Hexagons and 3pt functions

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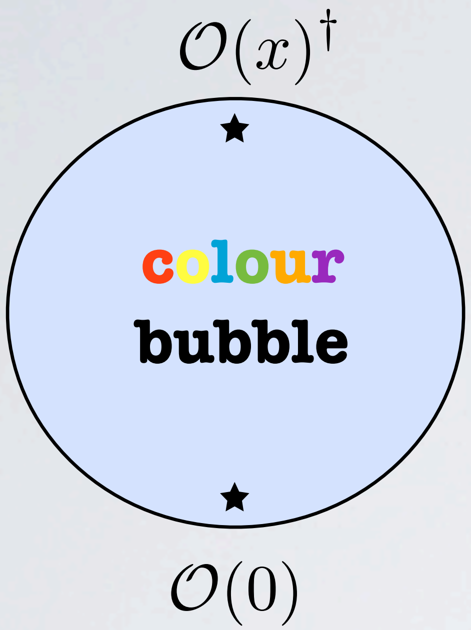
Current Themes in Holography
NBI Copenhagen 2016

based on work with
Vasco Goncalves, Shota Komatsu and Pedro Vieira

The spectral problem is solved

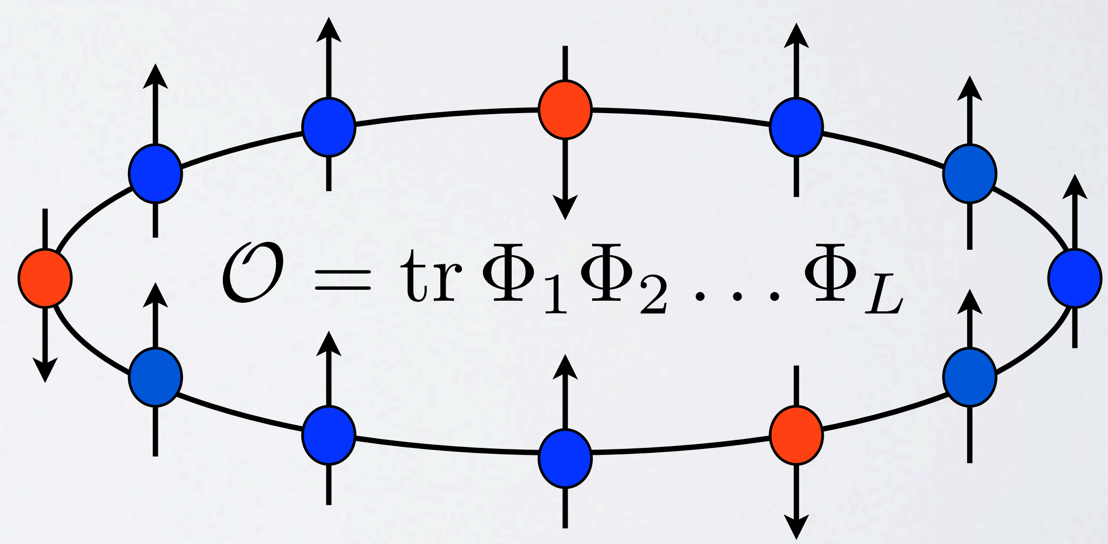


$$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \frac{1}{x^{2\Delta}}$$

All N=4 SYM planar 2pt functions
are known
@ any value of ‘t Hooft coupling

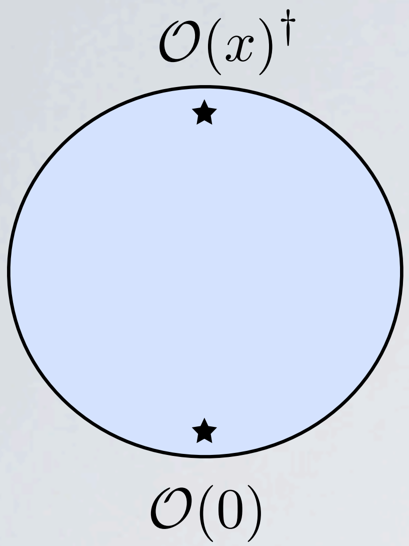
Fundamental working assumption :

Mixing problem for local
(single trace) operator is
equivalent to an integrable
spin chain problem



$\Delta =$ Scaling dimension
String energy
Spin chain energy

2pt function history



2009

Thermodynamic Bethe Ansatz

2005

Symmetries
Beisert S-matrix

Beisert-Staudacher
Asymptotic Bethe Ansatz

Quantum Spectral Curve

BMN Vacuum
Spin Chain Picture

2002

Final word (?)
Two Years Ago

QCD Story
Perturbative
Integrability

1995-1998

Full set of equations : see [Gromov,Kazakov,Leurent,Volin'14]

It leads to a wealth of amazing results / predictions for the gauge / string theory

Perturbative predictions

Example : Scaling dimension of shortest unprotected operator (so-called Konishi) = lightest massive string state

$$\mathcal{O} \sim \mathrm{tr}\, DZDZ$$

$$\begin{aligned}
\Delta \;=\;& 4 + 12g^2 - 48g^4 + 336g^6 + g^8\left(-2496 + 576\,\zeta_3 - 1440\,\zeta_5\right) \\
&+g^{10}\left(15168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30240\,\zeta_7\right) \\
&+g^{12}\left(-7680 - 262656\,\zeta_3 - 20736\,\zeta_3^2 + 112320\,\zeta_5 + 155520\,\zeta_3\,\zeta_5 + 75600\,\zeta_7 - 489888\,\zeta_9\right) \\
&+g^{14}\left(-2135040 + 5230080\,\zeta_3 - 421632\,\zeta_3^2 + 124416\,\zeta_3^3 - 229248\,\zeta_5 + 411264\,\zeta_3\,\zeta_5\right. \\
&\quad \left. - 993600\,\zeta_5^2 - 1254960\,\zeta_7 - 1935360\,\zeta_3\,\zeta_7 - 835488\,\zeta_9 + 7318080\,\zeta_{11}\right) \\
&+g^{16}\Big(54408192 - 83496960\,\zeta_3 + 7934976\,\zeta_3^2 + 1990656\,\zeta_3^3 - 19678464\,\zeta_5 - 4354560\,\zeta_3\,\zeta_5 \\
&\quad - 3255552\,\zeta_3^2\,\zeta_5 + 2384640\,\zeta_5^2 + 21868704\,\zeta_7 - 6229440\,\zeta_3\,\zeta_7 + 22256640\,\zeta_5\,\zeta_7 \\
&\quad + 9327744\,\zeta_9 + 23224320\,\zeta_3\,\zeta_9 + \frac{65929248}{5}\,\zeta_{11} - 106007616\,\zeta_{13} - \frac{684288}{5}\,Z_{11}^{(2)}\Big) \\
&+g^{18}\Big(-1014549504 + 1140922368\,\zeta_3 - 51259392\,\zeta_3^2 - 20155392\,\zeta_3^3 + 575354880\,\zeta_5 \\
&\quad - 14294016\,\zeta_3\,\zeta_5 - 26044416\,\zeta_3^2\,\zeta_5 + 55296000\,\zeta_5^2 + 15759360\,\zeta_3\,\zeta_5^2 - 223122816\,\zeta_7 \\
&\quad + 34020864\,\zeta_3\,\zeta_7 + 22063104\,\zeta_3^2\,\zeta_7 - 92539584\,\zeta_5\,\zeta_7 - 113690304\,\zeta_7^2 - 247093632\,\zeta_9 \\
&\quad + 119470464\,\zeta_3\,\zeta_9 - 245099520\,\zeta_5\,\zeta_9 - \frac{186204096}{5}\,\zeta_{11} - 278505216\,\zeta_3\,\zeta_{11} - 253865664\,\zeta_{13} \\
&\quad + 1517836320\,\zeta_{15} + \frac{15676416}{5}\,Z_{11}^{(2)} - 1306368\,Z_{13}^{(2)} + 1306368\,Z_{13}^{(3)}\Big)
\end{aligned}$$

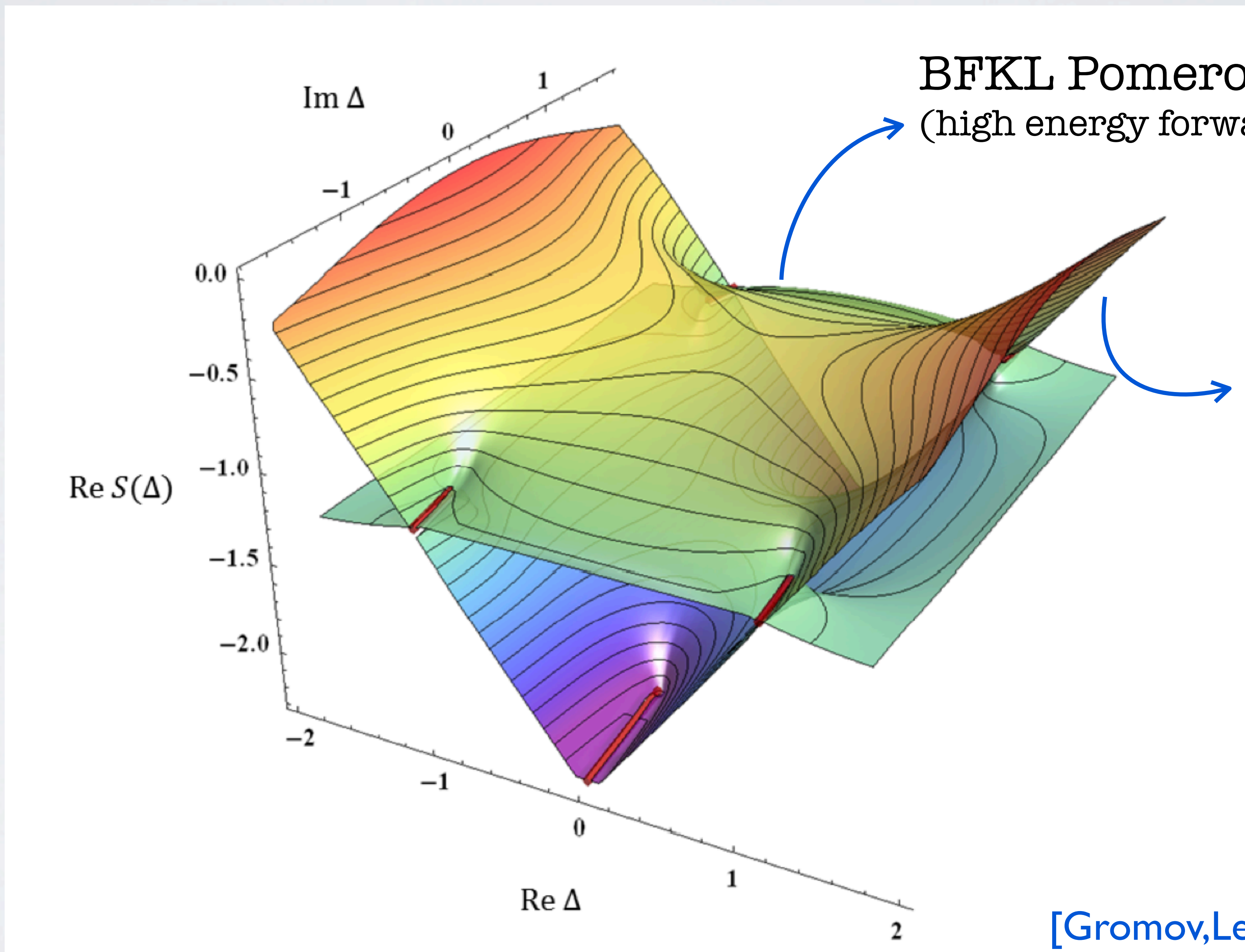
Comments :
- Z.. stand for single valued multiple zeta values
- Could get more loops if needed

[Marboe,Volin'14]

Exploring non-perturbative territories

Example : Scaling dimension of twist two operator for complex spin = leading Regge trajectory

$$\mathcal{O} \sim \mathrm{tr}\, Z D^S Z$$

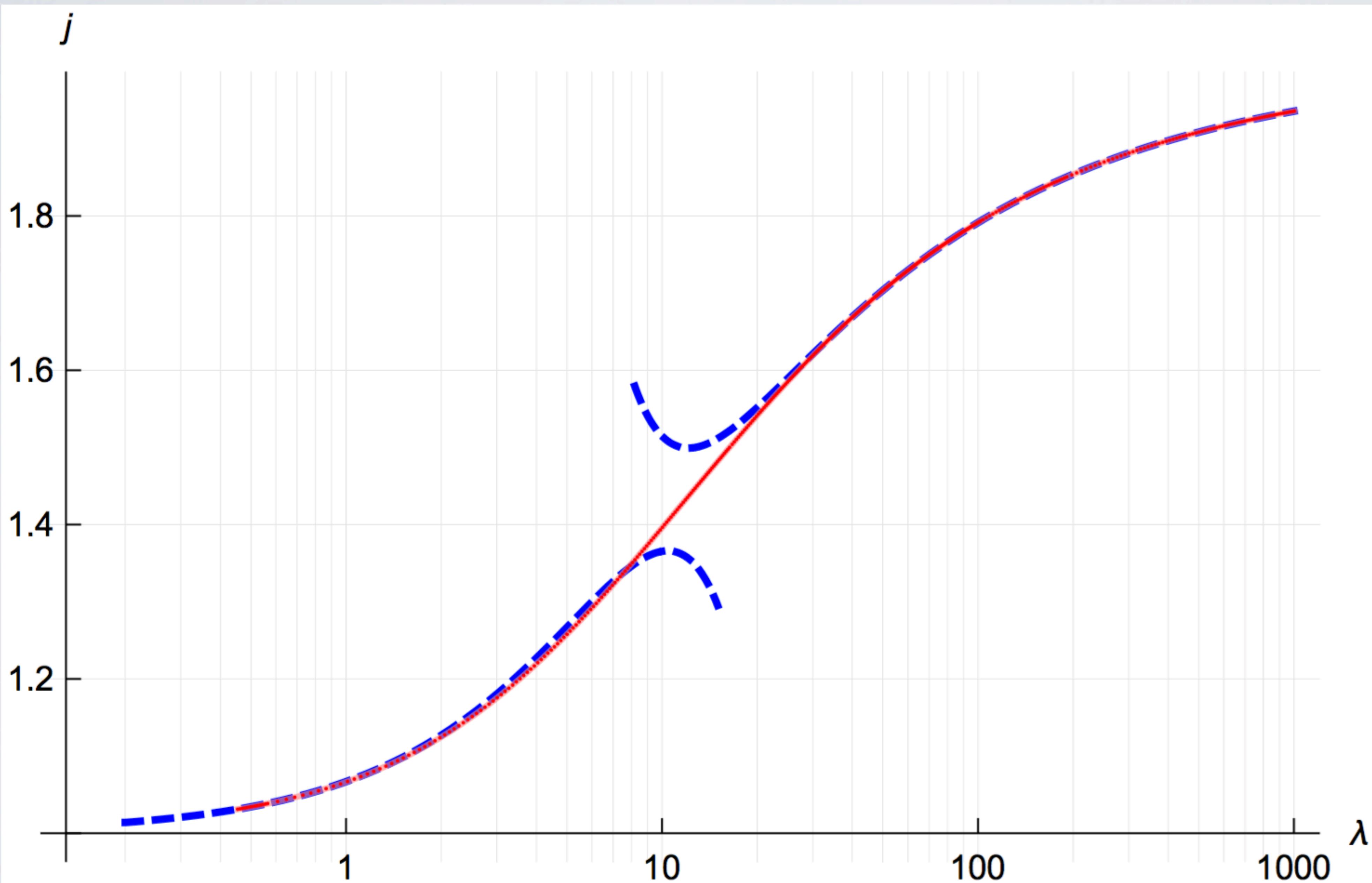


[Gromov,Levkovich-Maslyuk,Sizov'15]

