- Maximum attained for circular orbit:

$$j_{\max}(E) = \frac{\lambda}{8\pi\sqrt{-E/(2\mu)}}$$

Equating to an integer yields familiar spectrum:

(semi-classical quantization)

$$-E_n = \frac{\mu\lambda^2}{32\pi^2 n^2}$$

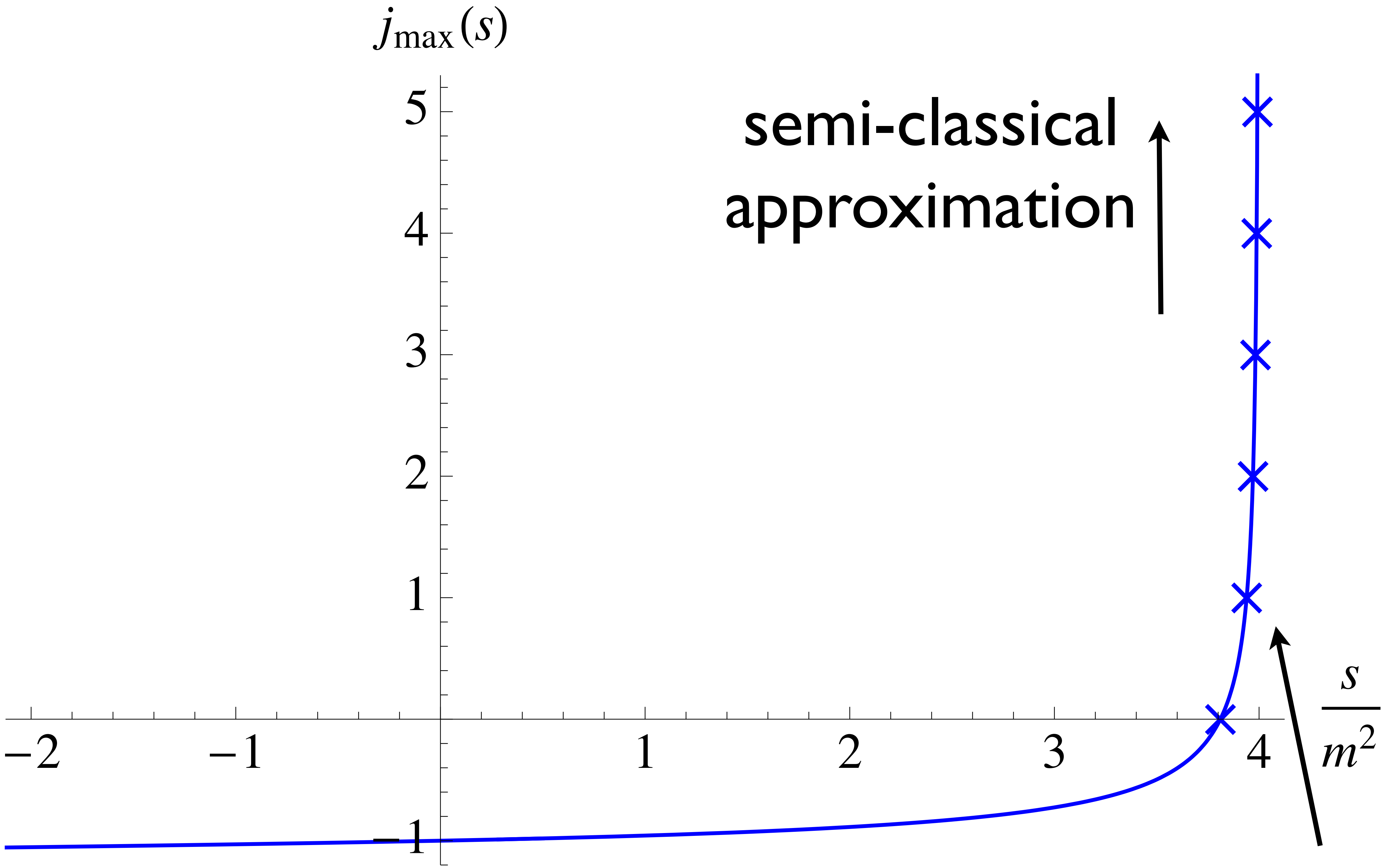
- The trajectory $j_{max}(s)$ is well-defined in any quantum field theory
- Defined from high-energy behavior of amplitudes:

$$\lim_{t\to\infty} A(s,t) = a(s)t^{j_{\max}(s)}$$

- [Physically, the spin $j_{max}(s)$ exists as a smooth function because Lorentz boosts are not quantized]
- Regge theory tells us we get bound states where

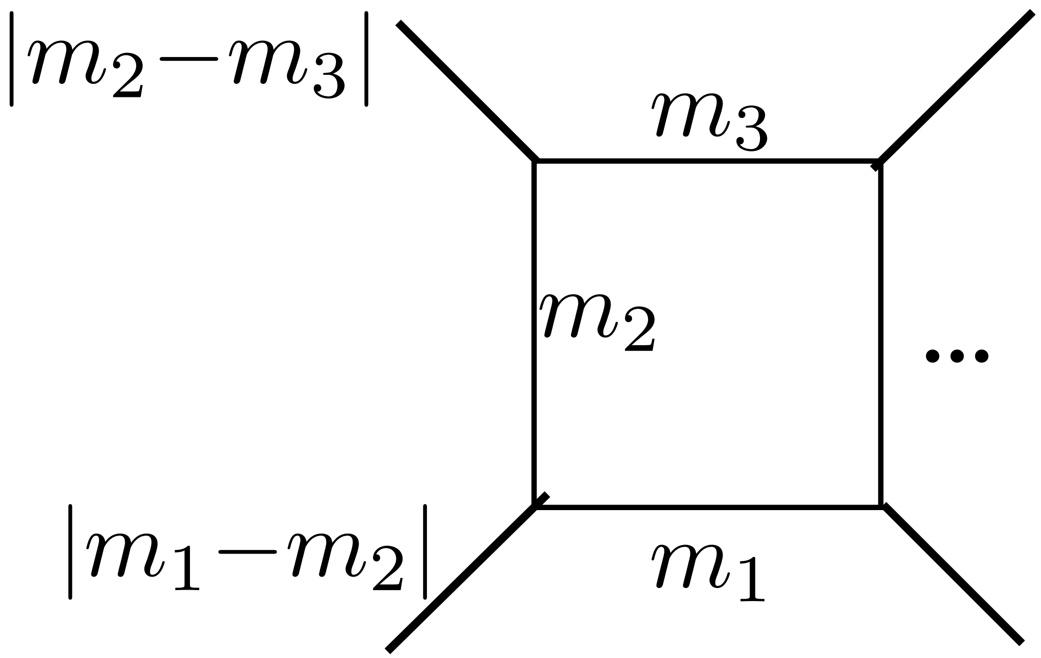
$$j_{\max}(s) = \text{integer}$$

Regge theory for H atom, I:



$$j_{\max}(E) + 1 = \frac{\lambda}{8\pi\sqrt{-E/m}}$$

- How to compute $j_{max}(s)$ in general?
- Recall Cutkowski's finding about 'reduced mass'



$$A_4(s, t, m_1, \ldots, m_4) = A_4^{\text{tree}} \times M(u, v)$$

$$v = \frac{4m_2 m_4}{-t + (m_2 - m_4)^2}$$

Amplitude depends on only two cross-ratios!

