Analytic bootstrap for higher spin operators

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This talk:

- Study **the anomalous dimension** and the **three point functions** of single trace operators in weakly coupled gauge conformal field theories in 4d in the limit when the spin ℓ is large.
- Consider φ to be a scalar field, the operators are of the form

$$\boxed{\mathcal{O}_\ell = Tr(\varphi^i \partial^\ell \varphi^{L-i})}$$

and the three point functions are (up to normalization) of the form

$$\boxed{\langle Tr(\varphi^p) Tr(\varphi^q) \mathcal{O}_\ell \rangle}$$

Conformal algebra

The conformal group in $d > 2$ contains

- Translations: P_μ
- Lorentz transformations: $M_{\mu\nu}$
- Scale transformations: D
- Special conformal transformations: K_μ

Part of the conformal algebra is

$$\begin{aligned}
[D, K_\mu] &= -K_\mu \\
[D, P_\mu] &= P_\mu \\
[P_\mu, K_\nu] &= \eta_{\mu\nu} D - iM_{\mu\nu}
\end{aligned}$$

How it acts on fields

In a CFT the fields belong to particular representations of the conformal algebra. The states in a CFT are defined by acting with the fields on the vacuum state, which is annihilated by all the generators of the conformal group.

The behaviour of $\phi(0)$ under dilatations and special conformal transformations is

$$
\begin{aligned}
[M_{\mu\nu}, \phi(0)] &= \Sigma_{\mu\nu}\phi(0) \rightarrow \text{SPIN} \\
[D, \phi(0)] &= -\Delta\phi(0) \rightarrow \text{DIMENSION} \\
[K_{\mu}, \phi(0)] &= 0 \rightarrow \text{PRIMARY FIELD}
\end{aligned}
$$

Primary fields

- P_μ raises the scaling dimension while K_μ lowers it. In unitary CFT there is a lower bound on the dimensions of the fields.
- Each representation of the conformal algebra must have some operator of lowest dimension, which must then be annihilated by $K_\mu \to$ PRIMARY OPERATOR
- By acting with P_μ on a primary $\to$ DESCENDANTS
- Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k c_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\mathcal{O}_k(0)$$

- on the rhs there are primaries and all the descendants.

2 and 3 pt functions

- All the information of a CFT is encoded in the set of dimensions and structure constants of local operators
- Conformal symmetry fixes the space-time dependence of 2 and 3 point functions. If we consider primary scalar operators:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{2\Delta}}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_{123}}|x_{23}|^{\Delta_{231}}|x_{13}|^{\Delta_{132}}}$$

where $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$

- For 2 and 3 point functions of different classes of operators there are tensorial structures to be taken into account, but still fully fixed by conformal symmetry.

Method

- The idea is to use the constraints imposed from the associativity of the operator product expansion, or equivalently crossing symmetry.
- Let us consider the four point function of non identical scalar operators

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{ij}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{kl}}{2}} \frac{G_{ijkl}(u,v)}{x_{12}^{\Delta_i+\Delta_j} x_{34}^{\Delta_k+\Delta_l}}$$

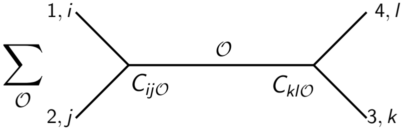
where $\boxed{\Delta_{ij} = \Delta_i - \Delta_j}$ and $\boxed{u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}}$ $\boxed{v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}}$

Method

OPE DECOMPOSITION

Conformal blocks

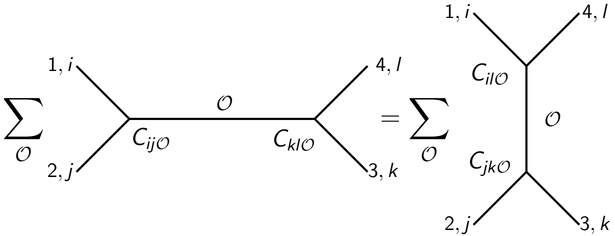
$$G_{ijkl}(u,v) = \sum_{\mathcal{O}} c_{ij\mathcal{O}} c_{kl\mathcal{O}} \boxed{g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v)}$$



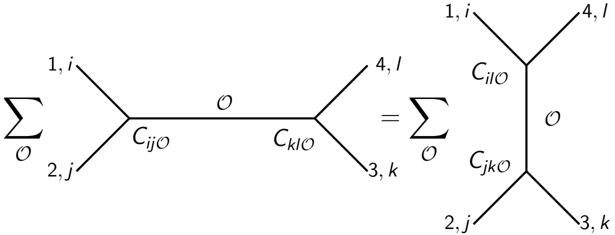
Method

- The correlator must be invariant under the exchange $(1, i) \leftrightarrow (3, k)$

$$v^{\frac{\Delta_j+\Delta_k}{2}} G_{ijkl}(u, v) = u^{\frac{\Delta_i+\Delta_j}{2}} G_{kjil}(v, u)$$