Plot of real part of the spin S as a function of the scaling dimension Δ for 't Hooft coupling = 6.3

Precision test of the gauge / gravity interpolation

Example : Pomeron (= Reggeized graviton) intercept

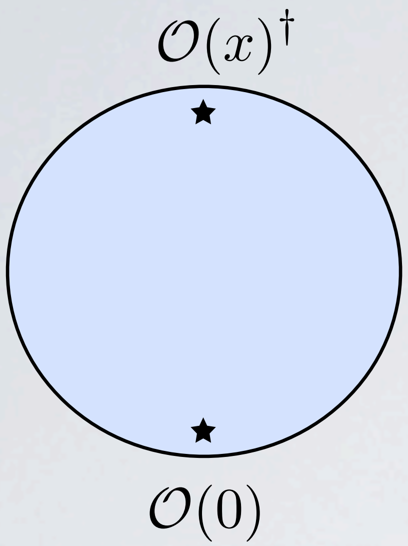


strong coupling
string prediction
graviton has j = 2

weak coupling
gauge prediction
Pomeron has j = 1

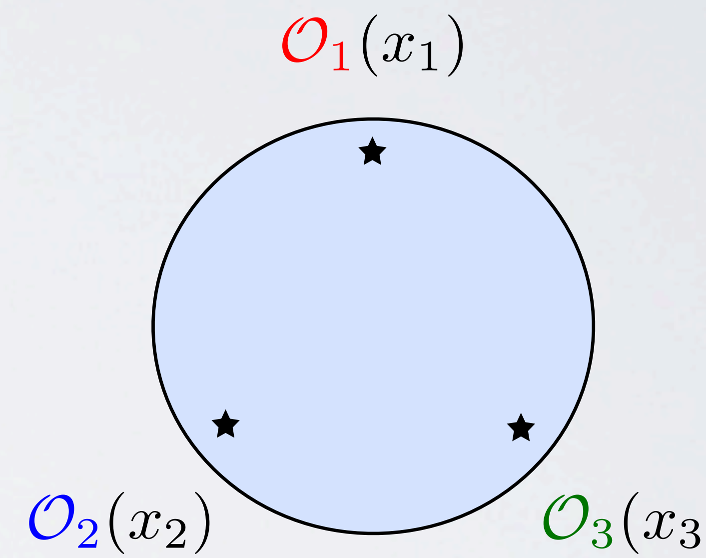
[Gromov,Levkovich-Maslyuk,Sizov'15]

Towards solving planar N=4 SYM theory



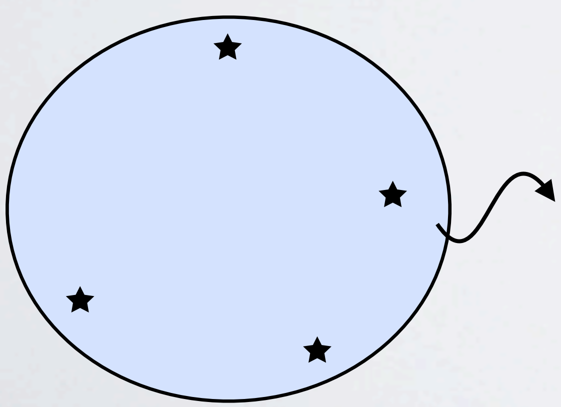
2pt functions
Solved [Many people here]

3pt functions
Wanted



[Here also]

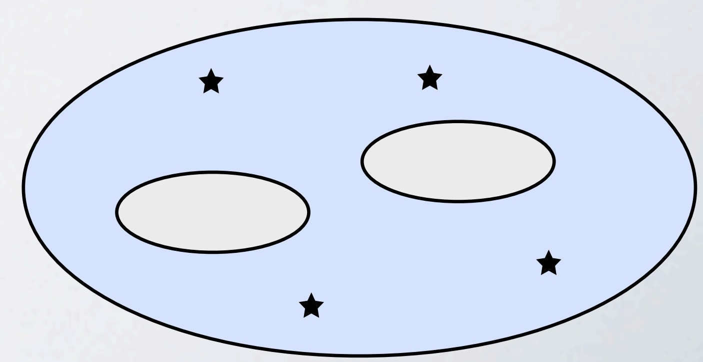
Looking forward...



Two cross ratios
Use OPE / conformal bootstrap or better?

... and farther

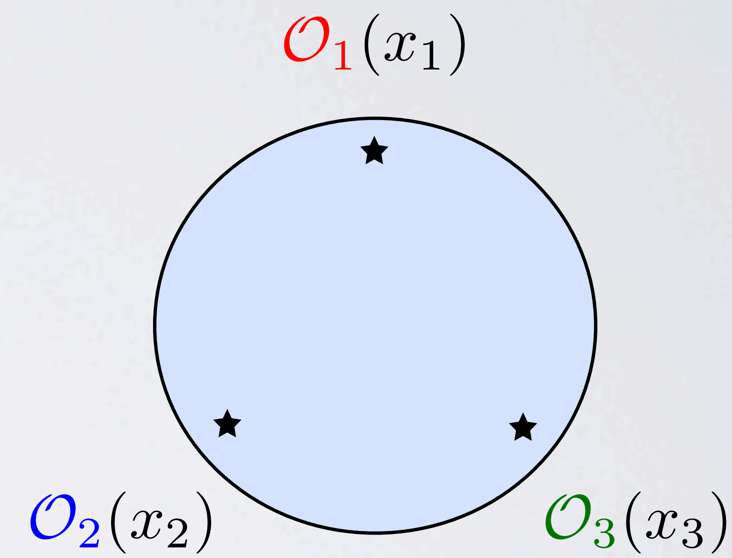
Can we also understand
handles = string loops?

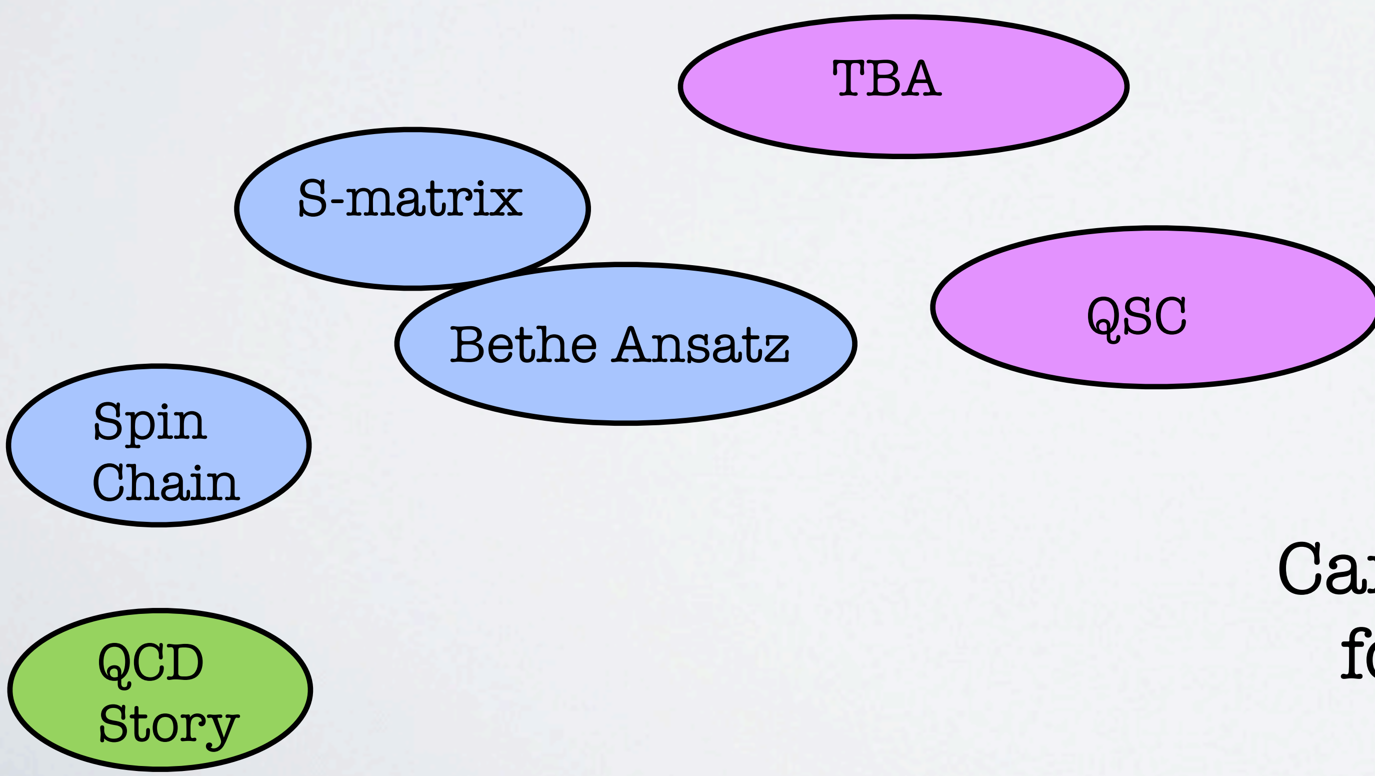


Plan / Goal / Question

Can we find structure constants of single trace operators at finite coupling in planar N=4 SYM theory?

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}}\, x_{23}^{\Delta_{23}}\, x_{13}^{\Delta_{13}}}$$



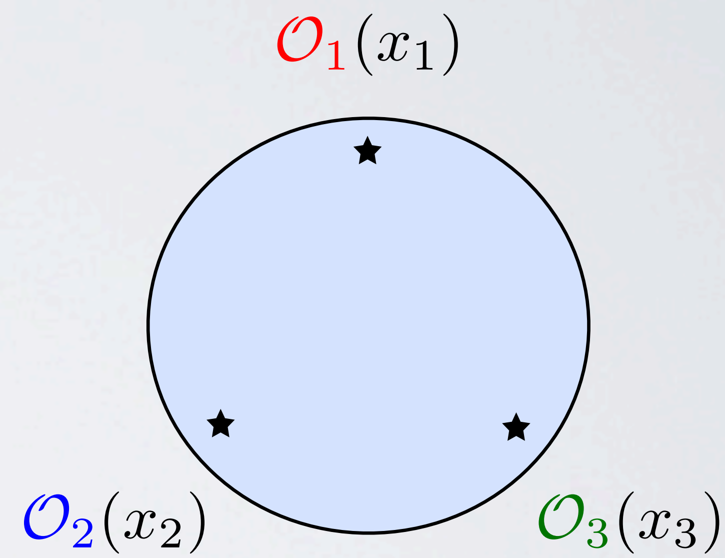


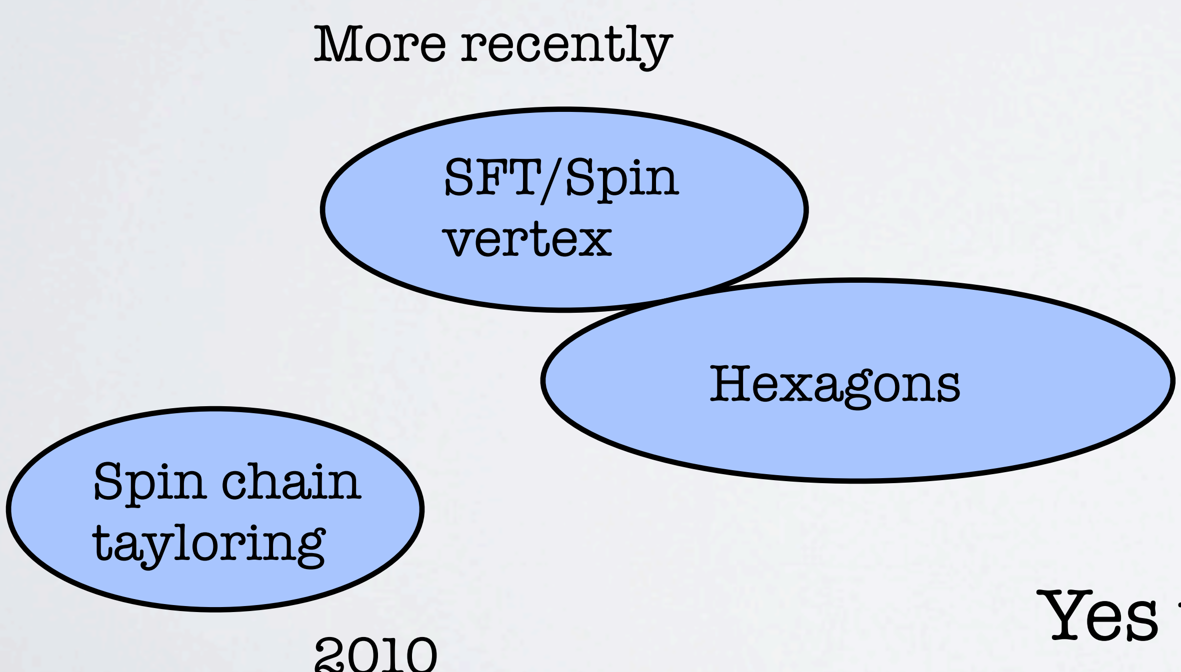
Can we follow a similar path for structure constants?

