# Entanglement Holography [part 2]

Based on:

arXiv:1509.00113 + arXiv:1605.nnnnn

with Michal Heller, Rob Myers, Yasha Neiman, Felix Haehl

Jan de Boer, Amsterdam

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## Basic idea:

(holography): Try to reorganize the degrees of freedom of quantum gravity in an interesting way. This may shed light on the (non)local nature of the underlying fundamental degrees of freedom. May help understand bulk reconstruction. Sheds new light on perturbative bulk computations.

(CFT): Try to reorganize the degrees of freedom of CFT's in an interesting way. Define new quantum information theoretic quantities. May shed new light on the structure of correlation functions. May be useful to study which CFT's have weakly coupled gravitational duals and for other applications.

# Observation 1:

Field theory locality

↕

Factorization $\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{\bar{D}}$ exists for any spatial domain D. Can compute entanglement entropy.

↕

Ryu-Takayanagi: Same quantity can be computed using bulk *extremal* surfaces.

↕

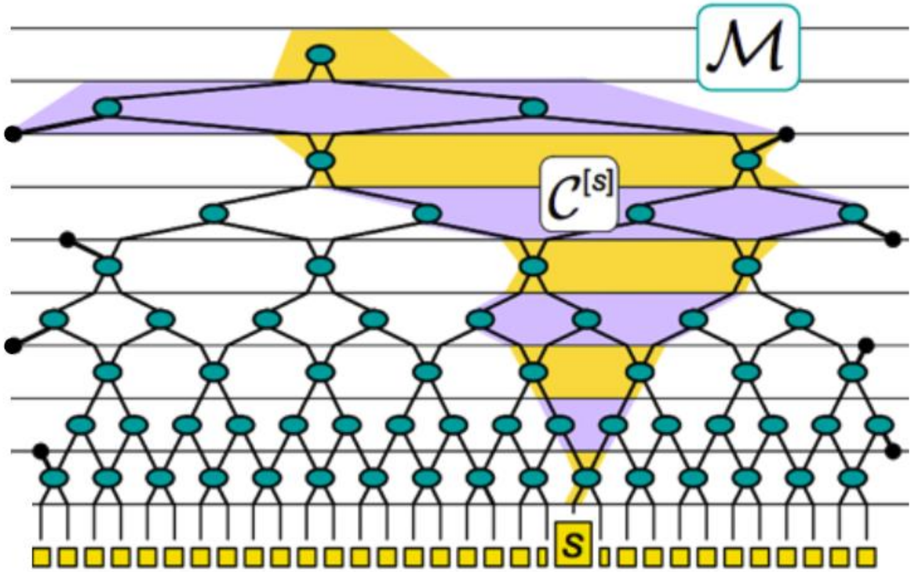
Hilbert space Factorization is only across bulk extremal surfaces. *Preferred role for extremal surfaces vs locality*?

# Observation 2:

Connection between renormalization group flow and entanglement entropy?

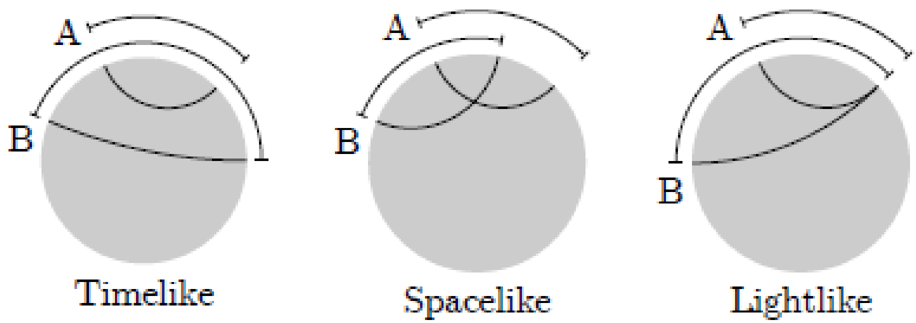
It has been argued that MERA has a Lorentzian causal structure (Bény, Czech et al).

The natural geometry to support this causal structure appears to be de Sitter space.