Method



$$\boxed{v^{\frac{\Delta_j+\Delta_k}{2}} \sum_{\mathcal{O}} c_{ij\mathcal{O}} c_{kl\mathcal{O}} g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v) = u^{\frac{\Delta_i+\Delta_j}{2}} \sum_{\mathcal{O}} c_{kj\mathcal{O}} c_{il\mathcal{O}} g_{\Delta,\ell}^{\Delta_{kj},\Delta_{il}}(v,u)}$$

- Put constraints on the dimensions Δ and on the OPE coefficients c
- Warning: Each operator on the lhs is mapped into an infinite combination on the rhs

Small u limit

- The conformal blocks are fully fixed by conformal symmetry and in 4d are known in a closed form
- The small u limit of a conformal block is

$$\boxed{g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v) = u^{\frac{\Delta-\ell}{2}} f_{coll\,(\Delta,\ell)}^{\Delta_{ij},\Delta_{kl}}(v)}$$

where

$$f_{coll\,(\Delta,\ell)}^{\Delta_{ij},\Delta_{kl}}(v) = (1-v)^{\ell}_{2}F_1\left(\frac{1}{2}(\Delta+\ell)-\frac{1}{2}\Delta_{ij}, \frac{1}{2}(\Delta+\ell)+\frac{1}{2}\Delta_{kl}, \Delta+\ell; 1-v\right)$$

- The small u limit is controlled by the twist $\tau = \Delta - \ell$ of the intermediate operator.
- Note: $f_{coll\,(\Delta,\ell)}^{\Delta_{ij},\Delta_{kl}}(v)$ has a logarithmic divergence as $v \to 0$

Small u and v limit

- Let us study for simplicity the case of $\Delta_i = \Delta_j = \Delta_k = \Delta_l = 2$, the intermediate operators are of the form $Tr(\phi\partial^\ell\phi)$
- Consider the small u and v limit of the crossing equation
- It focuses on leading twist operators with dimension $\Delta = \ell + 2 + \gamma_\ell$, where γ_ℓ is small in pert theory

$$\boxed{\frac{1}{u^2}\sum_\ell c_\ell^2 u^{1+\frac{\gamma_\ell}{2}} f^{(0,0)}_{coll\,(\ell+2,\ell)}(v) = \frac{1}{v^2}\sum_\ell c_\ell^2 v^{1+\frac{\gamma_\ell}{2}} f^{(0,0)}_{coll\,(\ell+2,\ell)}(u)}$$

- $v \to 0$ limit: each term in lhs diverges logarithmically $\to$ need to sum infinite many terms
- The divergence comes from the region $\ell \gg 1 \to \sum_{\ell=0}^{\infty}$ and $\sum_{\ell=\ell_0}^{\infty}$ produces the same singularity

Single trace operators of HS_2 are mapped into HS_2.

Results

- In this case it is possible to get constraints for the anomalous dimension and the OPE coefficients for leading twist operators for large value of the spin

$$\boxed{\gamma_\ell^{(2)} = g \log \ell + b(g) + \cdots}$$

$$\boxed{\frac{c^2(\ell)}{c_0^2(\ell)} = \kappa(g) 2^{-\gamma_\ell^{(2)}} e^{-b(g)\log \ell} \Gamma^2 \left(1 - \frac{\gamma_\ell^{(2)}}{2}\right)}$$

- These constraints are valid at all loop in perturbation theory and they come only from imposing crossing symmetry!

More general case

- Consider the correlator of non identical operators $G_{ppqq}(u, v)$, related by crossing to

$$G_{qppq}(v, u) = \frac{u^{\frac{p+q}{2}}}{v^p} G_{ppqq}(v, u)$$

$$G_{ppqq}(u, v) \xrightarrow[u,v \to 0]{} 1 + \frac{u}{v} c_{11}(\log(u), \log(v)) + \cdots$$

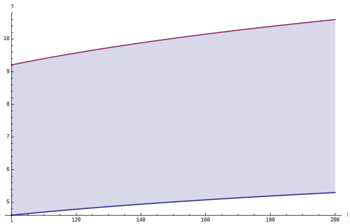
$$G_{qppq}(u, v) \xrightarrow[u,v \to 0]{} \frac{u^{\frac{p+q-2}{2}}}{v^{p-1}} c_{11}(\log(v), \log(u)) + \cdots$$

Single trace operators of HS_2 are mapped into HS_{p+q-2} under crossing.