An amazing equality

$$u = \frac{4m^2}{-s}, \quad v = \frac{4m_2 m_4}{-t + (m_2 - m_4)^2}$$

- Take $m_1 = m_3 = m$, without loss of generality.
- Keeping *u* fixed, two ways to make *v* small:

1. Regge limit: $t \to \infty$: $M \propto t^{j(s)+1} \propto v^{-j(s)-1}$
2. Small mass limit: $m_4 \to 0$: $M \propto m_4^{\Gamma_{\rm cusp}(\delta)} \propto v^{\Gamma_{\rm cusp}(\delta)}$

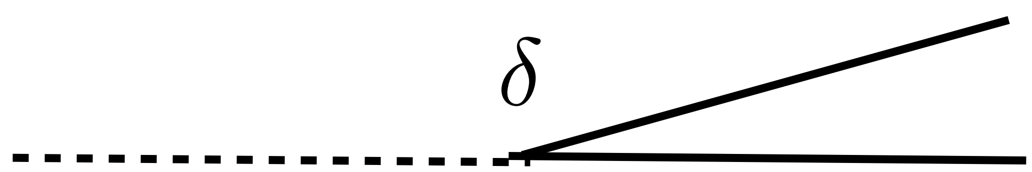
Conclusion: $$j(s) + 1 = -\Gamma_{\rm cusp}(\delta) \qquad s = 4m^2 \cos^2 \frac{\theta}{2}$$

[Correa,Henn,Maldacena&Sever]

'cusp anomalous dimension'

- Combining Cutkowski's formula with Regge theory, gives recipe for findind bound states:

1. Compute IR divergence for two particles going out at angle δ :



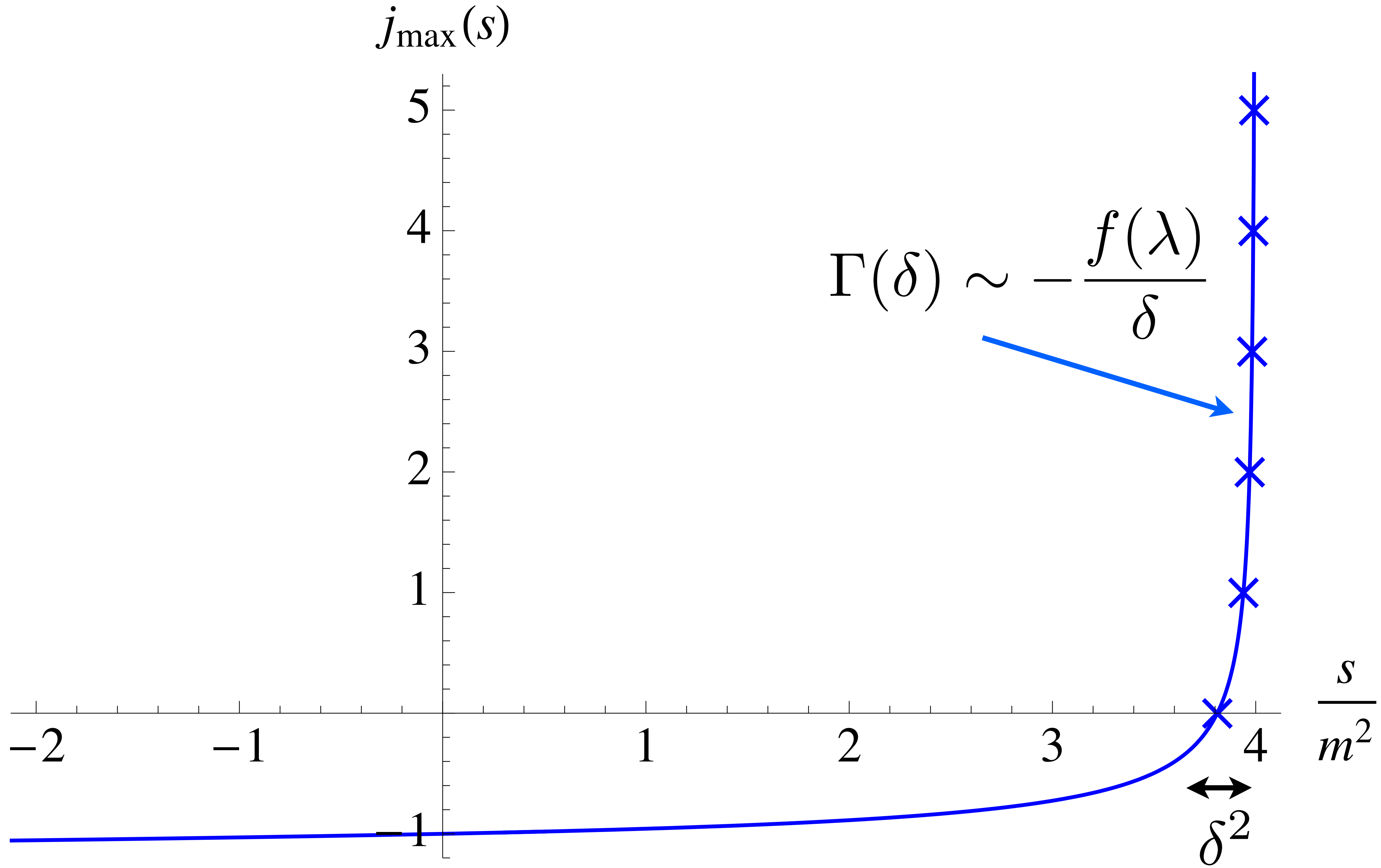
$$\Gamma_{\text{cusp}}(\delta)$$

2. Find angles δ for which $\Gamma_{\text{cusp}}(\delta) = -1, -2, \ldots$

3. Convert angle to energy:

$$E_n = 2m \cos \frac{\delta_n}{2}$$

Leading Regge trajectory at weak coupling:



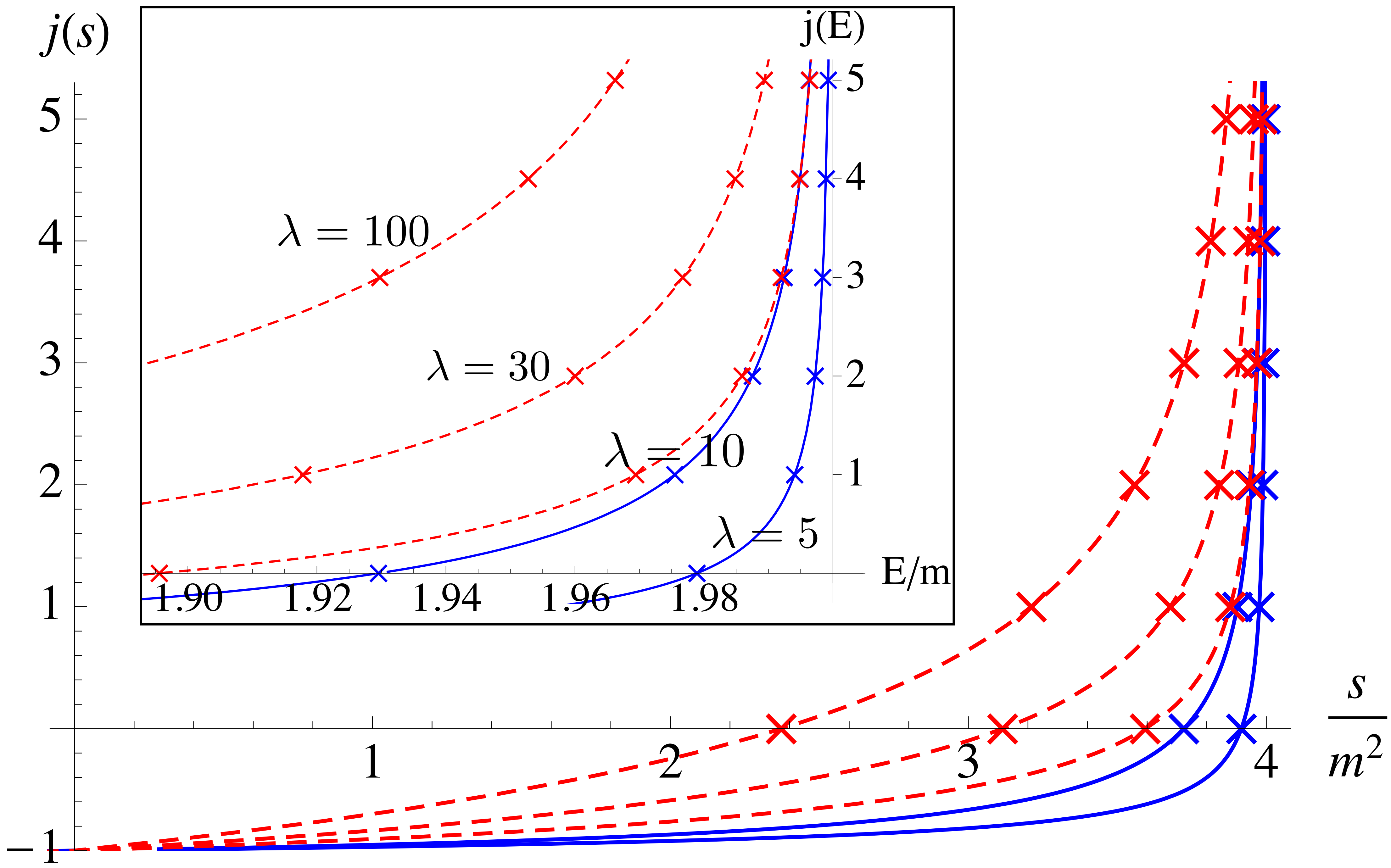
- Expansion parameter, after using equality, becomes distance from the right (δ<<1), instead of `how far up' (j>>1).