Plan / Goal / Question

Can we find structure constants of single trace operators at finite coupling in planar N=4 SYM theory?

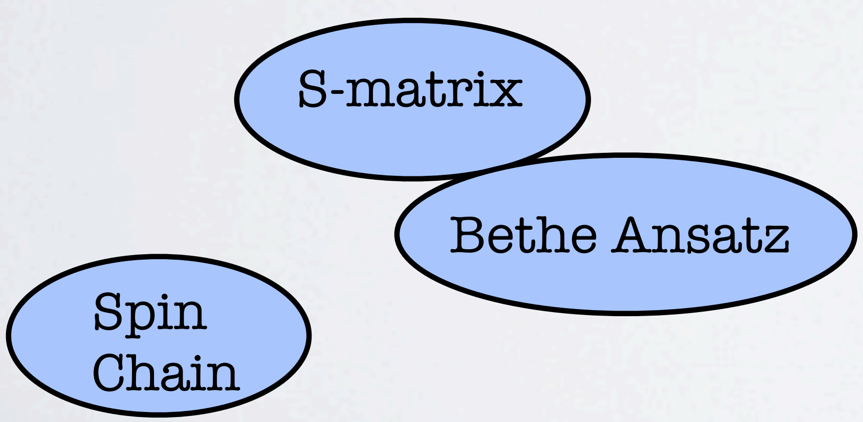
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}}\,x_{23}^{\Delta_{23}}\,x_{13}^{\Delta_{13}}}$$





Yes we can!... but it will take time...
In this talk I will show you how one can start off
using the hexagon bootstrap program

2-pt functions

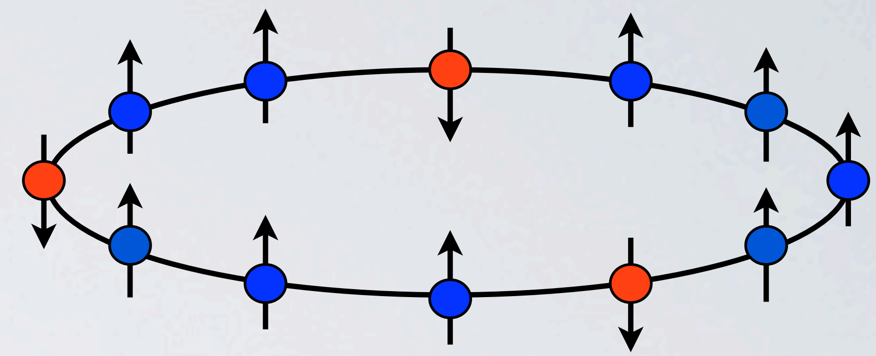


Focus on long spin chain states

Bethe States

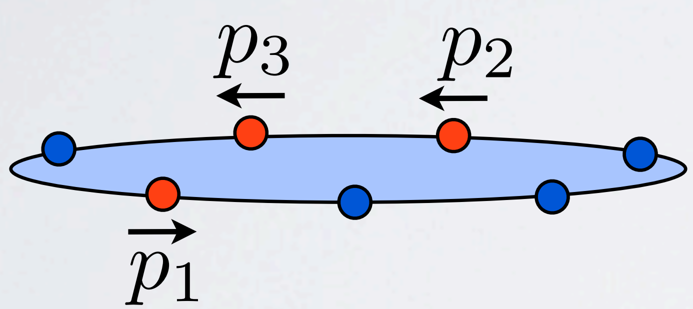
0) Pick BMN (= ferromagnetic) vacuum with very big length L

0) Add magnons

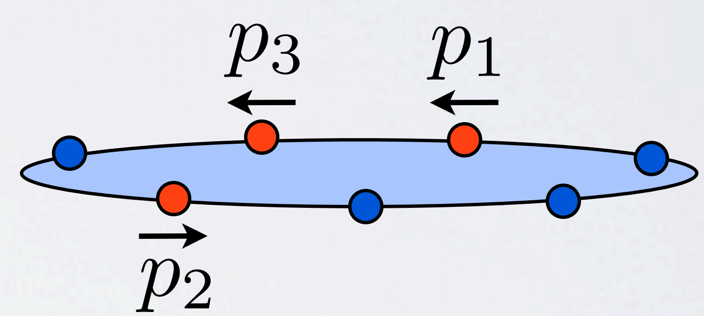


$$\mathcal{O} \sim \mathrm{tr} \ldots ZYZ \ldots ZYZ \ldots$$

1) Write Bethe wave function



$$+S(p_1, p_2)$$



$$+\ldots$$

2) Imposing periodicity conditions gives the Bethe ansatz equations : (i.e. quantization conditions for the magnon momenta)

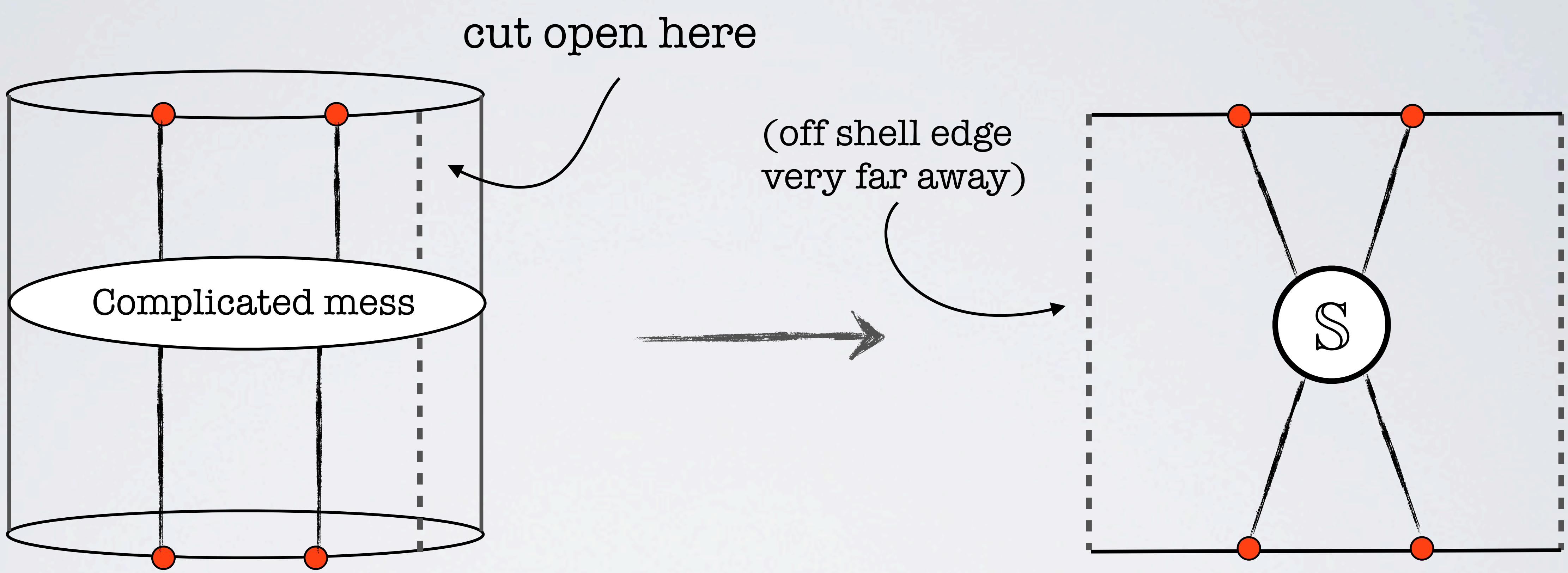
$$e^{ip_i L} \prod_{j \neq i} S(p_i, p_j) = 1$$

3) Get the energies :

$$E = \sum_i E(p_i)$$

Asymptotic solution I

It's a cutting procedure of sort :



Sort of dilute gas approximation :
Zoo of interactions reduces to 2-by-2 elastic scattering events

Geodesics to
asymptotic solution :
Magnon S-matrix

Thanks to integrability :
This description is correct up to exponentially small in system length (so called wrapping) corrections

[Ambjorn,Janik,Kristjansen'05][Bajnok,Janik'08]

$$e^{-L\times E} \sim O(g^{2L})$$

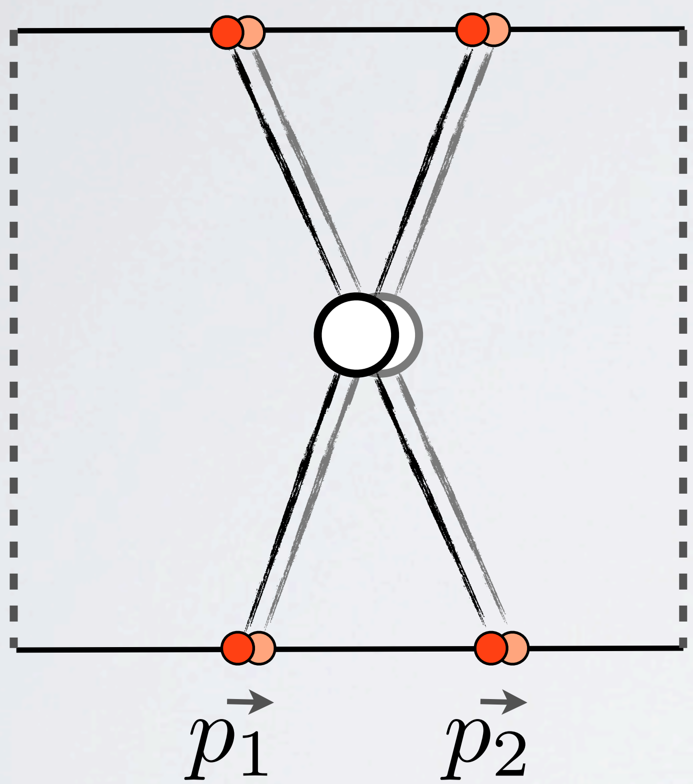
Power of symmetry

Way to go? Let the symmetries do the job
Residual symmetry group of BMN (ferro) vacuum :

[Beisert'05]

$$\underbrace{PSU(2|2)}_{\text{Left}} \times \underbrace{PSU(2|2)}_{\text{Right}} \ltimes \mathbb{R}^3$$

Central extensions : contain **energy** (and coupling constant)



Each magnon transforms in bi-fundamental irrep

$$\underbrace{\mathbf{2}|\mathbf{2}}_{\text{Left}} \otimes \underbrace{\mathbf{2}|\mathbf{2}}_{\text{Right}}$$

(Dimension = 16 = 8 bosons + 8 fermions)

Dispersion relation

$$E = \sqrt{1 + 16\, g^2 \sin^2\left(\frac{p}{2}\right)}$$

Power of symmetry

Way to go? Let the symmetries do the job
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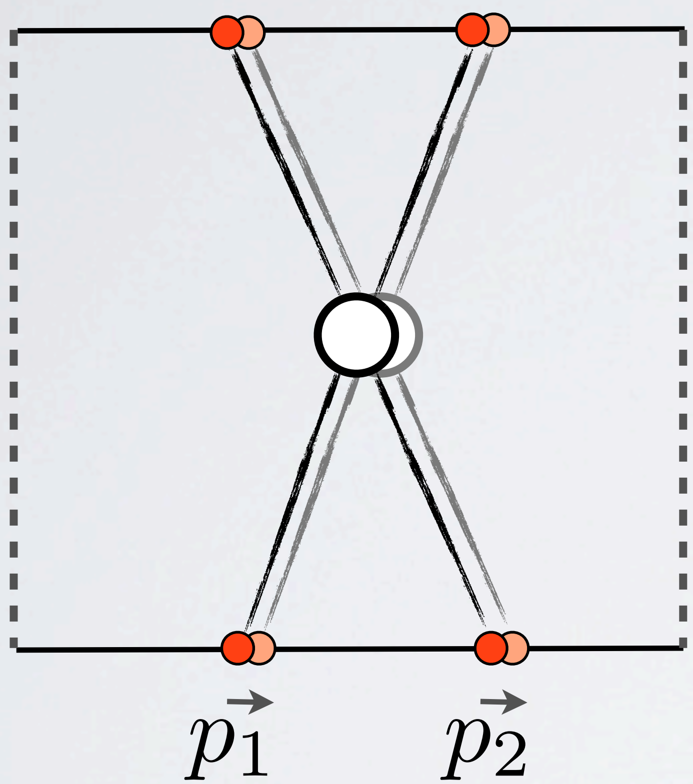
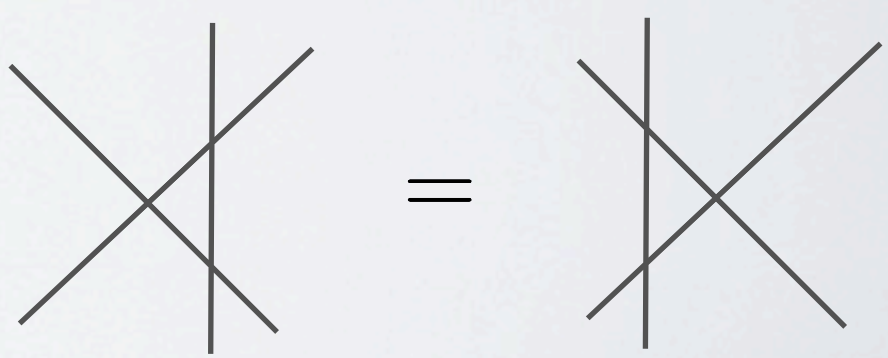
[Beisert'05]