Intermediate operators

- The intermediate operators on one channel will have twist τ greater than 2, if $p \neq q \neq 2$.
- These primary operators are highly degenerate at large spin. For $\tau = 3$ there are ℓ operators for large ℓ.
- The anomalous dimensions of such operators grows logarithmically within a band. For $\tau = 3$

$$g \log \ell + \cdots \leq \gamma^{(3)}_{I,\ell} \leq 2g \log \ell + \cdots$$



Crossing equation

- The dimension of the intermediate operators is
  1. $\Delta_{\mathcal{O}^{(2)}} = \ell + 2 + \gamma_\ell$
  2. $\Delta_{\mathcal{O}^{(p+q-2)}} = \ell + p + q - 2 + \gamma_\ell$

$$\frac{1}{v} \sum_{\ell} c_{pp\mathcal{O}^{(2)}_\ell} c_{qq\mathcal{O}^{(2)}_\ell} u^{\gamma^{(2)}_\ell/2} f_{coll\,(\Delta_\ell,\ell)}(v) =$$

$$\frac{1}{u^{p-1}} \sum_{\ell,I} c_{qp\mathcal{O}^{(p+q-2)}_{I,\ell}} c_{pq\mathcal{O}^{(p+q-2)}_{I,\ell}} v^{\gamma^{(p+q-2)}_{I,\ell}/2} f^{(q-p,p-q)}_{coll\,(\Delta_{I,\ell},\ell)}(u)$$

- The index I identifies each of the degenerate operators.
- Note that operator with twist 2 are NOT degenerate.

Divergence

- In order to solve the constraints given by the bootstrap equation, we would like to compute the divergent contribution as $v \to 0$ of

$$\sum_\ell a_\ell f^{\Delta_{ij},\Delta_{kl}}_{coll\,(\Delta,\ell)}(v) \quad a_\ell = \frac{\ell^\kappa}{4^\ell} + \cdots$$

where $\Delta = \Delta_0 + \ell + \gamma_\ell$

- The divergence arises from the large ℓ region and can be captured by focusing in the small v/large ℓ region
  - Take $v \to 0$ keeping $x = \ell\sqrt{v}$ fixed
  - use the integral representation for the hypergeometric function
  - $\sum_\ell \to \frac{1}{2}\int dx$

$$\boxed{\sum_{\ell\in 2\mathbb{Z}} \frac{\ell^\kappa}{4^\ell} f^{\Delta_{ij},\Delta_{kl}}_{coll\,(\Delta_0+\gamma_\ell+\ell,\ell)}(v) = \frac{1}{v^{(3+2\kappa-\Delta_{ij}+\Delta_{kl})/4}} \int_0^\infty dx x^{\kappa+\frac{1}{2}} \frac{2^{\Delta_0+\gamma}}{\sqrt{\pi}} K_{\frac{\Delta_{kl}-\Delta_{ij}}{2}}(2x)}$$

Final equation

- We can write then the crossing equation in the form

$$\Gamma(p-1)\Gamma(q-1)\int_0^\infty \hat{a}^{(2)}\left(\frac{x}{\sqrt{v}}\right) 2^{\gamma^{(2)}\left(\frac{x}{\sqrt{v}}\right)} u^{\frac{\gamma^{(2)}\left(\frac{x}{\sqrt{v}}\right)}{2}} x K_0(2x)dx =$$

$$\int_0^\infty x^{p+q-3}\hat{a}^{(p+q-2)}\left(\frac{x}{\sqrt{u}}\right)\langle 2^{\gamma^{(p+q-2)}\left(\frac{x}{\sqrt{u}}\right)} v^{\frac{\gamma^{(p+q-2)}\left(\frac{x}{\sqrt{u}}\right)}{2}}\rangle K_{p-q}(2x)dx$$

- the notation

$$\langle f(\ell)\rangle = \frac{\sum_{\ell,I} c_{qp\mathcal{O}^{(p+q-2)}_{I,\ell}} c_{pq\mathcal{O}^{(p+q-2)}_{I,\ell}} f_I(\ell)}{\sum_{\ell,I} c_{qp\mathcal{O}^{(p+q-2)}_{I,\ell}} c_{pq\mathcal{O}^{(p+q-2)}_{I,\ell}}}$$

has been used to keep track of degeneracies.