- Success of the semi-classical quantization built-in

Increasing the coupling

- Trajectory always linear around t=0
- The slope increases with coupling constant λ
- At large coupling the slope is large -> get states on a linear trajectory
- Simple way to see strings appear
- Agree precisely with string theory in AdS!

The leading trajectory from weak to strong coupling:



Some applications of dual conformal symmetry

- Massless limit governed by basically 3 numbers:

$$\mathcal{A}_{2\to 2}^{\text{BDS}} = \mathcal{A}_{2\to 2}^{\text{tree}} \times e^{-\frac{1}{4} f(g) \log \frac{-s}{c_1 m^2} \log \frac{-t}{c_1 m^2} + c_2}$$

- Using integrability of the theory, a lot is known about these numbers, as a function of the coupling

- Obtained by solving an integral equation (in which the coupling appears as a parameter)

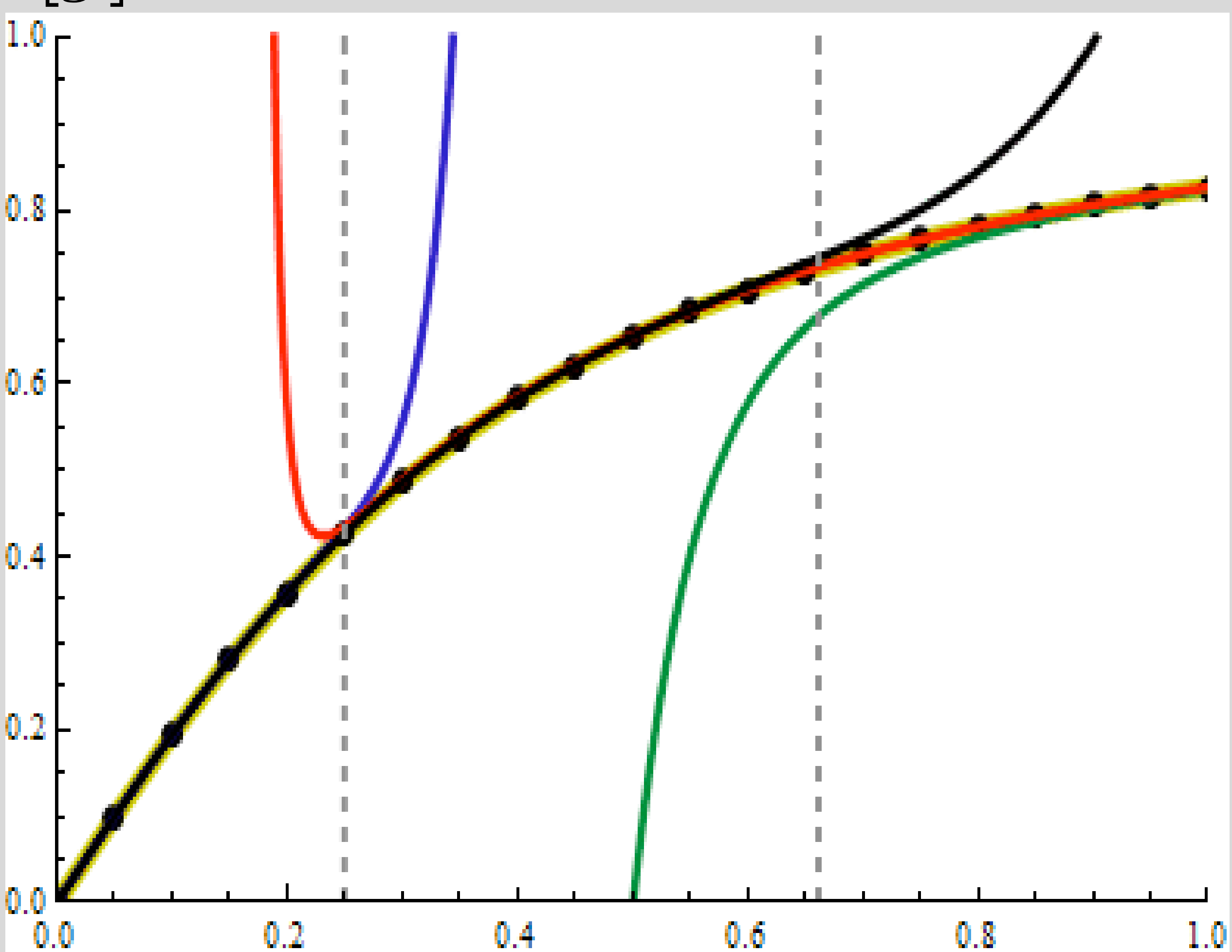
(Beisert, Eden&Staudacher '07)

State of art explicit results for **infinite** volume case.

Cusp anomalous dimension $Tr\, Z\, D^S Z$

$$\Delta - S = f[g] \log S + \ldots, \quad S \to \infty$$

$f[g]$



$$\frac{\lambda}{4\pi^2} \equiv 4g^2$$

Weak coupling:

[Moch, Vermaseren, Vogt, 04]
[Lipatov et al., 04]
[Bern et al., 06]
[Cachazo et al., 06]

$$f[g]=8g^2-\frac{8}{3}\pi^2g^4+\frac{88}{45}\pi^4g^6-\left(\frac{584}{315}\pi^6+64\zeta(3)^2\right)g^8+\ldots$$

[Beisert, Eden, Staudacher, 06]

Numerics: [Benna, Benvenuti, Klebanov, Scardicchio, 06]

Strong coupling:

[Gubser, Klebanov, Polyakov, 02]
[Frolov, Tseytlin, 02]
[Roiban, Tseytlin, 07]

$$f[g] = 4g - \frac{3\log 2}{\pi} - \frac{1}{4g}\frac{K}{\pi^2} - \ldots$$

[Klebanov et al, 06]
[Kotikov,Lipatov, 06]
[Alday et al, 07]
[Kostov, Serban, D.V., 07]
[Beccaria, Angelis, Forini, 07]

[Casteill, Kristjansen, 07]
[Belitsky, 07]

[Basso, Korchemsky, Kotanski, 07]
[Kostov, Serban, D.V., 08]

Nonperturbative corrections: [Basso, Korchemsky, 09]

Slide taken from D.Volin

Conclusions

- A very special quantum field theory exists in D=4:

planetary orbits ⟶ Hydrogen ⟶ (planar) N=4 SYM

- Can be analyzed using sophisticated methods from integrability, or using general methods
  -> ideal toy model for computing amplitudes, and testing ground for new methods!