Symmetry fixes S-matrix
(up to overall scalar factor)

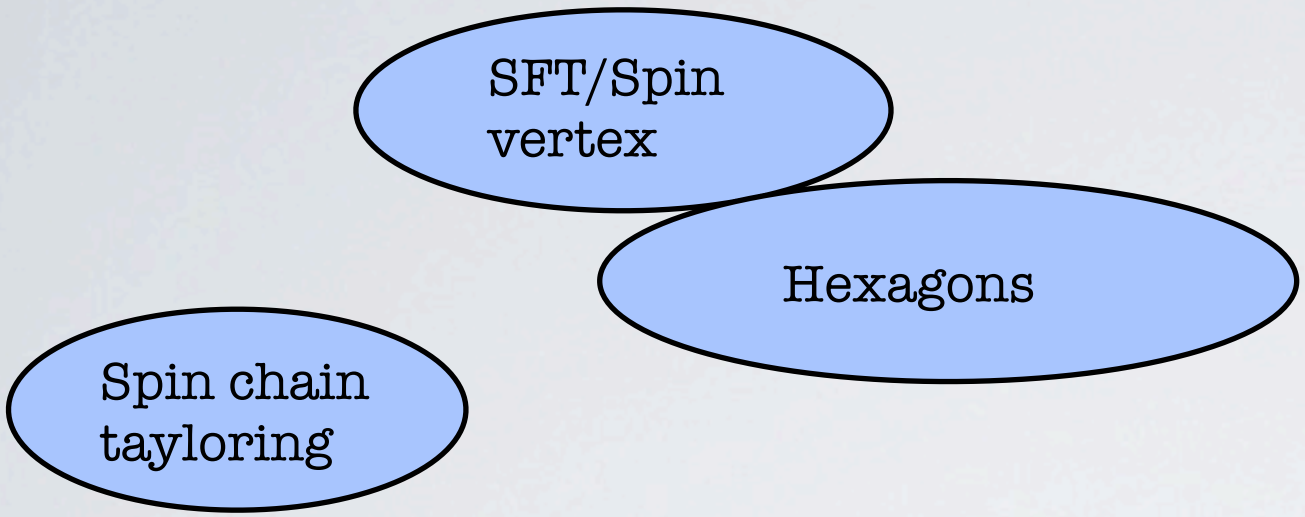
$$\mathbb{S}_{12} \sim S^0_{12}\, \mathcal{S}_{12} \times \dot{\mathcal{S}}_{12}$$



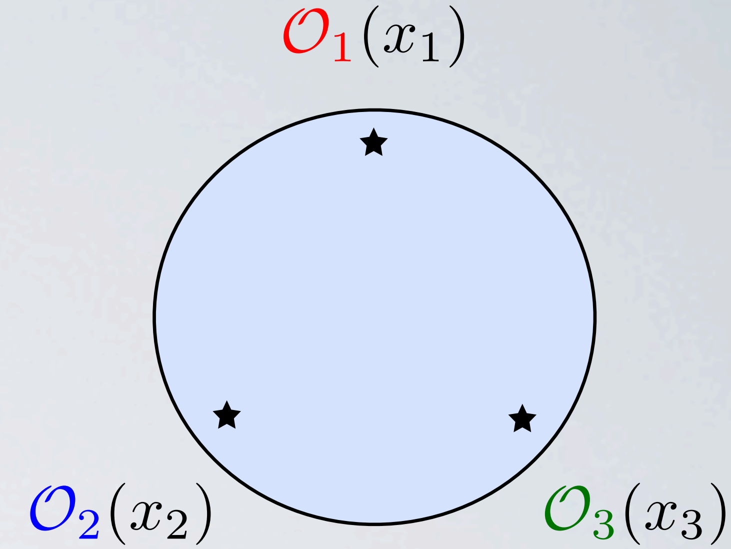
✓ Fulfills **Yang-Baxter** equation

✓ Scalar factor constrained by crossing symmetry [Janik'05]

More recently



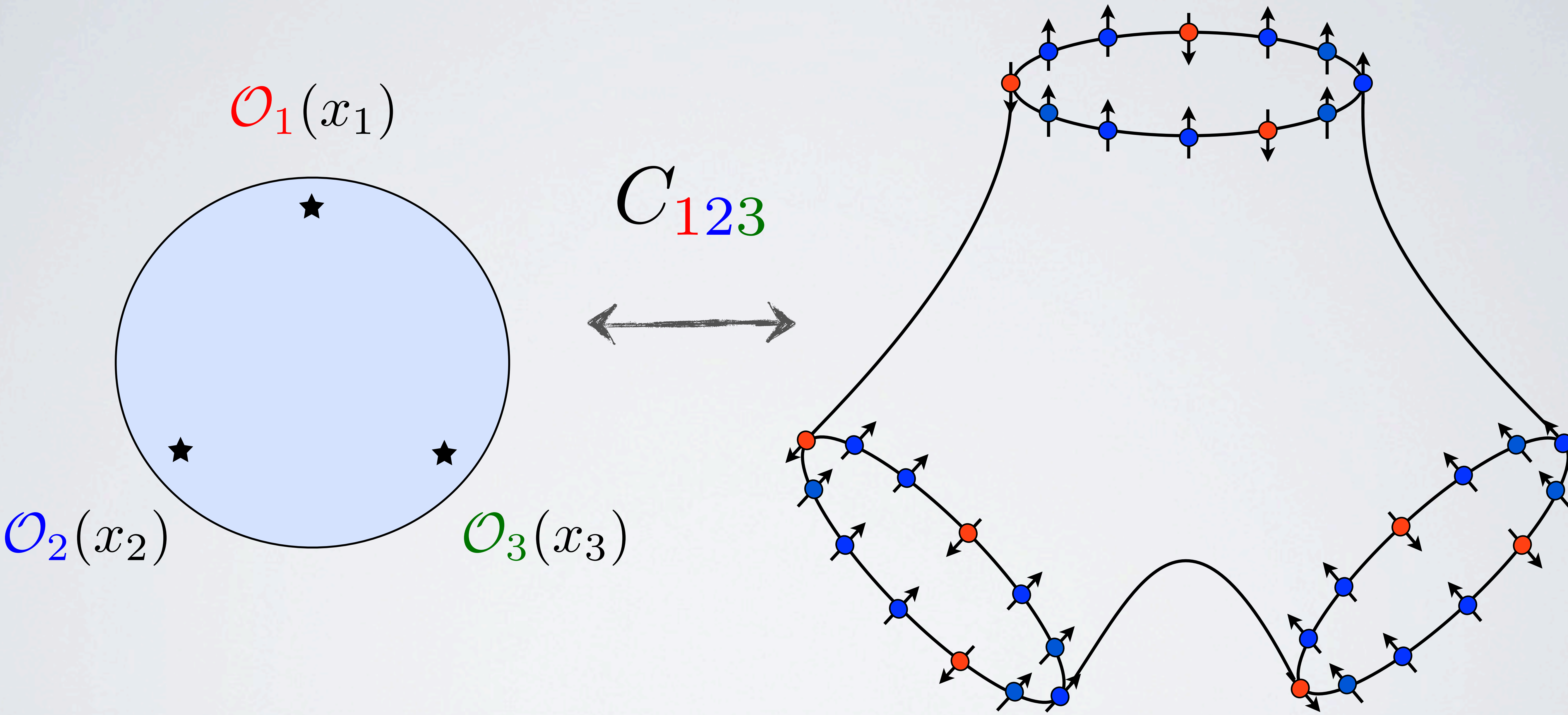
2010



From 3-pt functions to hexagons

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_{12}}\,x_{23}^{\Delta_{23}}\,x_{13}^{\Delta_{13}}}$$

Gauge / String definition



3-punctured sphere

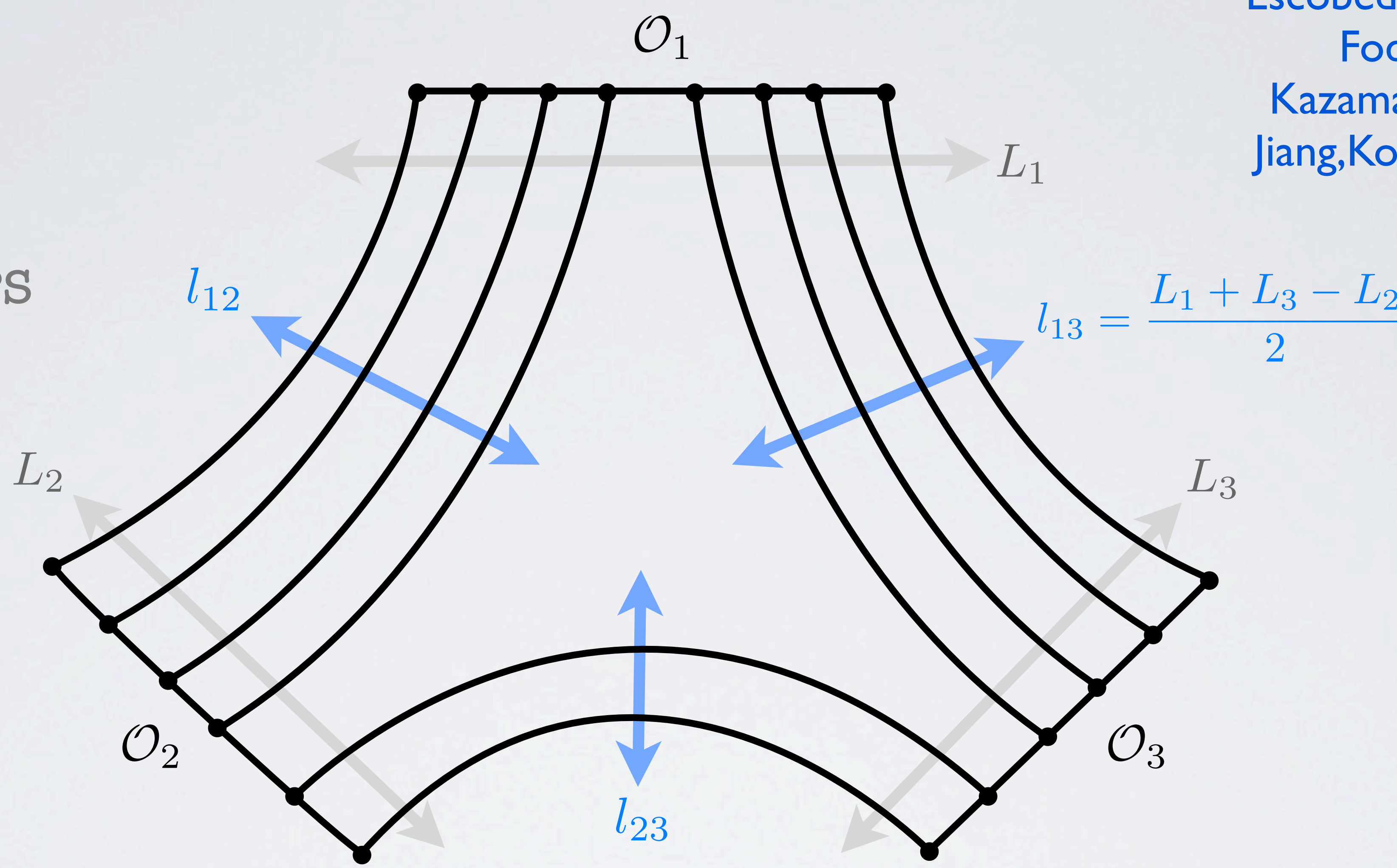
pair of pants
ending on 3 spin chains
at the boundary

Spin chain tayloring

[Many people, see e.g. Escobedo,Gromov,Sever,Vieira, Foda,Fleury,Caetano, Kazama,Komatsu,Nishimura, Jiang,Kostov,Petrovskii,Serban, etc.]

Topology :

- 3 operators
- 3 bridges



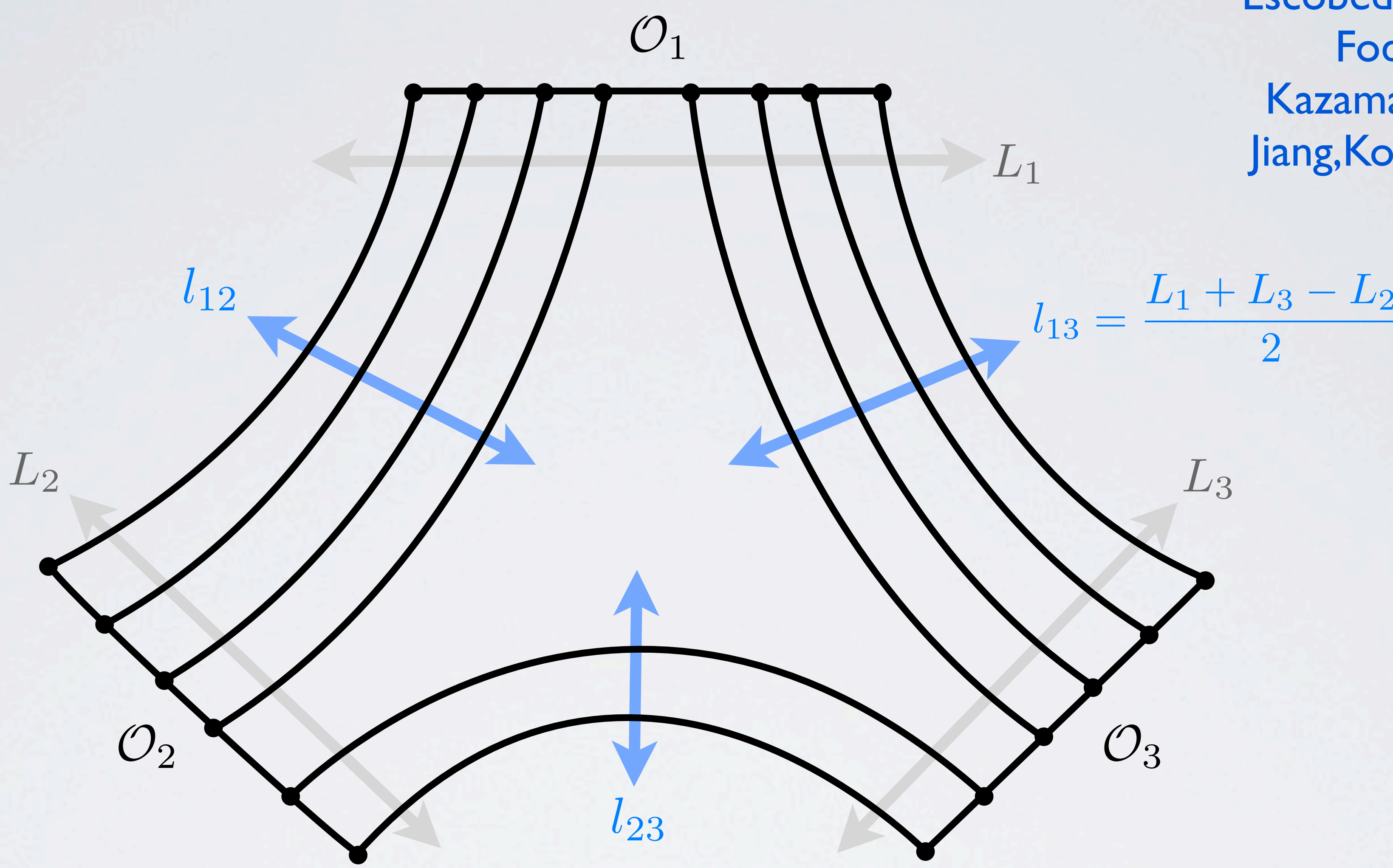
Example : 3 BPS states (= 3 spin chain vacua)

$$C_{123}^{\circ\circ\circ} = \frac{\sqrt{L_1 L_2 L_3}}{N}$$

- Contract scalar fields as indicated above
- Count number of inequivalent Wick contractions
- Normalize by norms

Spin chain tayloring

[Many people, see e.g. Escobedo, Gromov, Sever, Vieira, Foda, Fleury, Caetano, Kazama, Komatsu, Nishimura, Jiang, Kostov, Petrovskii, Serban, etc.]



