Seminar @ Niels Bohr Institute, Oct.30th, 2015

Gravity Dual of Quantum Information Metric

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Mainly based on the paper arXiv:1507.07555 written with

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Also partially based on arXiv:1506.01353
[Phys. Rev. Lett. 115, 171602 (2015)].

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① Introduction

Holographic Principle (or AdS/CFT)

⇒ ``Geometrization'' of Quantum States in QFTs

algebraically very complicated

⇒ a geometry of quantum information.

$$\left|\Psi(t)\right\rangle = \sum_{\{i_k\}} c_{\{i_k\}}(t)\ \left|i_1\right\rangle \otimes \left|i_2\right\rangle \ldots \otimes \left|i_N\right\rangle$$

[MERA: Vidal 2005, Swingle 2009]
[Raamsdonk 2009]
[Bianchi-Myers 2012]
[Miyaji-TT 2015],...

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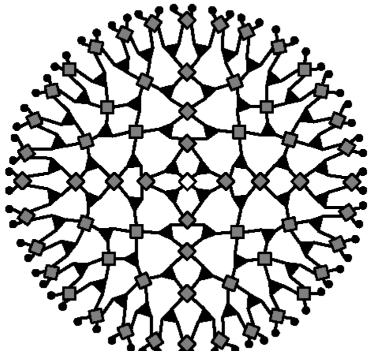

Emergent spacetime from Tensor Networks

One quantity which characterizes this duality is the entanglement entropy (EE) S_A .

[Another approach: Rob Myers's talk on Entanglement Holography]

The holographic entanglement entropy (HEE) relates the EE to the area of minimal surfaces: [Ryu-TT 2006]

$$S_A = \underset{\substack{\gamma_A \\ \partial\gamma_A \sim \partial A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_A)}{4G_N} \right].$$

In this talk, we propose another (approx.) formula which connects between geometry and quantum information:

$$G_{\lambda\lambda} \sim \underset{\Sigma}{\mathrm{Max}} \left[\frac{\mathrm{Vol}(\Sigma)}{R^{d+1}} \right].$$

Quantum information metric for a d+1 dim. CFT

Time slice in AdS

This is partially motivated by the recent Susskind's conjecture [Susskind 2014] :

$$C \sim \frac{\mathrm{Vol}(\Sigma)}{G_N R} \quad .$$

Computational complexity
~ Min[# of unitary transformations]
which realizes the transformation
$|0\rangle|0\rangle\ldots|0\rangle \rightarrow |\Psi\rangle$

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② Quantum Information Metric in CFTs

(2-1) Definition

Consider two different pure states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. We define the distance (called **Bures distance**) between them as

$$D(|\Psi_1\rangle, |\Psi_2\rangle) = 1 - |\langle\Psi_1|\Psi_2\rangle| \quad .$$

For mixed states we can generalize this to

$$D(\rho_1, \rho_2) = 1 - \underbrace{\mathrm{Tr}\left[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right]}_{\text{Fidelity}} \quad .$$

Fidelity

~How much is it difficult to distinguish two states by POVM measurement.

Consider pure states with parameters $\left|\Psi(\lambda_1, \lambda_2, \cdots)\right\rangle$.

We define the **information metric G** as follows

$$D\left(\left\langle\Psi(\lambda)\middle|\Psi(\lambda+d\lambda)\right\rangle\right) = 1 - \left|\left\langle\Psi(\lambda)\middle|\Psi(\lambda+d\lambda)\right\rangle\right|$$
$$= G_{\lambda_i\lambda_j}(d\lambda_i)(d\lambda_j) + O\left((d\lambda)^3\right).$$

Motivation of information metric ⇒ Quantum Estimation Theory

A quantum version of ***Cramer-Rao bound*** argues

[Helstrom 76]

$$\left\langle(\delta\lambda)^2\right\rangle \geq \frac{1}{G_{\lambda\lambda}}.$$

Mean square error

Note: Two definitions of Information Metric

$$\text{Bures}: \qquad G^{(B)}_{\lambda\lambda} d\lambda^2 = B[\rho(\lambda + d\lambda), \rho(\lambda)]$$

$$\text{Relative Entropy}: \qquad G^{(F)}_{\lambda\lambda} d\lambda^2 = S[\rho(\lambda + d\lambda) \| \rho(\lambda)]$$

$$\textit{where} \qquad B[\rho, \sigma] = 1 - \mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}],$$

$$\text{in particular,} \quad B\big[|x\rangle\langle x|, |y\rangle\langle y|\big] = 1 - |\langle x|y\rangle|,$$

$$S[\rho \| \sigma] = \mathrm{Tr}[\rho(\log\rho - \log\sigma)].$$

Note: G(B) and G(F) are equivalent only classically.