# Third observation:

Motivated by among others by differential entropy (Balasubramanian, JdB, Chowdhury, Czech, Heller), Czech, Lamprou, McCandlish and Sully introduced "*kinematic space*", the space of geodesics, and came up with a natural metric




If the endpoints are $u = \theta - \alpha$ and $v = \theta + \alpha$ then

$$\omega = \partial_u \partial_v S du \wedge dv$$

$$ds^2 \sim \frac{1}{\sin^2 \alpha}(-d\alpha^2 + d\theta^2)$$

indeed: de Sitter

Length in AdS = Volume in dS

# Fourth observation:

The first law of entanglement entropy states that

$$\delta S(B) = 2\pi \int_B d^{d-1}x' \; \frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R} \, \langle T_{tt}(\vec{x}') \rangle$$

This contains the “bulk-boundary” propagator of de Sitter space and hence

$$\left(\nabla_a \nabla^a - m^2\right) \, \delta S = 0 \qquad\qquad m^2 L^2 = -d$$

in the spacetime with the metric

$$ds^2 = \frac{L^2}{R^2}\left(-dR^2 + d\vec{x}^2\right)$$

Size of ball B = scale = time of de Sitter

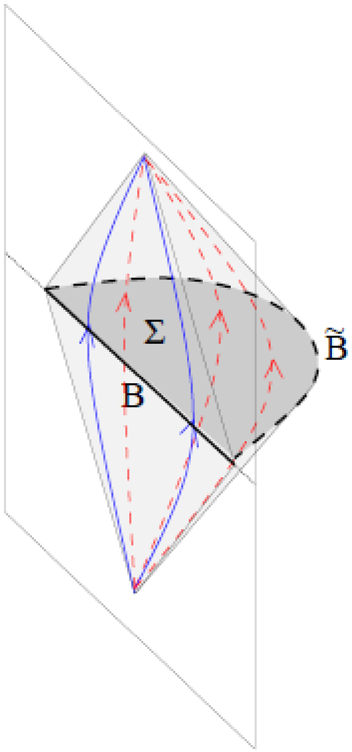
Valid in any CFT

Similar equations but different perspective in:
Nozaki, Numasawa, Prudenziati, Takayanagi;
Bhattacharya, Takayanagi

How to bring these pieces of information together?

- Preferred role for extremal surfaces
- Tensor networks that carry a (natural?) Lorentzian metric
- Space of geodesics in Euclidean AdS = dS
- First law of entanglement entropy seems to obey a free field equation on some de Sitter space.
- Reorganization of degrees of freedom.

Note:
Variation of entanglement entropy is computed to first order by the minimal surface as well:




$$\delta S(B) \sim \int_{\tilde{B}} d^{d-1}x E^{ij} \delta h_{ij}$$

To bring this information together, it will be useful to characterize “spherical” minimal surfaces in Euclidean and Lorentzian AdS.

Spherical = minimal surfaces preserving a maximal amount of symmetry, i.e. having spherically symmetric cross sections.

# Some geometry (1)

Euclidean $\mathrm{AdS}_d$ is the equation

$$\langle X, X \rangle \equiv -X_0^2 + X_1^2 + \ldots + X_d^2 = -1$$

A “spherical” minimal surface is given by

$$\langle X, U \rangle = 0, \qquad \langle U, U \rangle = +1$$

The space of U’s is de Sitter space (or rather $\mathrm{dS}/\mathbb{Z}_2$ )

Some geometry (2)

Lorentzian $AdS_{d+1}$ is the equation

$$\langle X, X\rangle \equiv -X_0^2 - X_1^2 + X_2^2 + \ldots + X_{d+1}^2 = -1$$

A “spherical” minimal surface is given by

$$\{\langle X, U\rangle = 0 \cap \langle X, V\rangle = 0\}, \qquad \langle U, U\rangle = +1, \langle V, V\rangle = -1$$

The space of spherical minimal surfaces is

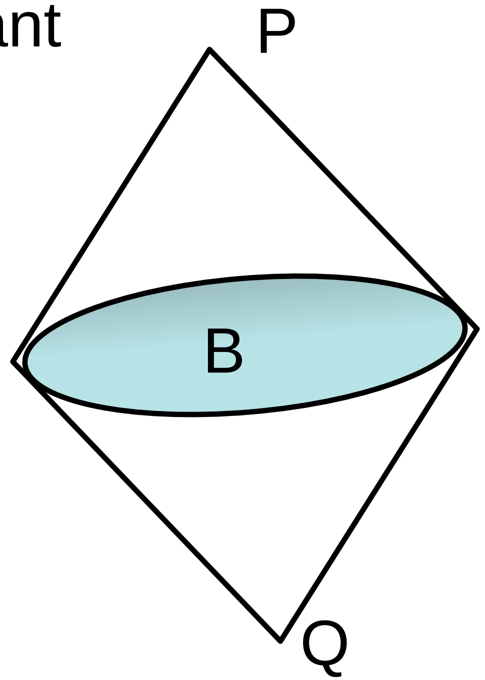
$$M = \frac{SO(2,d)}{SO(1,d-1) \times SO(1,1)}$$

This is a space of dimension 2d with metric with signature (d,d).