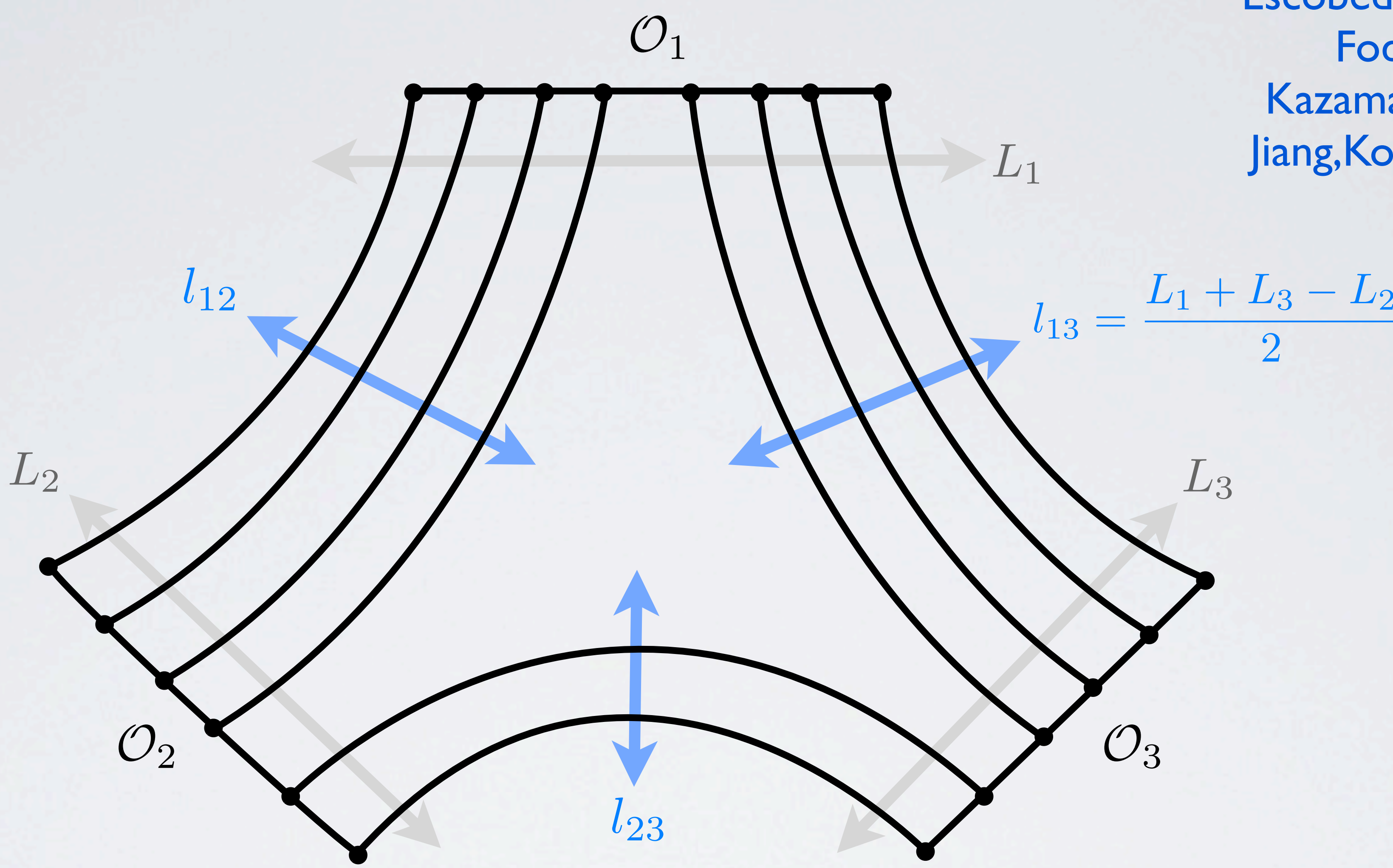
More complicated : 3 non-BPS states

- Same as before
- But split each spin chain $\mathcal{H}_i = \mathcal{H}_i^{(a)} \otimes \mathcal{H}_i^{(b)}$

$$C_{123}^{\text{tree}} = \frac{\sqrt{L_1 L_2 L_3}\sum_{\text{p}} \langle \Psi_3^{(a)} | \Psi_1^{(b)} \rangle \langle \Psi_1^{(a)} | \Psi_2^{(b)} \rangle \langle \Psi_2^{(a)} | \Psi_3^{(b)} \rangle}{N\sqrt{\langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle \langle \Psi_3 | \Psi_3 \rangle}}$$

Spin chain tayloring

[Many people, see e.g. Escobedo,Gromov,Sever,Vieira, Foda,Fleury,Caetano, Kazama,Komatsu,Nishimura, Jiang,Kostov,Petrovskii,Serban, etc.]

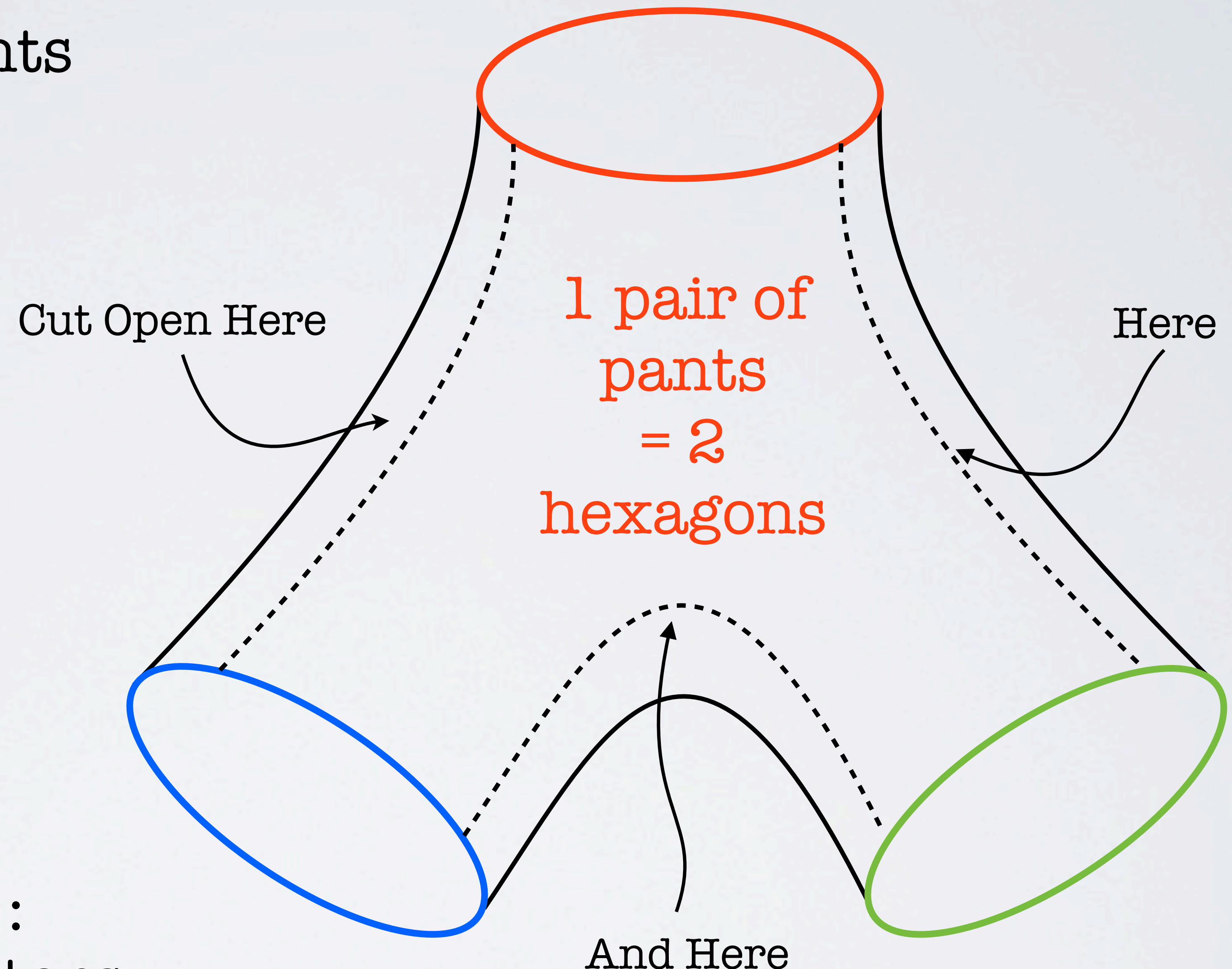


Recipe : Cut spin chain states and compute their overlap following the Wick contractions
Use integrability to evaluate partial wave function overlaps

How to go to higher loops? Hard... spin chain wave functions are unknown, as well as corrections to splitting vertex

Cutting /asymptotic procedure

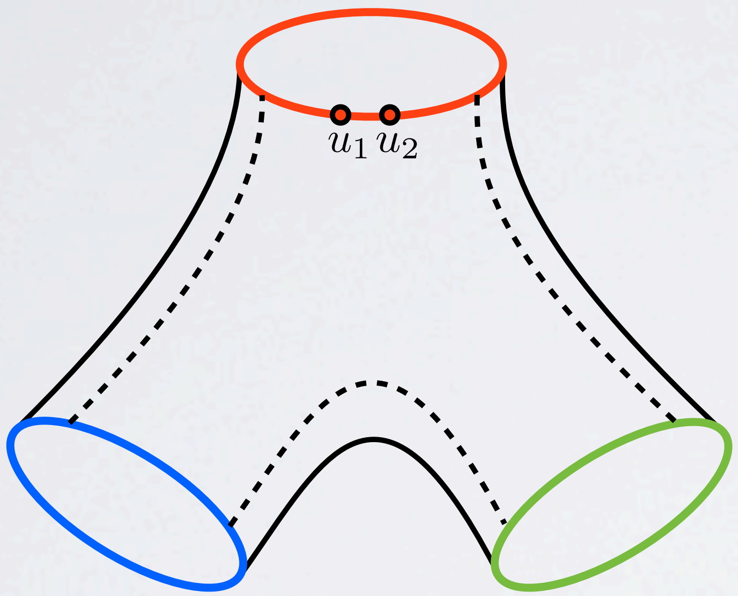
- Start with pair of pants
- Cut open 3 times
- Get 2 hexagons



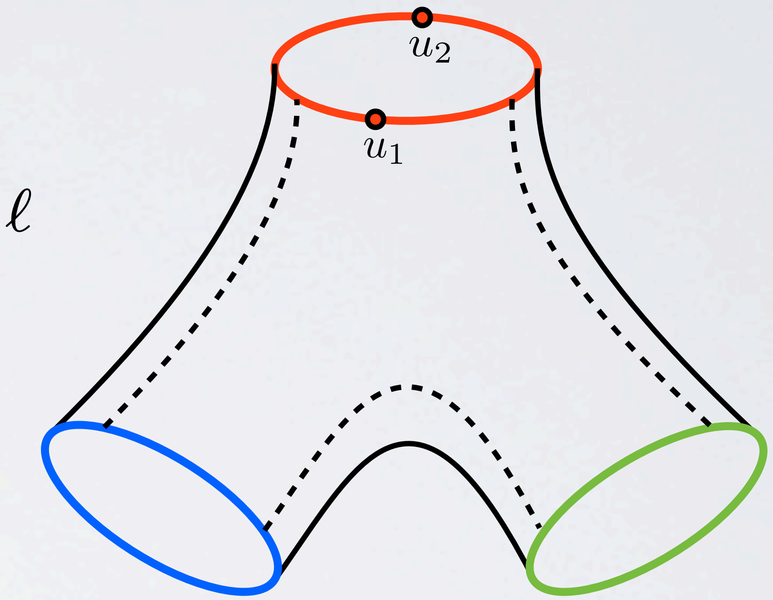
Bottom line :
For large operators
the 3pt function factorizes into 2 disjoint hexagons