We will employ the Bures metric G(F) below.

[For the Fisher metric G(F), refer to Lashkari-Raamsdonk 2015].

Example 1: Free boson (-) and fermion (+)

$$|\Psi(\lambda)\rangle = \sqrt{1 \mp |\lambda|^2} \cdot e^{-\lambda a^+ b^+} |0\rangle,$$

$$\langle \Psi(\lambda') | \Psi(\lambda) \rangle = \frac{\sqrt{(1 \mp |\lambda'|^2)(1 \mp |\lambda|^2)}}{1 - \lambda'^* \lambda}.$$

$$\Rightarrow ds^2 = \frac{d\lambda d\lambda^*}{(1 \mp |\lambda|^2)^2}.$$

Free Boson: 2d hyperbolic space H2

Free Fermion: 2d sphere S^2

Example 2: Spacetime metric from information metric ?

[Miyaji-Numasawa-Shiba-Watanabe-TT 2015]

Consider a free scalar field (with a mass) in a (d+1) dimensional curved spacetime.

It is clear that the two point function $\langle \varphi(x)\varphi(y) \rangle$ behaves as follows when D(x,y) is very small:

$$\langle \varphi(x)\varphi(y) \rangle \sim \frac{1}{D(x,y)^{d-2}}.$$

To define a normalized state $\left|\varphi(x)\right\rangle \propto \varphi(x)\left|0\right\rangle$,

we need a UV regularization, which leads to

$$\left\langle\varphi(x)\middle|\varphi(y)\right\rangle = \frac{\varepsilon^{d-1}}{\left(\varepsilon^2 + D(x,y)^2\right)^{\frac{d-1}{2}}} \quad .$$

Then the information metric reads

$$ds^2 \propto \frac{1}{\varepsilon^2} g_{ij} dx^i dx^j , \quad \Rightarrow \text{spacetime metric}$$

It is natural to choose ε to be a length of order Planck scale. ⇒ The information metric measures a distance in the unit of Planck length.

In AdS3/CFT2 we can take ε~1/c ⇒ R_{AdS}~c.

The main purpose of this talk is to consider a (d+1) dim. CFT and perform one parameter deformation:

$$S(\lambda) = S_{CFT} + \lambda \int dt dx^d O(x,t).$$

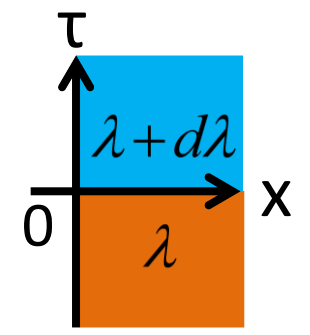
We choose $|\Psi(\lambda)\rangle$ as the ground state of the deformed QFT defined by $S(\lambda)$.

We are interested in the corresponding information metric $G_{\lambda\lambda}$.   [or called fidelity susceptibility Shi-Jian Gu 2010]

(2-2) Information Metric in CFT

In the path-integral formalism (τ=Euclidean time),

$$\langle \Psi(\lambda+d\lambda) | \Psi(\lambda) \rangle$$
$$= \frac{1}{\sqrt{Z_1 Z_2}} \int D\phi \exp\left[-\int dx^d \left(\int_{-\infty}^{0} d\tau \, L(\lambda) + \int_{0}^{\infty} d\tau \, L(\lambda+d\lambda) \right) \right].$$



Since we encounter UV divergences at τ=0, we regulate by a point splitting or equally by replacing $|\Psi(\lambda+d\lambda)\rangle$ with

$$|\Psi(\lambda+d\lambda)\rangle_\varepsilon = \frac{e^{-\varepsilon H(\lambda)} |\Psi(\lambda+d\lambda)\rangle}{\sqrt{\langle \Psi(\lambda+d\lambda) | e^{-2\varepsilon H(\lambda)} | \Psi(\lambda+d\lambda) \rangle}}.$$

Finally we obtain the following expression:

$$G_{\lambda\lambda} = \frac{1}{2}\int dx^d \int dx'^d \int_{\varepsilon}^{\infty} d\tau \int_{-\infty}^{-\varepsilon} d\tau' \langle O(x,\tau) O(x',\tau') \rangle.$$

Comments: It only involves a two point function.