Spherical minimal surfaces in Lorentzian signature can be characterized by two points P,Q that bound a causal diamond.

There is a unique conformally invariant metric on the space of two points M.

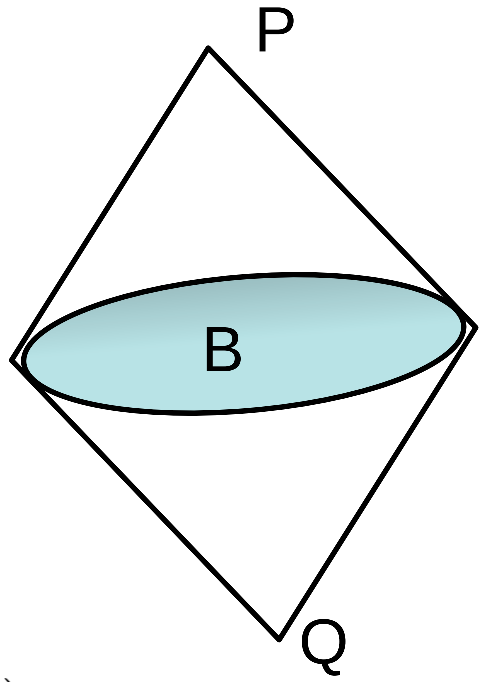
The geodesic distance between two pairs of points is a simple function of their cross ratio.




$$ds^2 = \frac{1}{(P-Q)^2}\left(\eta_{\mu\nu} - 2\frac{(P-Q)_\mu (P-Q)_\nu}{(P-Q)^2}\right) dP^\mu dQ^\nu$$

What is the general picture?

Conjecture: For every bulk field $\phi$ (or boundary operator) there is a corresponding entanglement-like quantity $S_\phi(P,Q)$ where P,Q are two points in Minkowski space where the field theory lives. P and Q define a ball B:



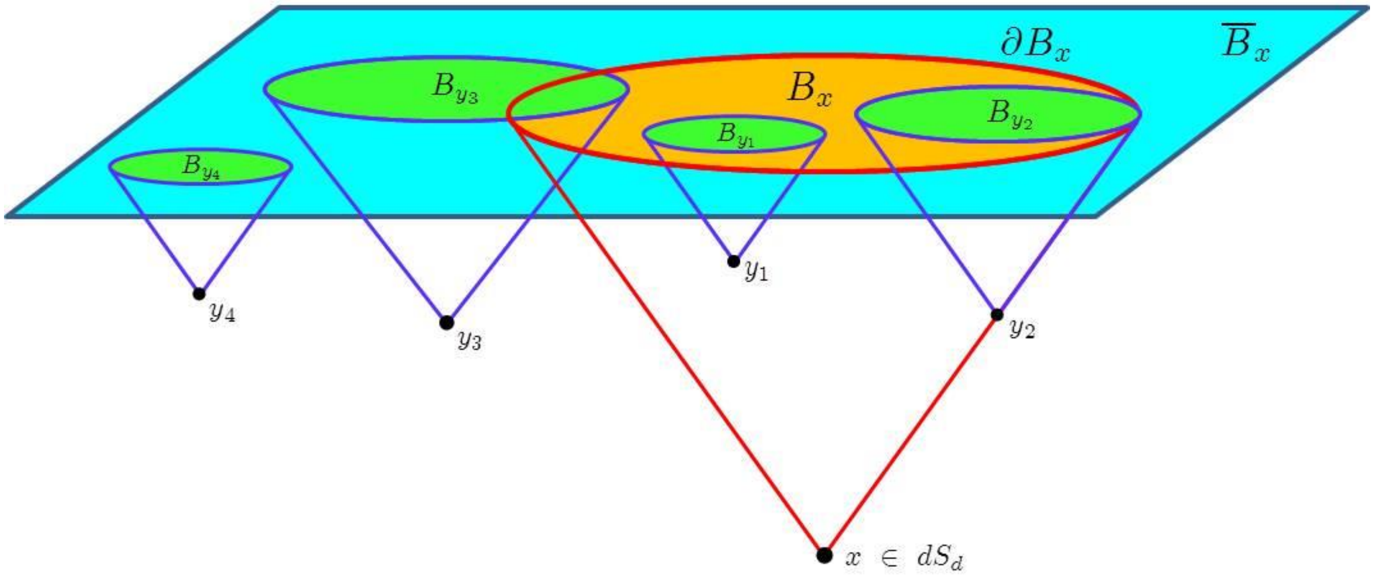

Standard case is entanglement entropy of B.

The coset space M carries a natural metric which is invariant under the conformal group.

Wild conjecture: All quantities $S_\phi(P,Q)$ obey local, dynamical field equations on M!

If we restrict P and Q such that B is part of a given fixed time slice then this defines a submanifold of M which is de Sitter space by construction.

Relation between balls and points in de Sitter:



How to define $S_\phi(P,Q) \equiv S_\phi(B)$ ?

First define to first order for perturbations around the ground state. To leading order all quantities $S_\phi(P,Q)$ vanish unless $\phi$ corresponds to the metric.

Proposal: for scalar operators

$$\delta S_\phi(B) = C \int_{D(P,Q)} d^d x \left( \frac{|P-x||x-Q|}{|P-Q|} \right)^{\Delta_{\mathcal{O}}-d} \langle \mathcal{O}(x) \rangle$$

$\mathcal{O}$: CFT operator dual to $\phi$

where

$D(P,Q)$: causal diamond bounded by P and Q.

## Features:

- It obeys a Klein-Gordon equation on M.
- It is reminiscent of the first law of entanglement entropy, except that the integral is over a full causal diamond.
- The kernel that appears in the integral intertwines the SO(2,d) actions on AdS and M.
- It has an OPE interpretation

$$A(z)B(w) = \frac{C(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_C}} + \frac{\partial C(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_C-1}} + \ldots$$

resum all derivatives and take A=B:

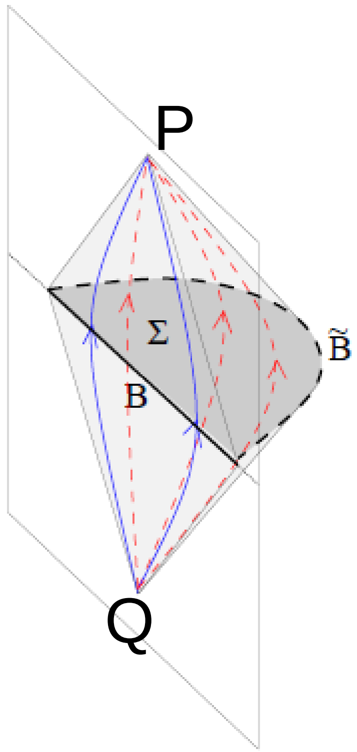
$$(z-w)^{2\Delta_A} A(z)A(w) = C' \int_{D(z,w)} d^d x \left( \frac{|z-x||x-w|}{|z-w|} \right)^{\Delta_c - d} C(x)$$

which has exactly the same form!

## Features (continued):

It has an interesting bulk interpretation in AdS/CFT:

$$\delta S(B) = C'' \int_{\tilde{B}} d^{d-1}x \;\; \phi(x)$$




Related to work of
Hijano, Kraus, Perlmutter, Snively.

Equivalence of bulk quantity

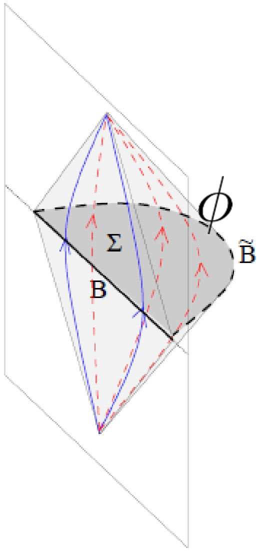
$$\delta S(B) = C'' \int_{\tilde{B}} d^{d-1}x \;\; \phi(x)$$

and boundary quantity

$$\delta S_\phi(B) = C \int_{D(P,Q)} d^d x \left( \frac{|P-x||x-Q|}{|P-Q|} \right)^{\Delta_{\mathcal{O}}-d} \langle \mathcal{O}(x) \rangle$$

follows from conformal invariance and entanglement wedge reconstruction.