Cutting /asymptotic procedure

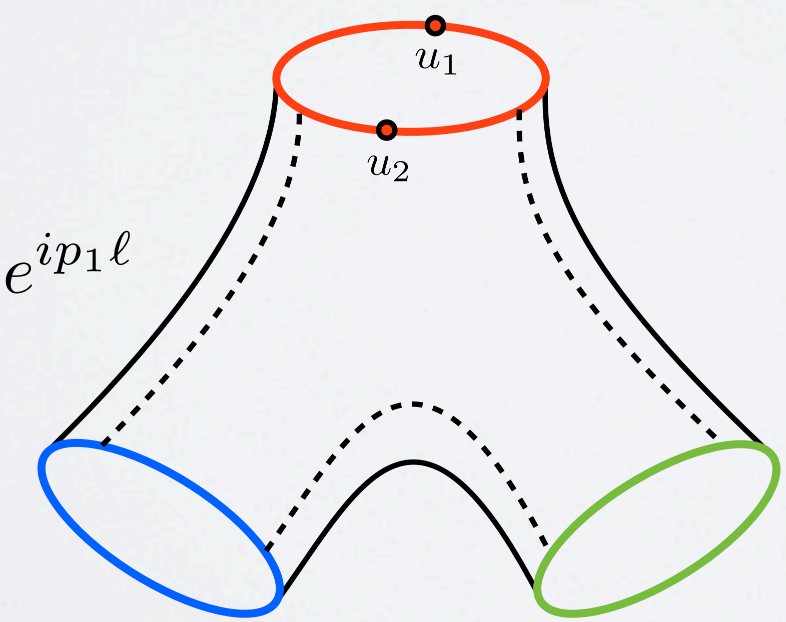
Hexagon factorization
with magnons



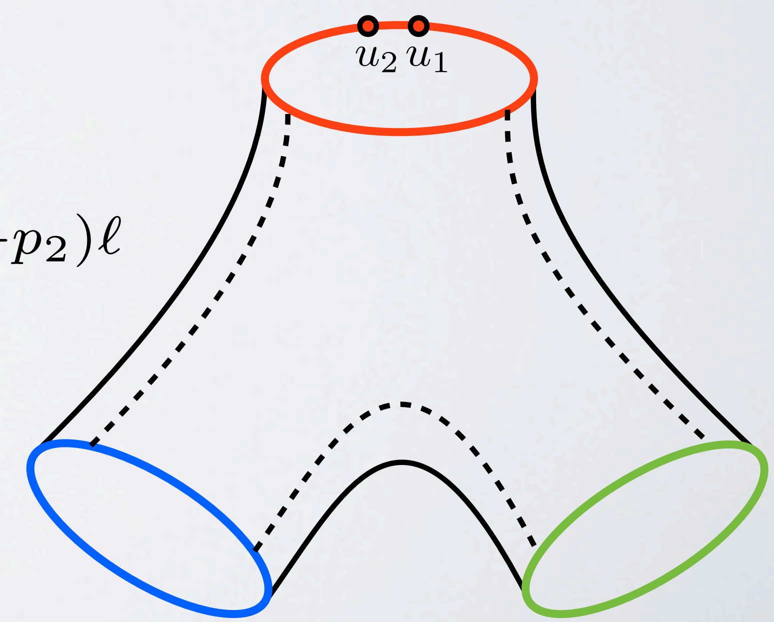
$$+ \quad e^{ip_2\ell}$$



$$+ \quad S(u_1, u_2)e^{ip_1\ell}$$



$$+ \quad e^{i(p_1+p_2)\ell}$$



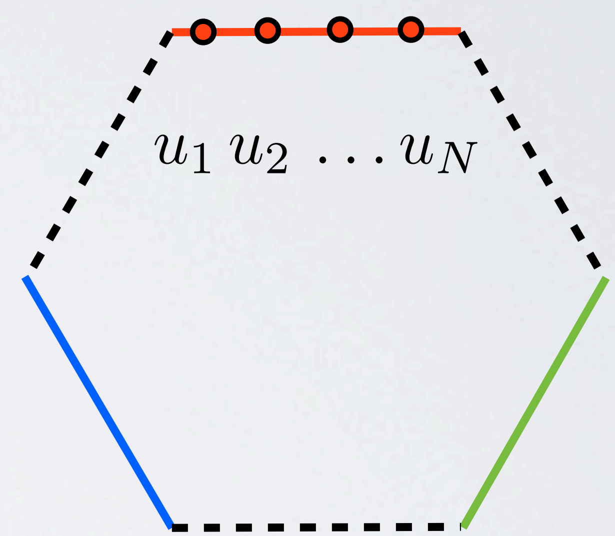
Hexagon factorization

[BB,Komatsu,Vieira'15]

- 3pt function = sum of products of 2 hexagons

- Leftover information about spin chain state is in the sum over bipartite partition of Bethe roots

- Elementary block = hexagon form factor



Amplitude for creating magnons on the edges of an hexagon

$$\mathfrak{h}^{A_1\dot{A}_1,\ldots,A_N\dot{A}_N}(u_1,\ldots,u_N) = \langle\mathfrak{h}|\big(|\chi_1^{A_1\dot{A}_1}\cdots\chi_N^{A_N\dot{A}_N}\rangle_1\otimes|0\rangle_2\otimes|0\rangle_3\big)$$

Apply integrable bootstrap to determine it at finite coupling

Use super-symmetry

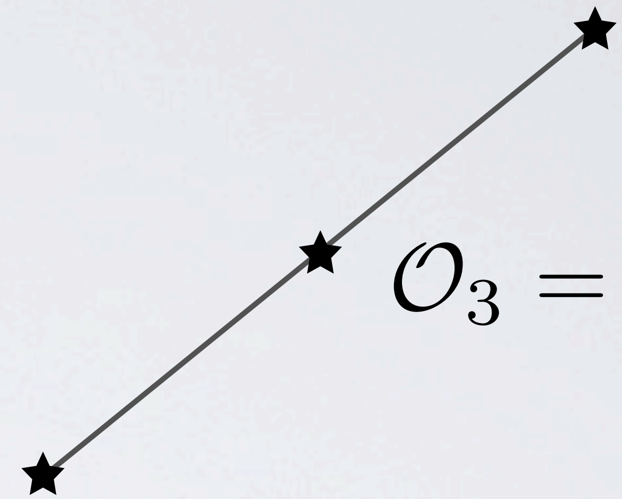
3pt function = (BMN)^3

2 BMN vacua + 1 twisted BMN vacuum

(for BPS condition)

$$\tilde{Z} = Z + \bar{Z} + Y - \bar{Y}$$

(for overlap ops 1 and 2)



$$\mathcal{O}_2 = \mathrm{tr}\, \bar{Z}(\infty)^{L_2}$$

$$\mathcal{O}_3 = \mathrm{tr}\, \tilde{Z}(1)^{L_3}$$

$$\mathcal{O}_1 = \mathrm{tr}\, Z(0)^{L_1}$$

part of family of twisted correlators

see [Drukker,Plefka'09]

Residual symmetry : $O(3) \times O(3)$

fix a line in spacetime

fix three (real) scalars out of six

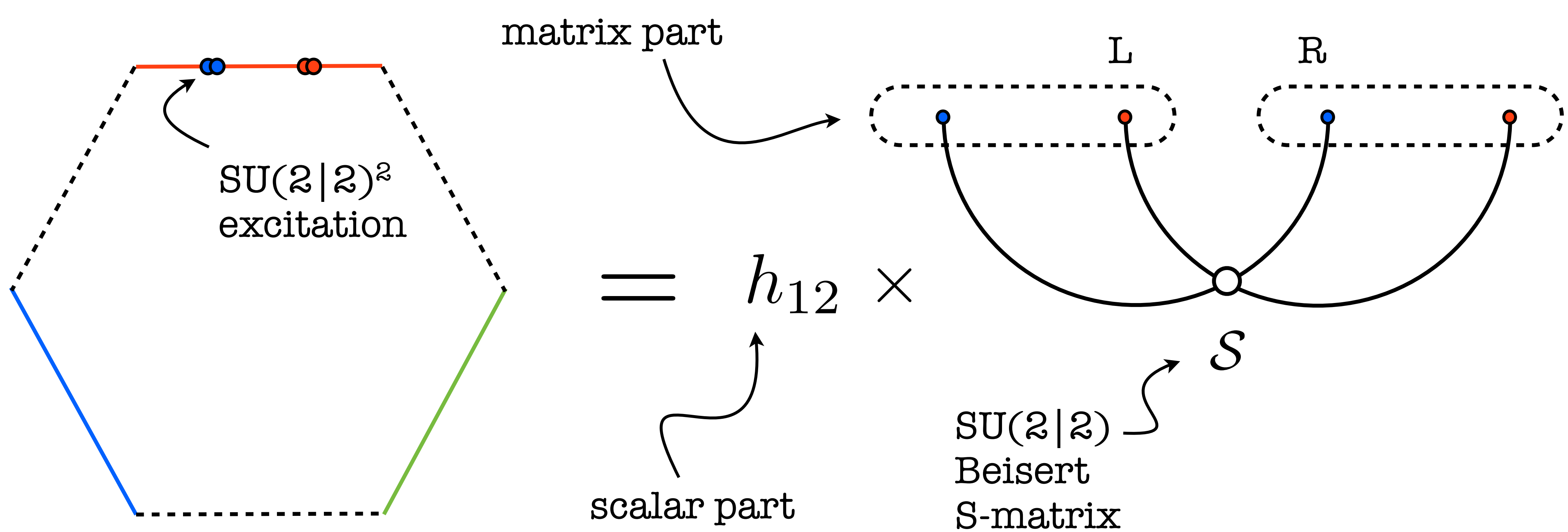
+ 8 Supercharges : $\mathcal{Q}^a{}_\alpha + \epsilon^{ab}\epsilon_{\alpha\beta}\dot{\mathcal{S}}^\beta{}_b$ (and dotted version)

Total symmetry of hexagon: $PSU(2|2)$

= diagonal subgroup of $PSU(2|2)_{\text{Left}} \times PSU(2|2)_{\text{Right}}$

Power of symmetry

Two-magnon hexagon form factor fixed

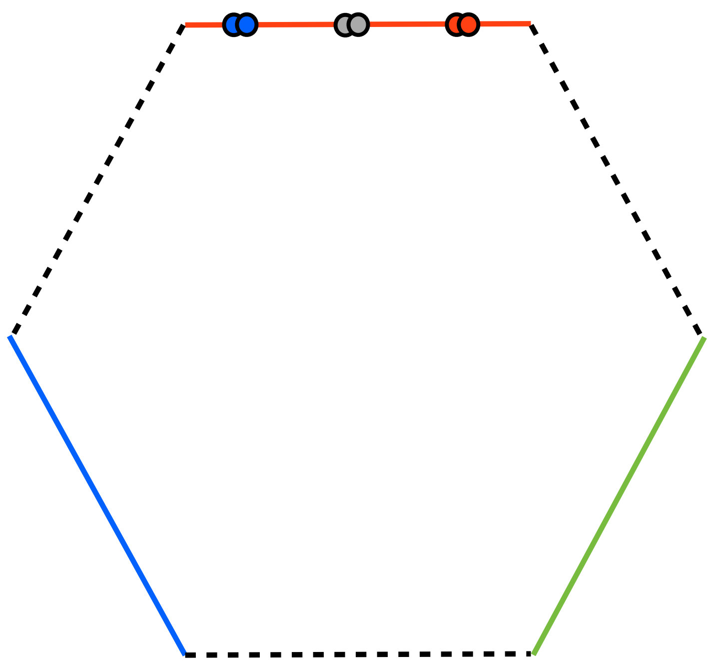


up to a scalar factor

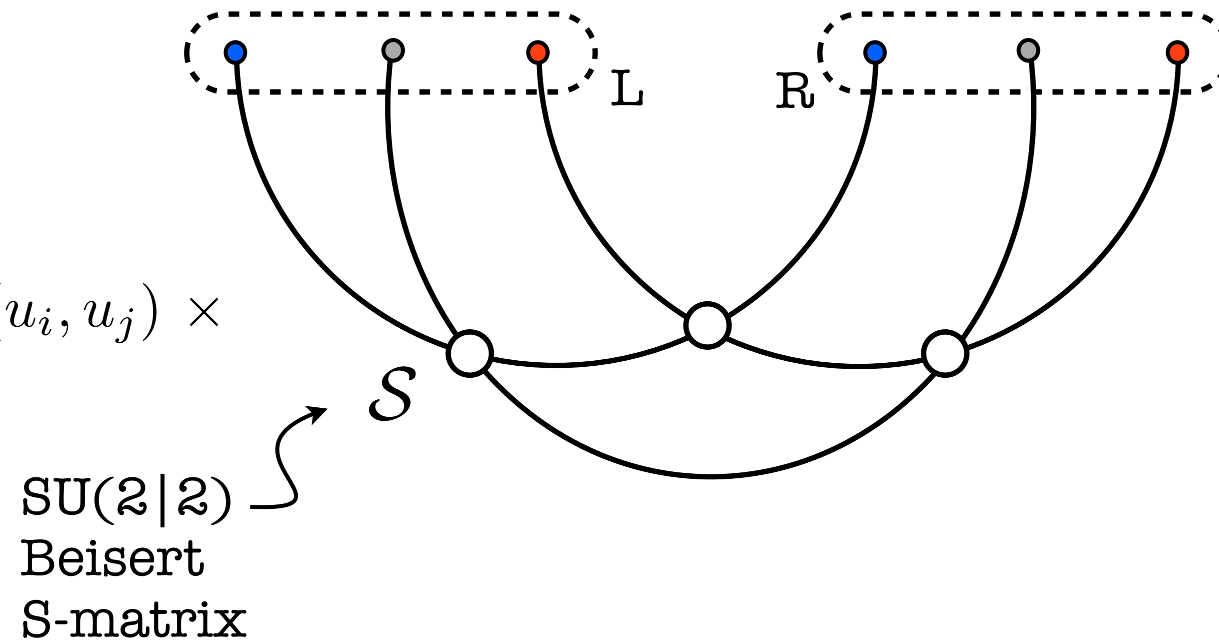
$$\mathfrak{h}^{A_1\dot{A}_1,A_2\dot{A}_2} = (-1)^{\dot{f}_1 f_2} \times h_{12} \times \langle \chi_2^{\dot{A}_2} \chi_1^{\dot{A}_1} | \mathcal{S}_{12} | \chi_1^{A_1} \chi_2^{A_2} \rangle$$