Thus it is universal for CFTs at λ=0 when space is R^d.

$G_{\lambda\lambda}$ is an universal information theoretic quantity to characterize CFT ground states.

$G_{\lambda\lambda}$ at λ=0 (CFT point)

O(x,t) is a primary with conformal dim. Δ

$$\Rightarrow \quad \langle O(x,\tau)O(x',\tau')\rangle = \frac{1}{\left((\tau-\tau')^2+(x-x')^2\right)^{\Delta}}.$$

After integration, we find the simple scaling (UV div.):

$$G_{\lambda\lambda} = N_d \cdot V_d \cdot \varepsilon^{d+2-2\Delta} \quad (\text{when } d+2-2\Delta<0).$$

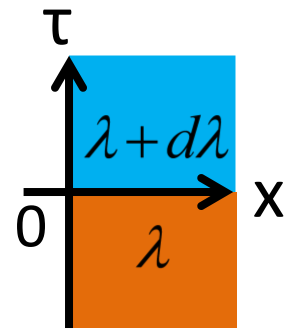
$$N_d = \frac{2^{d-2\Delta}\pi^{d/2}\Gamma(\Delta-d/2-1)}{(2\Delta-d-1)\Gamma(\Delta)}.$$

For $d+2-2\Delta>0$, $G_{\lambda\lambda} \propto V_d \cdot L^{d+2-2\Delta}$. (IR div.)

③ A Gravity Dual Proposal of Information Metric

We focus on an exactly marginal perturbation i.e. Δ=d+1.

(3-1) Exact Gravity Dual via Janus Solutions



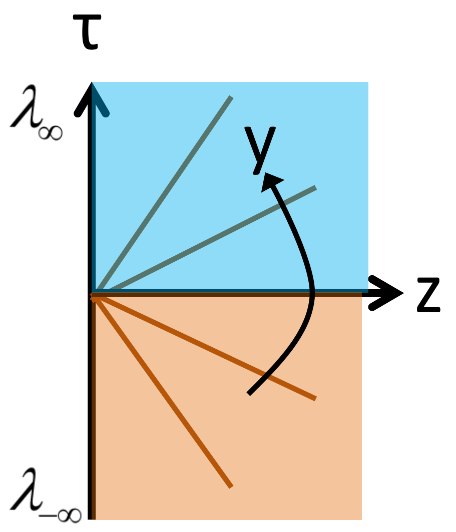
A gravity dual of the CFT with the interface is known as a **Janus solution**. [Bak-Gutperle-Hirano 03] [Clark-Freedman-Karch-Schnabl 04]

AdS3 Janus model [Bak-Gutperle-Hirano 03] :

$$S_{Janus} = -\frac{1}{16\pi G_N}\int dx^3 \sqrt{g}\left[R - g^{ab}\partial_a\lambda\partial_b\lambda + 2R_{AdS}^{-2}\right]$$

$$ds^2 = R_{AdS}^2\left(dy^2 + f(y)ds_{AdS2}^2\right), \quad \lambda(y) = \gamma\int_{-\infty}^{y}\frac{dy}{f(y)} + \lambda_{-\infty},$$

$$f(y) = \frac{1}{2}\left(1 + \sqrt{1 - 2\lambda^2}\cosh(2y)\right), \quad \lambda_{\infty} - \lambda_{-\infty} \approx \gamma + O(\gamma^3).$$



In this model, we can evaluate the classical on-shell action:

$$S_{Janus}(\gamma) - S_{Janus}(0) = \frac{R_{AdS} \cdot V_1}{16\pi G_N \varepsilon} \log \frac{1}{1-2\gamma^2} > 0,$$

where ε is the UV cut off in the AdS2.