We therefore have a simple bulk representation of an "OPE block"



Similarly, in Euclidean signature an OPE block is computed by a geodesic connecting P and Q.

Explains geodesic Witten diagram computations of Hijano, Kraus, Perlmutter, Snively.

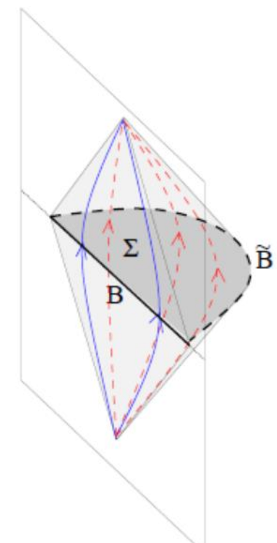
Boundary expression is less insightful due to absence of causal diamond.

Our claim is that to first order the OPE block is a local operator on the auxiliary space M.

Another example: higher spin conserved currents Integral is now only over B not over entire Causal diamond.

Consider the following first law

$$\delta S^{(s)}(B) = (2\pi)^{s-1} \int_B d^{d-1}x' \left( \frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R} \right)^{s-1} \langle T_{tt\dots t}(\vec{x}') \rangle$$




which can be used to define a linearized higher spin generalization of entanglement entropy. It obeys a dS Klein-Gordon equation with mass

$$m^2 L^2 = -(s-1)(d+s-2)$$

(Cf Belin, Hung, Maloney, Matsuura, Myers, Sierens; Hijano, Kraus)

So far have only defined these quantities to first order, both in the CFT as well as on the boundary……….

Challenge: extend these definitions to higher order?

- Use replica trick and generalization of twist fields as suggested by OPE? Modular Hamiltonian vs $\mathcal{O}$ ?
- Try to construct order by order in perturbation theory (what criteria to use?)
- Use structure of conformal blocks, OPE’s, etc?
- Use map to hyperbolic black hole, relate to partition functions of black holes with scalar fields?

Challenge: even if this can be done, do these quantities obey local field equations in dS or M?

There is one example where nonlinear definition is known and we can check for local field equations: standard EE

Consider a general metric of the form

$$ds^2 = \frac{dw^2 + dx^+ dx^-}{w^2} - \frac{6}{c} T(x^+) {dx^+}^2 - \frac{6}{c} \bar{T}(x^-) {dx^-}^2 + \frac{36}{c^2} w^2 T(x^+) \bar{T}(x^-) dx^+ dx^-$$

then the entanglement entropy is equal to

$$S = \frac{c}{6} \log \left( \frac{(f_+(x_1^+) - f_+(x_2^+))(f_-(x_1^-) - f_-(x_2^-))}{\epsilon^2 \sqrt{\partial_+ f_+(x_1^+) \partial_+ f_+(x_2^+) \partial_- f_-(x_1^-) \partial_- f_-(x_2^-)}} \right)$$

with

$$T(x^+) = \frac{c}{12} \frac{\partial_+^3 f_+}{\partial_+ f_+} - \frac{3}{2} \left( \frac{\partial_+^2 f_+}{\partial_+ f_+} \right)^2$$

This obeys, in any background,

$$\frac{\partial}{\partial x_1^+}\frac{\partial}{\partial x_2^+}S = \frac{c}{6\epsilon^2}e^{-12S/c}$$

This is like a Liouville equation, suggestion S defines a metric on de Sitter space in conformal gauge. Connection to quantum gravity on de Sitter space?

For this particular case, $M=dS_2 x dS_2$ corresponding to left- and right movers.

Entanglement entropy factorizes into left- and right-movers.

A more interesting test: take a higher spin theory in AdS3, for example with massless fields of spins 2 and 3.

Using the Chern-Simons formulation of the theory, and the relation between entanglement entropy and Wilson lines (JdB, Jottar; Ammon, Castro, Iqbal) we can compute both the ordinary entanglement entropy and its spin three generalization for arbitrary spin 2,3 backgrounds.