N-magnon form factor

Conjecture for N-magnon form factor :



$$= \prod_{i<j}^{N} h(u_i, u_j) \times$$



$$\mathfrak{h}^{A_1 \dot{A}_1 \cdots A_N \dot{A}_N} = (-1)^{\mathfrak{f}} \prod_{i<j}^{N} h_{ij} \langle \chi_N^{\dot{A}_N} \ldots \chi_1^{\dot{A}_1} | \, \mathcal{S} \, | \chi_1^{A_1} \ldots \chi_N^{A_N} \rangle$$

Bootstrap for scalar factor

Hexagon as a branch point twist field (with conical excess)

[Cardy,Castro-Alvaredo,Doyon'07]

Two main axioms :

I. Watson equation : one can permute magnons using S-matrix

$$h_{12}/h_{21} = S^0_{12} = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{1-1/x_1^- x_2^+}{1-1/x_1^+ x_2^-}\frac{1}{\sigma_{12}^2}$$

II. Decoupling/crossing equation : a pair of a magnon and anti-magnon with zero net charges and energy must decouple (also known as kinematical pole condition)

$$h(u_1^{2\gamma}, u_2)h(u_1, u_2) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\frac{1-1/x_1^+ x_2^-}{1-1/x_1^+ x_2^+}$$

One main solution (not unique) :

(same as Janik's crossing equation)

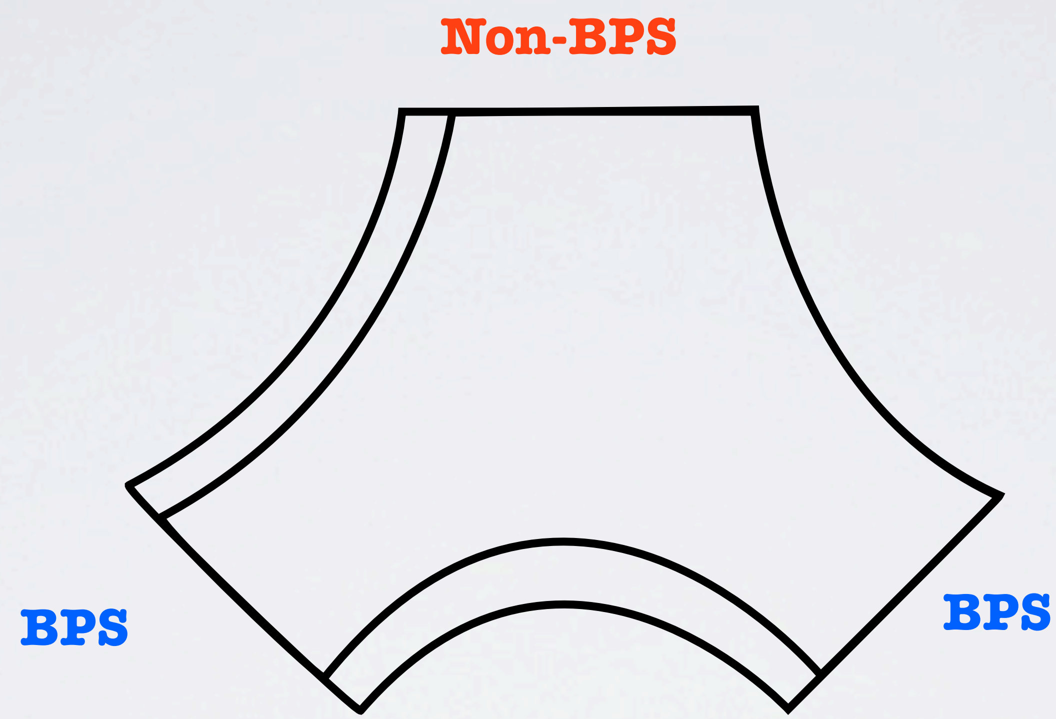
[BB,Komatsu,Vieira'15]

$$h_{12} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\frac{1-1/x_1^- x_2^+}{1-1/x_1^+ x_2^+}\frac{1}{\sigma_{12}}$$

Gluing hexagons into 3-pt functions

Asymptotic formula

Consider 2 BPS operators and 1 non-BPS operator, e.g.



i.e.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = C_{123}^{\bullet\circ\circ} \times \frac{\text{tensor}}{x_{12}^{\Delta_{12}} x_{23}^{\Delta_{23}} x_{31}^{\Delta_{31}}}$$

e.g. $\mathcal{O}_1 = \mathrm{tr} D^S Z^{L_1}$ with the rest BPS

Asymptotic formula

Hexagon prediction :

first factor has to do with normalization of spin chain state (i.e. conversion factor from infinite to finite volume normalization) see [Pozsgay, Takacs'08]

$$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2 = \frac{\prod_{k=1}^{S}\mu(u_k)}{\det\,\partial_{u_i}\phi_j\prod_{i<j}S(u_i,u_j)}\times\mathcal{A}^2$$

Hexagon part

sum over partitions of Bethe Roots

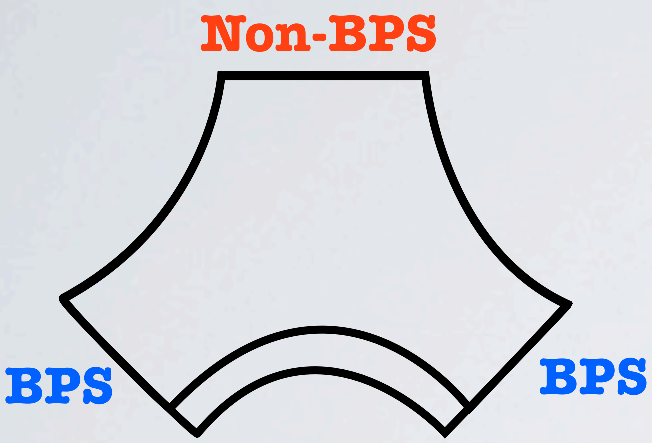
$$\mathcal{A} = \prod_{i<j} h(u_i,u_j)\sum_{\alpha\cup\bar{\alpha}=\mathbf{u}}(-1)^{|\bar{\alpha}|}\prod_{j\in\bar{\alpha}}e^{ip_j\ell}\prod_{i\in\alpha,j\in\bar{\alpha}}\frac{1}{h(u_i,u_j)}$$

Valid to all loops asymptotically (large enough operators)

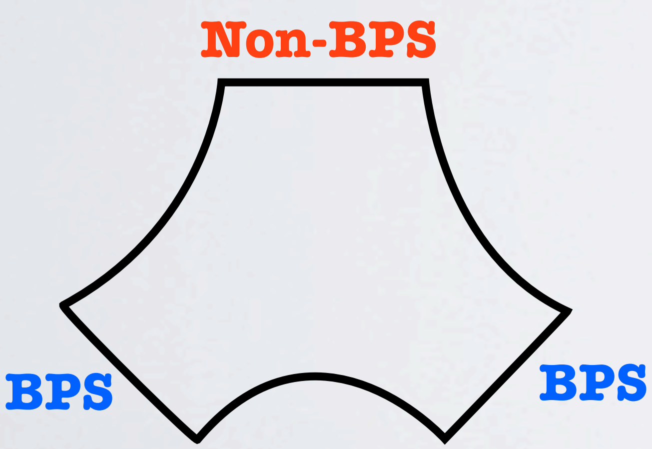
Comparison with data

from
[Eden,Heslop,Korchemsky,Sokatchev'11]
[Eden'12]
[Chicherin,Drummond,Heslop,Sokatchev'14]

$$\left(\frac{C_{123}^{\bullet\circ\circ}}{C_{123}^{\circ\circ\circ}}\right)^2$$



Spin	*"Long" Bridge i.e. length* $\ell = 2$
2	$\frac{1}{6} - 2g^2 + 28g^4 + \dots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \frac{36653}{9261}g^4 + \dots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \frac{826643623}{2156220000}g^4 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \frac{2748342985341731}{8530540523505000 0}g^4 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \frac{1564220341863916339 09}{622011694049832340800 00}g^4 + \dots$

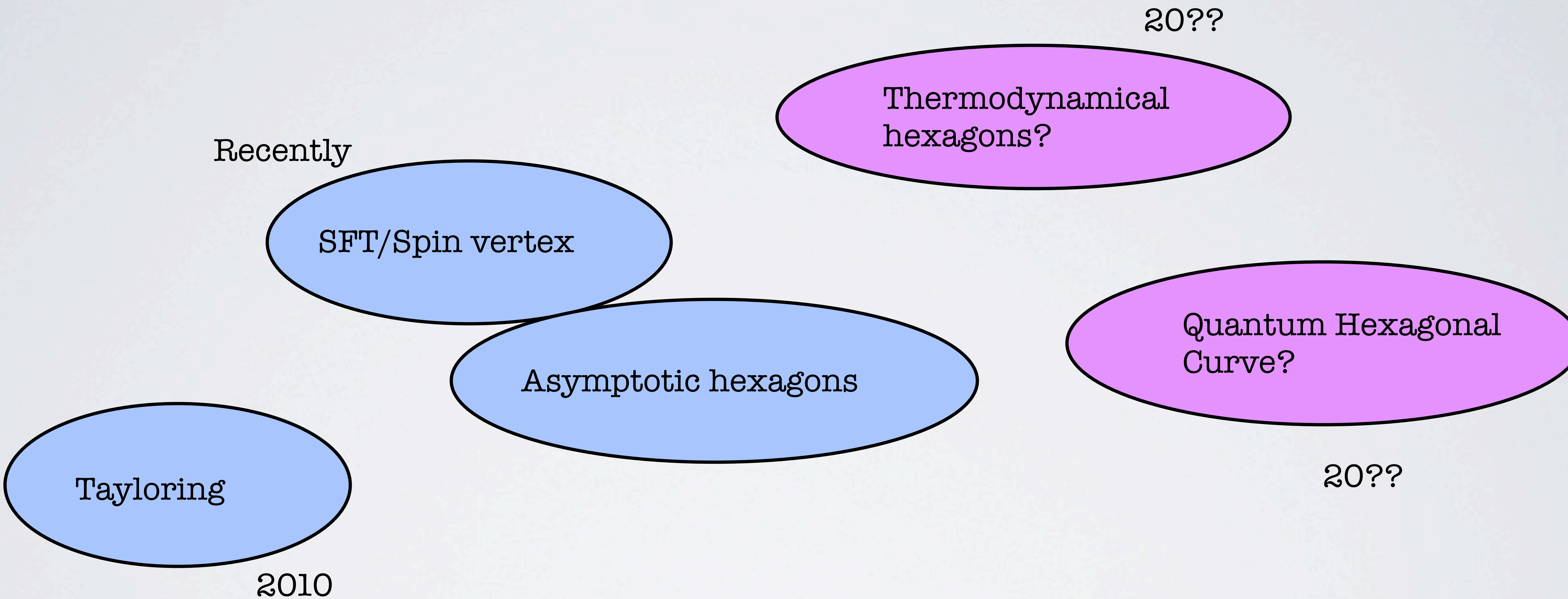


Spin	*"Short" Bridge i.e. length* $\ell = 1$
2	$\frac{1}{6} - 2g^2 + (28 + 12\zeta_3)g^4 + \dots$
4	$\frac{1}{70} - \frac{205}{882}g^2 + \left(\frac{76393}{18522} + \frac{10}{7}\zeta_3\right) g^4 + \dots$
6	$\frac{1}{924} - \frac{553}{27225}g^2 + \left(\frac{880821373}{2156220000} + \frac{7}{55}\zeta_3\right) g^4 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{9018009000}g^2 + \left(\frac{5944825782678337}{17061081047010000 0} + \frac{761}{75075}\zeta_3\right)g^4 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{27991929747600}g^2 + \left(\frac{171050793565932326659}{6220116940498323408 0000} + \frac{671}{881790}\zeta_3\right) g^4 + \dots$

2-loop mismatch

Perfect agreement between asymptotic hexagon description and data...
... up to zeta's

Full solution?

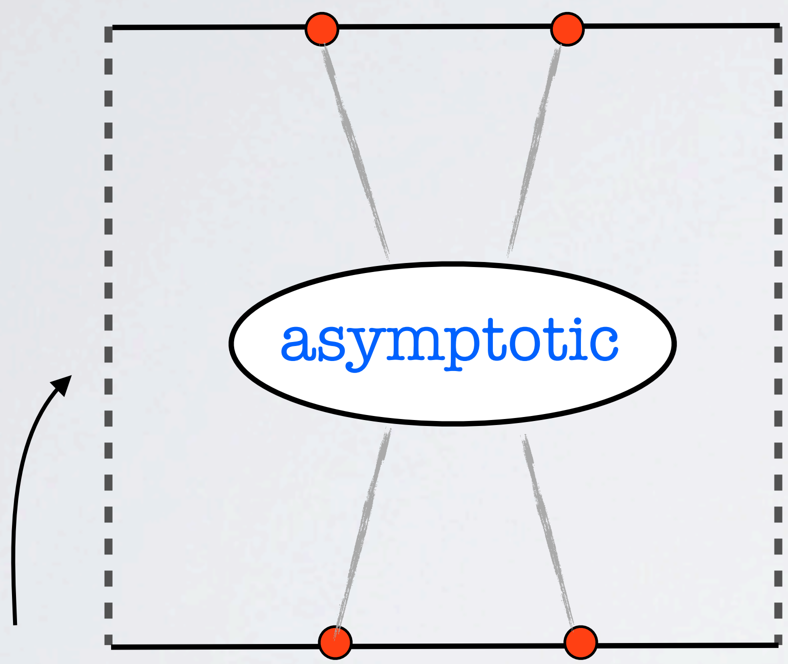


A complete solution must include finite size corrections (wrapping effect) because spin chains have finite lengths