Thus we can estimate the information metric as

$$\left| \left\langle \Psi(\gamma) \middle| \Psi(0) \right\rangle \right| = e^{-S_{Janus}(\gamma)+S_{Janus}(0)} \approx 1 - \frac{R_{AdS} V_1}{8\pi G_N \varepsilon} \gamma^2,$$

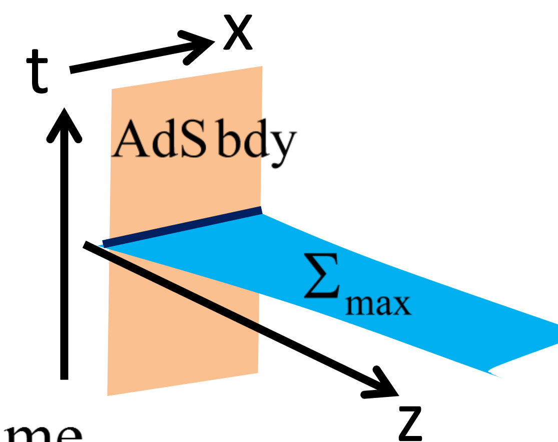
$$\Rightarrow \quad G_{\lambda\lambda} = \frac{c V_1}{12\pi\varepsilon}. \quad (c = \text{central charge}).$$

By noting the normalization $\lambda_{CFT} \propto \sqrt{c}\lambda_{AdS}$, we can confirm that this holographic result agrees with our previous CFT result.

(3-2) Gravity Dual Proposal for General Backgrounds

For generic setups (e.g. AdS BHs) with less symmetries, the construction of Janus solutions is difficult.

⇒Instead, we would like to propose a covariant formula which computes the information metric:



$$G_{\lambda\lambda} = n_d \cdot \frac{\mathrm{Vol}(\Sigma_{\max})}{R_{AdS}^{d+1}}.$$

$\Sigma_{\max}$: The bulk time slice with maximal volume

n_d : a certain O(1) coefficient

Note: This formula is based on a hard-wall approximation.

Similar to holography for BCFT

[Karch-Randall 2000,2001, TT 2011].

An explanation

Since we are interested in an infinitesimal exactly marginal deformation of a CFT, we can model the Janus interface

as a **probe defect brane** with an infinitesimally small tension T:

$$S_{Janus} \approx S_{gravity} + T\int_{\Sigma} \sqrt{g}\, dx^{d+1}.$$

The Einstein equation tells us

$$T \approx n_d \cdot \frac{(\delta\lambda)^2}{R^{d+1}},$$

as we can confirm in Janus solutions explcitly.

The standard probe approximation leads to the formula:

$$G_{\lambda\lambda} = n_d \cdot \frac{\mathrm{Vol}(\Sigma_{\max})}{R_{AdS}^{d+1}}.$$

Example 1 : Poincare AdSd+2

$$ds^2 = R_{AdS}^2 \frac{dz^2 + dx_\mu dx^\mu}{z^2}.$$

$$G_{\lambda\lambda} = n_d V_d \int_\varepsilon^\infty \frac{dz}{z^{d+1}} = \frac{n_d V_d}{d\varepsilon^d}.$$

Example 2 : Global AdSd+2

$$ds^2 = R_{AdS}^2 \left(-(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega_d^2 \right).$$

$$G_{\lambda\lambda} = n_d V_d \int_0^{1/\varepsilon} \frac{r^d dr}{\sqrt{r^2+1}} < G_{\lambda\lambda}\Big|_{\text{Poincare}}$$

Example 3 : AdSd+2 Schwarzschild BH

$$ds^2 = R_{AdS}^2 \left(-\frac{1-(z/z_0)^{d+1}}{z^2} dt^2 + \frac{dz^2}{z^2(1-(z/z_0)^{d+1})} + \frac{dx_i dx_i}{z^2} \right).$$

$$G_{\lambda\lambda} = n_d V_d \int_\varepsilon^\infty \frac{dz}{\sqrt{h(z)} z^{d+1}} = \frac{n_d V_d}{d} \left(\frac{1}{\varepsilon^d} + \frac{b_d}{z_0^d} \right).$$

$$b_1 = 0, \quad b_2 \approx 0.70, \quad b_3 \approx 1.31, \ldots$$

④ Dynamics of Information Metric and AdS BHs

In order to test our holographic information metric, we turn to a time-dependent example.

⇒ Consider thermofield doubled (TFD) CFTs $\left|\Psi^{(1)}_{TFD}\right\rangle$ under time evolutions. We assume 2d CFTs.