## Result

$$
\begin{aligned}
\Sigma_1 &= (\gamma_2(x_1)-\gamma_2(x_2))\gamma_1'(x_1)-(\gamma_1(x_1)-\gamma_1(x_2))\gamma_2'(x_1)\\
\Sigma_2 &= (\gamma_2(x_1)-\gamma_2(x_2))\gamma_1'(x_2)-(\gamma_1(x_1)-\gamma_1(x_2))\gamma_2'(x_2)\\
\Phi_1 &= (\gamma_1'(x_1)\gamma_2''(x_1)-\gamma_2'(x_1)\gamma_1''(x_1))\\
\Phi_2 &= (\gamma_1'(x_2)\gamma_2''(x_2)-\gamma_2'(x_2)\gamma_1''(x_2))
\end{aligned}
$$

$$
S_{EE}^2=\log\left(\frac{\Sigma_1\Sigma_2}{\Phi_1\Phi_2}\right), \qquad S_{EE}^3=\log\left(\frac{\Sigma_2^{1/2}\Phi_1^{1/6}}{\Sigma_1^{1/2}\Phi_2^{1/6}}\right).
$$

These obey in any background

$$\frac{\partial^2}{\partial x_1 \partial x_2} S^2_{EE} = \exp\left(-\frac{S^2_{EE}}{2} + 3S^3_{EE}\right) + \exp\left(-\frac{S^2_{EE}}{2} - 3S^3_{EE}\right)$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} S^3_{EE} = -\frac{1}{2}\exp\left(-\frac{S^2_{EE}}{2} + 3S^3_{EE}\right) + \frac{1}{2}\exp\left(-\frac{S^2_{EE}}{2} - 3S^3_{EE}\right)$$

These equations are those of SL(3) Toda theory, suggesting we are looking at higher spin gravity in de Sitter space in conformal gauge.

For interacting scalar fields, more difficult to find a good definition.

Idea is to try

$$S_\phi(B) = \int_D dx K(P, Q, x)\langle \mathcal{O}(x)\rangle + \int_D \int_D dx dy K(P, Q, x, y)\langle \mathcal{O}(x)\rangle \langle \mathcal{O}(y)\rangle + \dots$$

but so far progress has been limited as conformal invariance alone does not fix such an expansion.

Should we expect a CFT definition or only a holographic one?

$$S_\phi(B) = \int_{\tilde{B}} dx \phi(x) + \int_{\tilde{B}/\Sigma} \int_{\tilde{B}/\Sigma} dx dy K(x, y)\phi(x)\phi(y) + \dots$$

Constructive approach:

1. Compute perturbative corrections to ordinary entanglement entropy due to other fields.

2. Demand that these corrections arise from a local theory on M.

3. This should fix most of the theory on M, viewing entanglement entropy as being related to the conformal factor of a metric on M.

4. Finally, we construct $S_\phi(P,Q)$ perturbatively so that it agrees with the local theory on M.

Extra complications due to following issue:

Map from $EAdS_d$ to $dS_d$ preserves dimension.

Map from $AdS_{d+1}$ to $M_{2d}$ does not. Describing image is difficult (generalized Radon transformation) except when d=2. (but see Czech, Lamprou, McCandlish, Mosk, Sully)

For d=2 image of map obeys (to first order)

$$(\Box_1 - \Box_2) S_B(\phi) = 0$$

using $M=dS_2 x dS_2$.

Notice a striking similarity to doubled field theory since the two copies of de Sitter correspond to left and right-movers.

## Some recent related papers:

*Czech, Lamprou, McCandlish, Mosk, Sully, arXiv:1604.03110*
(significant overlap, restricted to first order)

*Asplund, Callebaut, Zukowski, arXiv:1604.02687*
(study of entanglement entropy in BTZ and conical defect)

*Beach, Lee, Rabideau, van Raamsdonk, arXiv:1604.05308*
(second order corrections to EE and explicit check with de Sitter perturbation theory)

*Carneiro da Cunha, Guica, arXiv:1604.07383*
(geodesic operators vs bulk reconstruction)

## Summary/open problems

- Found evidence for local interacting theories that describe the evolution of various entanglement-like quantities as one changes scale. *What is the fundamental meaning of this?*
- Generalization to arbitrary fields (charged fields, fermions, …)?
- Right way to think about interactions? Generalized twist fields?
- Not clear why this works? Is this a fundamental property of arbitrary CFT's, or only those with a weakly coupled gravitational dual?
- Connection to tensor networks?
- Does this shed light on a possible holographic dual description of de Sitter space?
- What is the meaning of the space M?
- Relation to OPE/conformal blocks/CFT data?