- normalized in such a way that $\hat{a}^{(i)} = 1 + \ldots$

Solution

- At tree level: $\gamma^i = 0$ and $\hat{a} = 1 \rightarrow$ eq is fulfilled
- At loop level: Both $\langle\gamma^{(p+q-2)}\rangle$ and $\hat{a}^{(p+q-2)}$ have a logarithmic behaviour

$$\boxed{\langle\gamma^{(p+q-2)}(\ell)\rangle = \langle\rho\rangle \log \ell + \langle\beta\rangle + \cdots}$$

$$\rho = \rho_1 g + \rho_2 g^2 + \cdots$$
$$\beta = \beta_1 g + \beta_2 g^2 + \cdots$$

$$\boxed{\hat{a}^{(2)}(\ell) = 1 + g(a_{10} + a_{11} \log \ell) + g^2(a_{20} + a_{21} \log \ell + a_{22} \log^2 \ell) + \cdots}$$

$$\boxed{\hat{a}^{(p+q-2)}(\ell) = 1 + g(a_{10}^{(pq)} + a_{11}^{(pq)} \log \ell) + g^2(a_{20}^{(pq)} + a_{21}^{(pq)} \log \ell + a_{22}^{(pq)} \log^2 \ell) + \cdots}$$

Results

$$\gamma^{(2)}(\ell) = g \log \ell + b$$

$$\hat{a}^{(2)}(\ell) = \alpha(g) 2^{-g \log \ell - b + \beta} \ell^{-\beta} \boxed{\Gamma\left(p - 1 - \frac{1}{2}\gamma^{(2)}(\ell)\right) \Gamma\left(q - 1 - \frac{1}{2}\gamma^{(2)}(\ell)\right)}$$

$$\langle \gamma^{(p+q-2)}(\ell) \rangle = \langle g \rangle \log \ell + \langle \beta \rangle$$

$$\hat{a}^{(p+q-2)}(\ell) = \alpha(g) 2^{-g \log \ell} \ell^{-b} \boxed{\Gamma\left(1 - \frac{1}{2}\gamma^{(p+q-2)}(\ell)\right)^2}$$

- $\hat{a}^{(2)}(\ell)$ comes from a factorized OPE coefficient, then $\langle \ell^{-\beta} \rangle = f(p) f(q)$
- universal behaviour of the OPE coefficients

$$\boxed{c_{pq\mathcal{O}_\ell^{(\tau)}} \sim \Gamma\left(\frac{\Delta_p + \Delta_q - \tau - \gamma_\ell^{(\tau)}}{2}\right)}$$

Comments

- The form of the OPE coefficient we found has a similar structure compared to results obtained for extremal correlators in supergravity computations, using Witten diagrams. Notice that it is important that the correlators that we consider are NON protected.
- Poles: when the full twist of the higher spin operator equals the sum of the dimensions of the other two. The appearance of these poles is believed to be related to operator mixing.

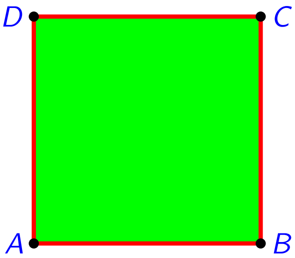
  [Bargheer, Minahan, Pereira-Minahan, Pereira-Korchemsky]
- Notice that in this specific limit, the OPE coefficients are fixed in terms of the dimensions.

Relation to Wilson Loops

- In the limit where $u, v \to 0$ at the same rate, the four-point function should reduce to the expectation value of a rectangular Wilson loop

[Alday, Eden,Korchemsky,Maldacena, Sokatchev]

$$\lim_{u,v\to 0} \frac{G_{conn}}{G^{tree}_{conn}} \sim e^{-\frac{\Gamma_{cusp}}{4}\log u \log v + \frac{b_1}{2}\log u + \frac{b_2}{2}\log v} J(u,v)$$



- Fast moving particles going between different vertices
  - Leading div proportional to the area of the rectangle
  - Subleading div proportional to the perimeter
- Back reaction of the color electric field on the propagation of the particles $J(u, v)$

Relation to Wilson Loops I

- Using the relation that we get for the dimensions and the OPE coefficients, we are able to compute the four point function in the same limit and compute $J(u, v)$, which is in general a very complicated object,

$$\boxed{J(u,v) = \int_0^\infty dxdy x^{1-\beta+\frac{g}{2}\log u} y^{p+q-3-b+\frac{g}{2}\log v} e^{-g\log x \log y} K_0(2x) K_{p-q}(2y)}$$

- Notice that $\Gamma_{cusp} = g$ in our conventions.

Conclusions

- We have found the form of the anomalous dimension and OPE coefficients of single trace operators, in weakly coupled gauge conformal field theories, in the large spin limit
- The method that we used relies only on crossing symmetry of the four point function and the structure of the OPE.
- We have seen that the OPE coefficients have a universal form, which have a specific pole structure
- We compute the four point function in the small u, v limit and compare to the WL/correlator correspondence.