TFD = a pure state description of thermal state.

$$\left|\Psi_{TFD}\right\rangle = Z(\beta)^{-1}\cdot\sum_n e^{-\beta E_n/2}\left|n\right\rangle_A\left|n\right\rangle_B$$

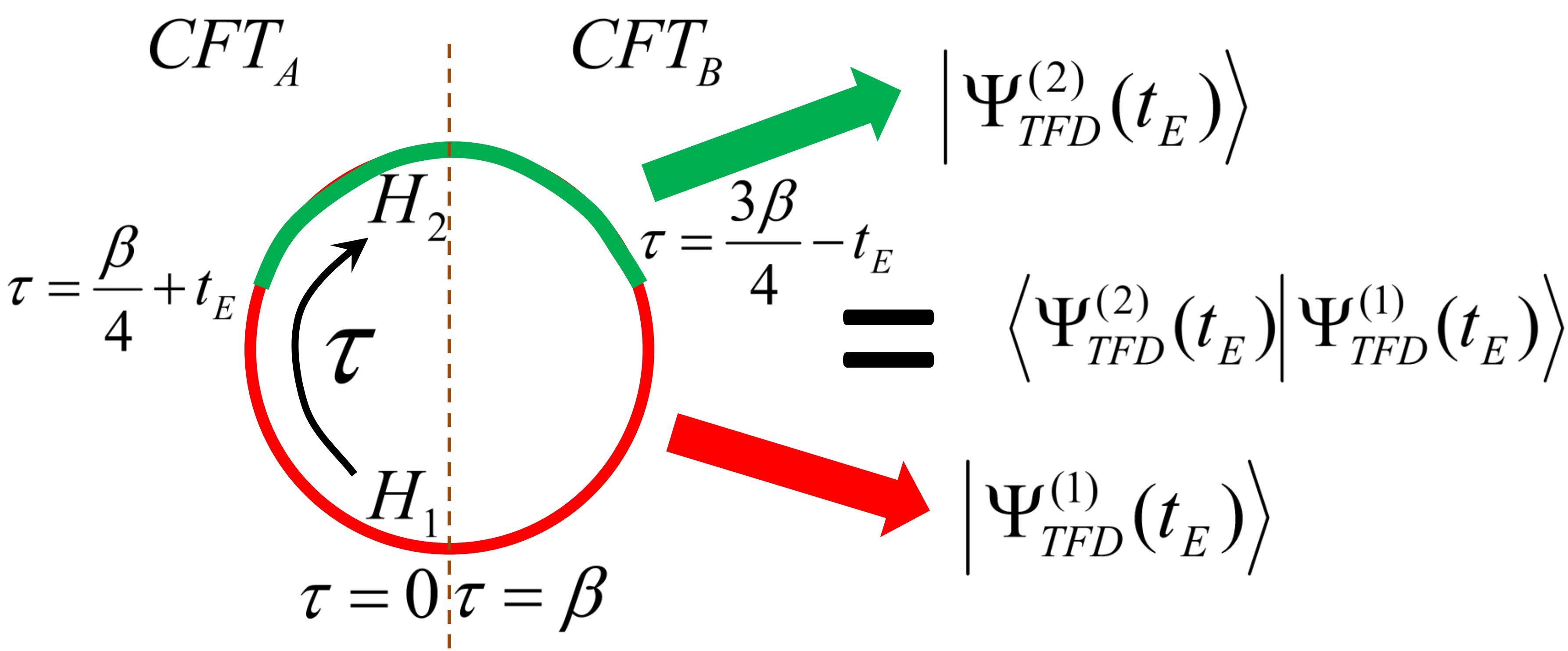
$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[\left|\Psi_{TFD}\right\rangle\left\langle\Psi_{TFD}\right|\right] = Z(\beta)^{-1}\cdot\sum_n e^{-\beta E_n}\left|n\right\rangle_A\left\langle n\right|_A = \rho_{thermal}\ .$$

$$\text{Time evolution: } \rho_{TFD}(t) = e^{i(H_A+H_B)t}\cdot\left|\Psi_{TFD}\right\rangle\left\langle\Psi_{TFD}\right|\cdot e^{-i(H_A+H_B)t}.$$

We consider another TFD state $\left|\Psi^{(2)}_{TFD}\right\rangle$ based on the CFT with an infinitesimal exactly marginal perturbation.

⇒ Compute the information metric for this deformation.

In the Euclidean path-integral description, we have



Thus we can calculate the information metric:

$$G_{\lambda\lambda}(t_E) = \frac{1}{2}\int dx_1 \int dx_2 \int_{\frac{\beta}{4}+t_E+\varepsilon}^{\frac{3\beta}{4}-t_E-\varepsilon} d\tau_1 \int_{-\frac{\beta}{4}-t_E+\varepsilon}^{\frac{\beta}{4}+t_E-\varepsilon} d\tau_2 \langle O(x_1,\tau_1)O(x_2,\tau_2)\rangle,$$

$$\langle O(x_1,\tau_1)O(x_2,\tau_2)\rangle = \frac{(\pi/\beta)^{2\Delta}}{\left(\sinh^2\frac{\pi(x_1-x_2)}{\beta}+\sin^2\frac{\pi(\tau_1-\tau_2)}{\beta}\right)^{\Delta}}.$$

Note: We assume the space direction is non-compact.

$\Rightarrow$ Our result is universal for any 2d CFTs.

We focus on $\Delta=2$ (exactly marginal).

Eventually, we get $$G_{\lambda\lambda}(t_E) = \frac{\pi V_1}{8\varepsilon} + \frac{2\pi^2 V_1}{\beta^2}\left(t_E \cdot \cot\frac{4\pi t_E}{\beta} - \frac{\beta}{4\pi}\right).$$

Real time behavior

By setting $t = -it_E$, we obtain

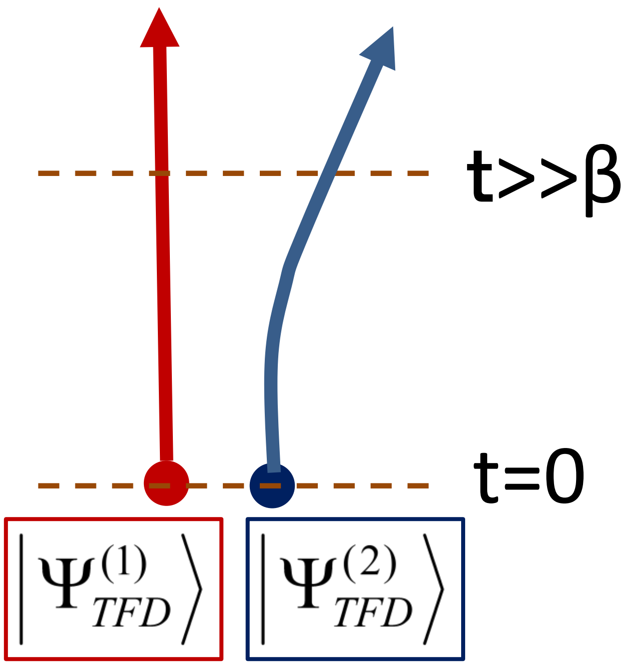
$$G_{\lambda\lambda}(t_E) = \frac{\pi V_1}{8\varepsilon} + \frac{2\pi^2 V_1}{\beta^2}\left(t \cdot \coth\frac{4\pi t}{\beta} - \frac{\beta}{4\pi}\right).$$

At late time $t >> \beta$,

we find a linear t behavior:

$$G_{\lambda\lambda}(t_E) \approx \frac{\pi V_1}{8\varepsilon} + \frac{2\pi^2 V_1}{\beta^2} \cdot t \quad .$$

(We expect a half of the above result for quantum quenches.)



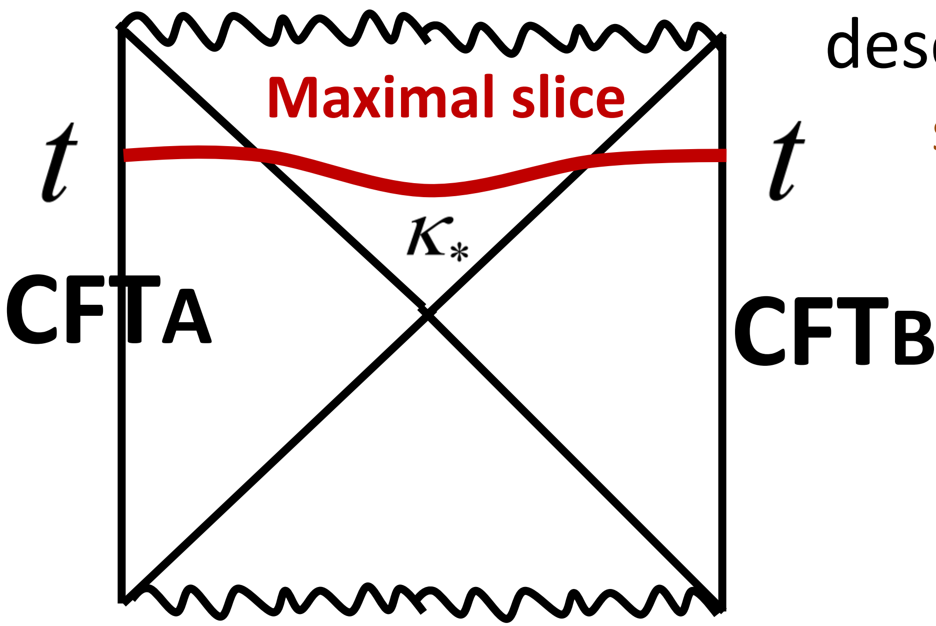
Holographic Dual

The TFD state is dual to the eternal BTZ BH. [Maldacena 2001]

The information metric is dual to the volume of the maximal slice which connects the two boundaries.

⇒ We can get a result V=V(t) , described by integrals.

Similar to [Hartman-Malcacena2013].



$$\frac{\mathrm{Vol}(\Sigma)}{R^{d+1}V_1} = 2\sinh\rho_0 + 2\int_0^{\kappa_*} d\kappa \frac{\cos\kappa}{\sqrt{\sin^2(2\kappa_*)/\sin^2(2\kappa) - 1}}$$

$$-2\int_0^{\rho_\infty} d\rho \frac{\cosh\rho\left(\sqrt{\sinh^2(2\rho)+\sin^2(2\kappa_*)} - \sinh^2(2\rho)\right)}{\sqrt{\sinh^2(2\rho)+\sin^2(2\kappa_*)}},$$

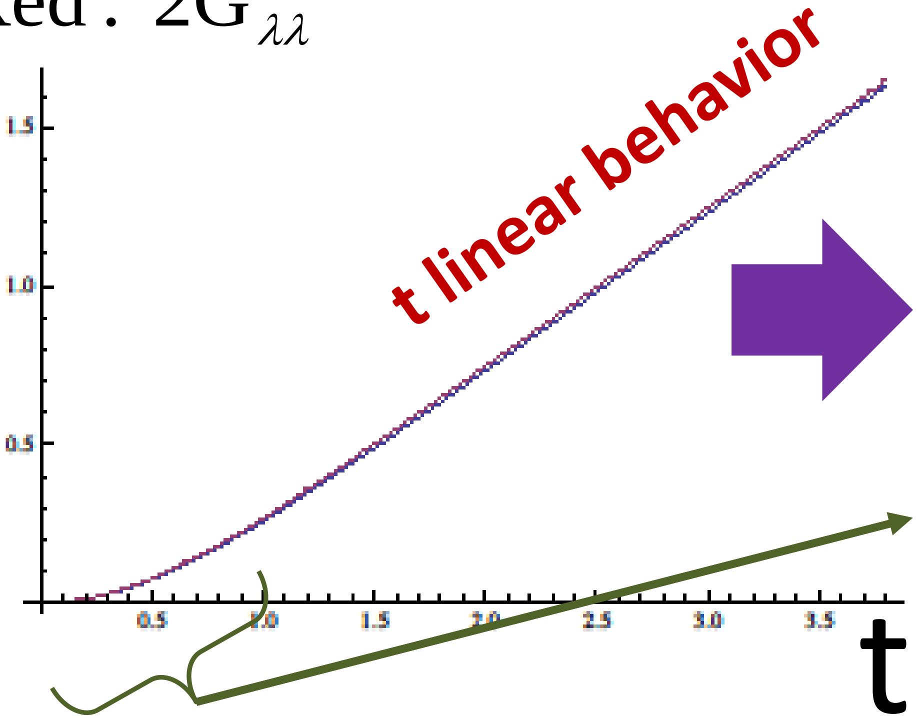
$$t = \int_0^{\kappa_*} \frac{d\kappa}{\sin\kappa\sqrt{1-\sin^2(2\kappa)/\sin^2(2\kappa^*)}}$$

$$-\int_0^{\rho_\infty} \frac{d\rho}{\sinh\rho\sqrt{1+\sinh^2(2\rho)/\sin^2(2\kappa^*)}},$$

Comparison between Holographic and CFT result

Blue : $\mathrm{Vol}(\Sigma)/R_{\mathrm{AdS}}^{d+1}$

Red : $2G_{\lambda\lambda}$



The functional form almost coincides up to a small discrepancy.

$$\mathrm{Vol}(\Sigma)/R_{\mathrm{AdS}}^{d+1} \approx \frac{2}{\pi} t^2,$$

$$2G_{\lambda\lambda} \approx \frac{2}{3} t^2.$$

⑤ Conclusions

- In addition to entanglement entropy, the quantum information metric is a useful quantity which connects between quantum information of a QFT and the geometry of its gravity dual.

- We conjectured the holographic formula of information metric (using a hard-wall approximation) .

$$G_{\lambda\lambda} = n_d \cdot \frac{\mathrm{Vol}(\Sigma_{\max})}{R_{AdS}^{d+1}}.$$

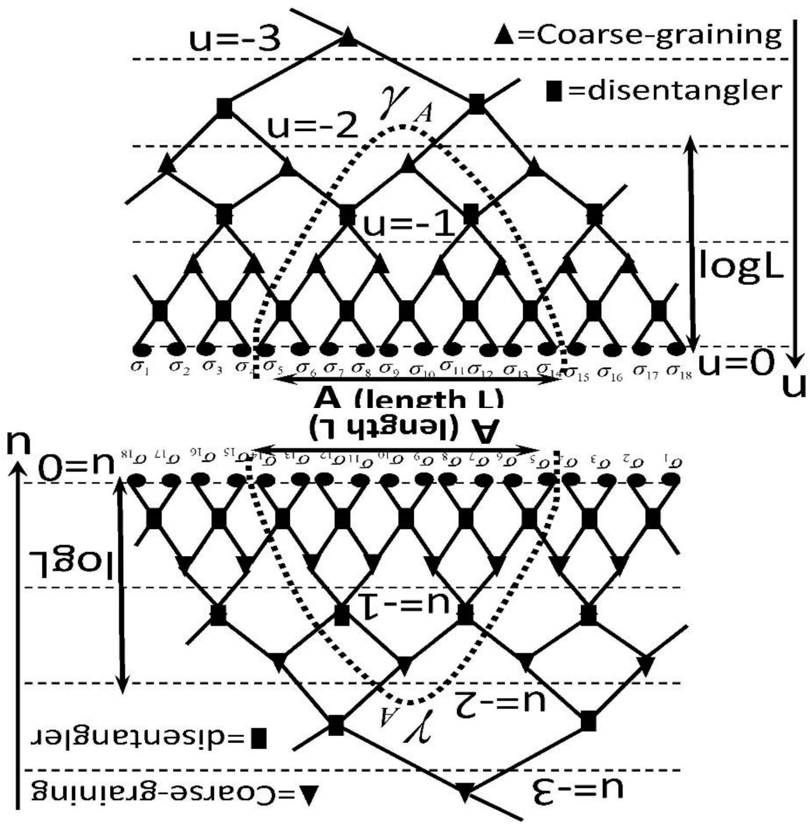
cf. Susskind's conjecture:
The volume is dual to complexity.
Any connection to our results ?

- We also computed the information metric purely in CFTs which nicely agree with our holographic formula.

$\Rightarrow$ $G_{\lambda\lambda} \propto t$ is universal for any CFT TFD states.

A connection to the tensor network description

$$\langle \Phi | \Phi' \rangle =$$



$$\langle \Phi |$$

$$| \Phi' \rangle$$

$$G_{\lambda\lambda} \propto \#\text{Vertices} \approx \text{Vol[time slice]}/R^{d+1}.$$

Note also $\#\text{Vertices} \approx \#\text{Unitary transf.} \sim \text{complexity}?$

Future problems

- CFTs on compact spaces

  ⇒ no universal behavior and the results depend on the spectrum of CFTs. Can we use large N limit ?

- More time-dependent examples of gravity duals, such as quantum quenches, local quenches etc.

- More direct connection to tensor networks (cMERA)..