Beyond asymptotic description

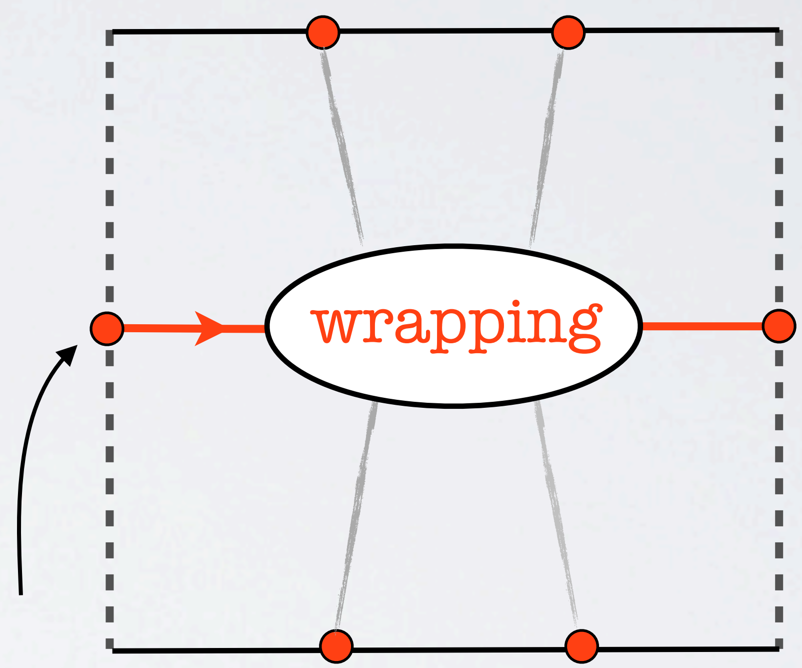
Include finite size effects = so-called wrapping effects

[Ambjorn,Janik,Kristjansen'05]
[Bajnok,Janik'08]



vacuum (in mirror = double Wick rotated theory)

+



virtual effect : exchange of 1 particle in mirror channel

+ ...

more mirror particles exchanged

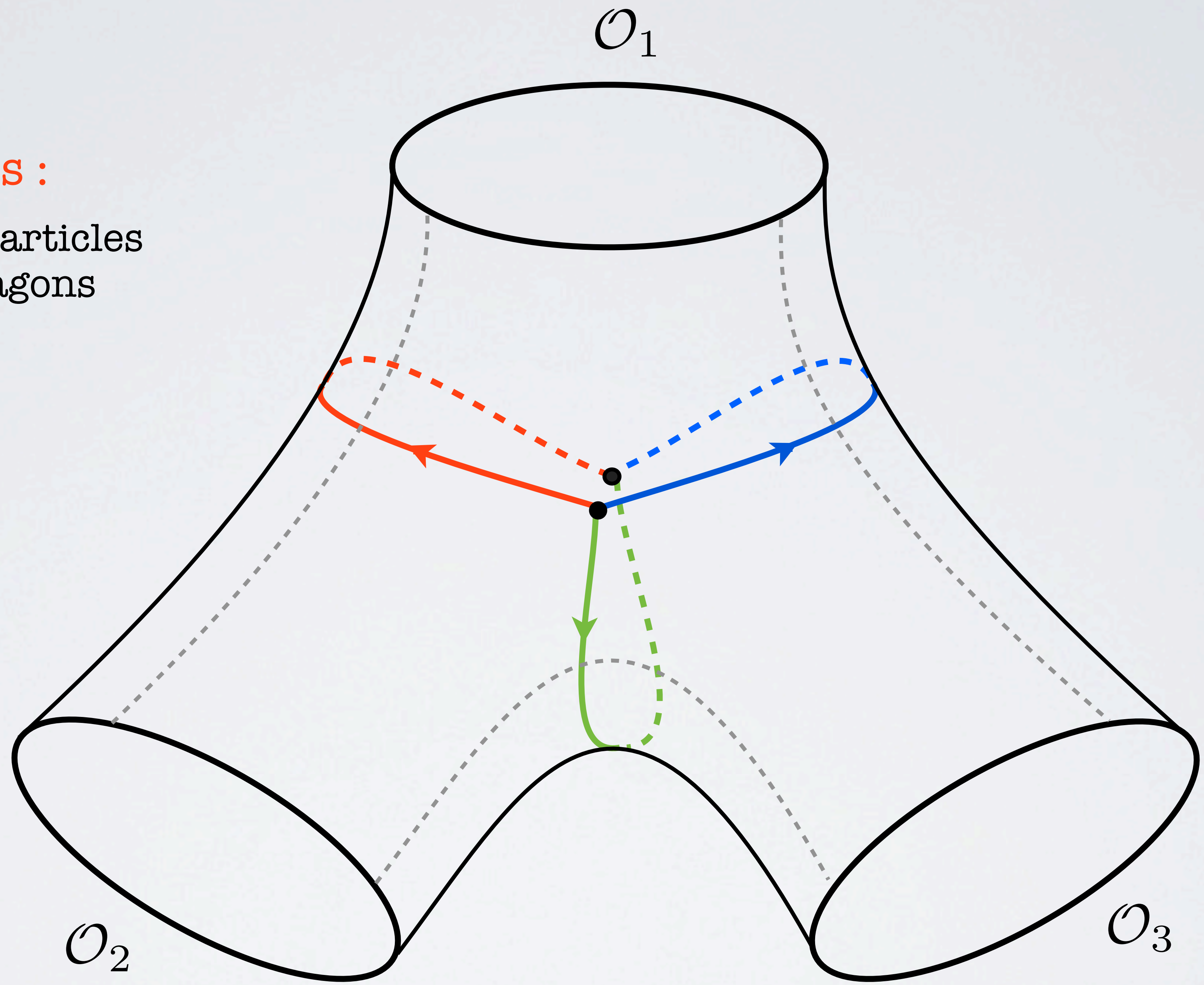
wrapping corrections : $$e^{-L\times E} \sim O(g^{2L})$$

(Resummation of all finite size corrections leads to TBA eqs and Quantum Spectral Curve)

Finite size effects for 3pt functions

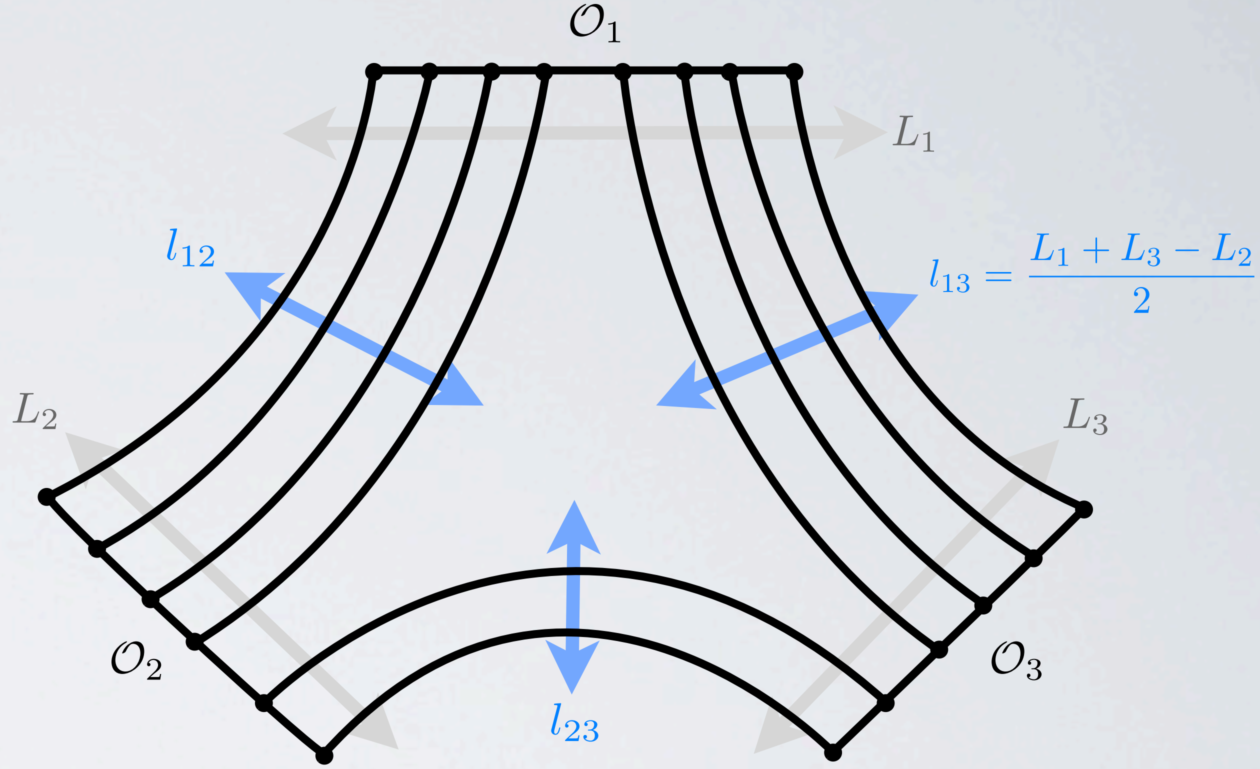
New virtual effects :

Exchange of mirror particles between the two hexagons



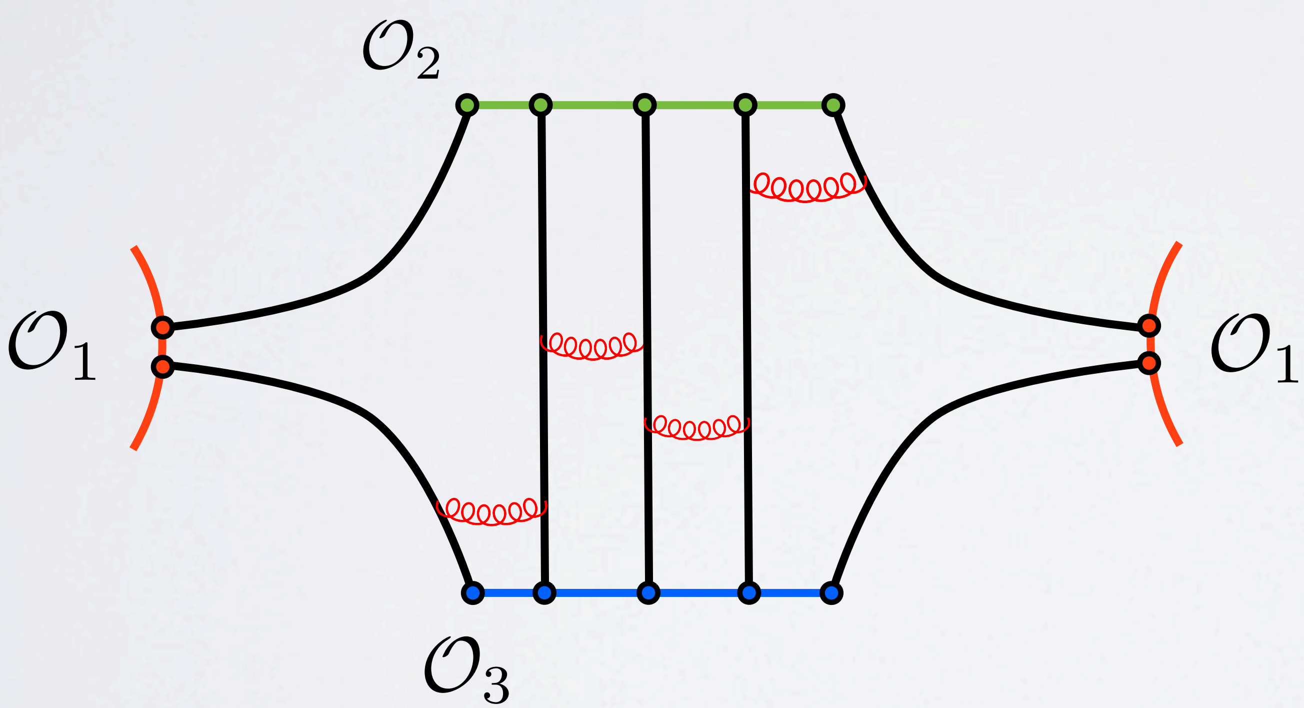
These new virtual effects come from the 3 mirror channels (= where we cut)

Finite size effects at weak coupling



New virtual effects :

Exchange of mirror particles between the two hexagons



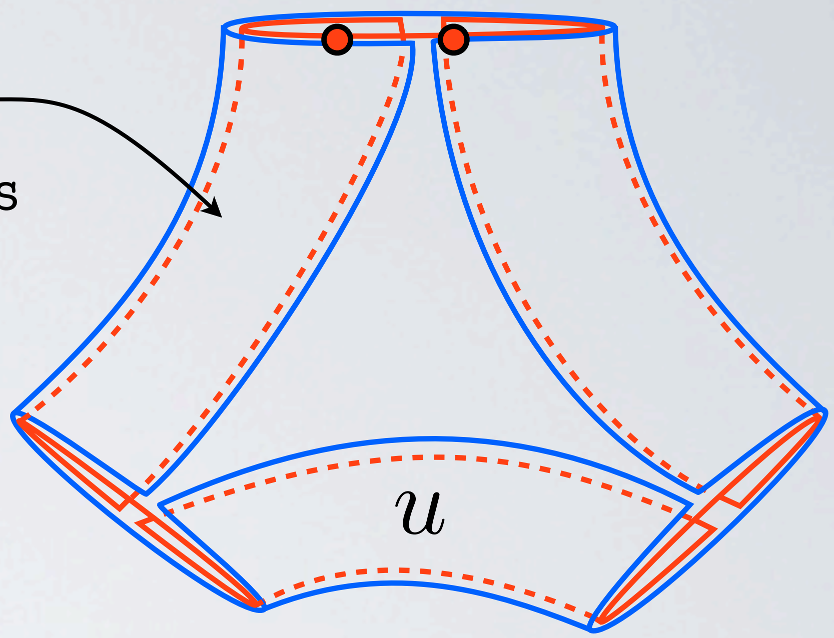
Here a gluon is passing through the bridge

Virtual effects suppressed with bridge size $O(g^{2\ell_{ij}})$

Mirror effects using hexagons

First finite size effect

Corrections coming from exchange of a single particle in the three mirror channels



$$\mathcal{A} \to \mathcal{A} + \delta\mathcal{A}_{12} + \delta\mathcal{A}_{23} + \delta\mathcal{A}_{31}$$

Asymptotic = vacuum contribution

Integral over momentum of exchanged particle

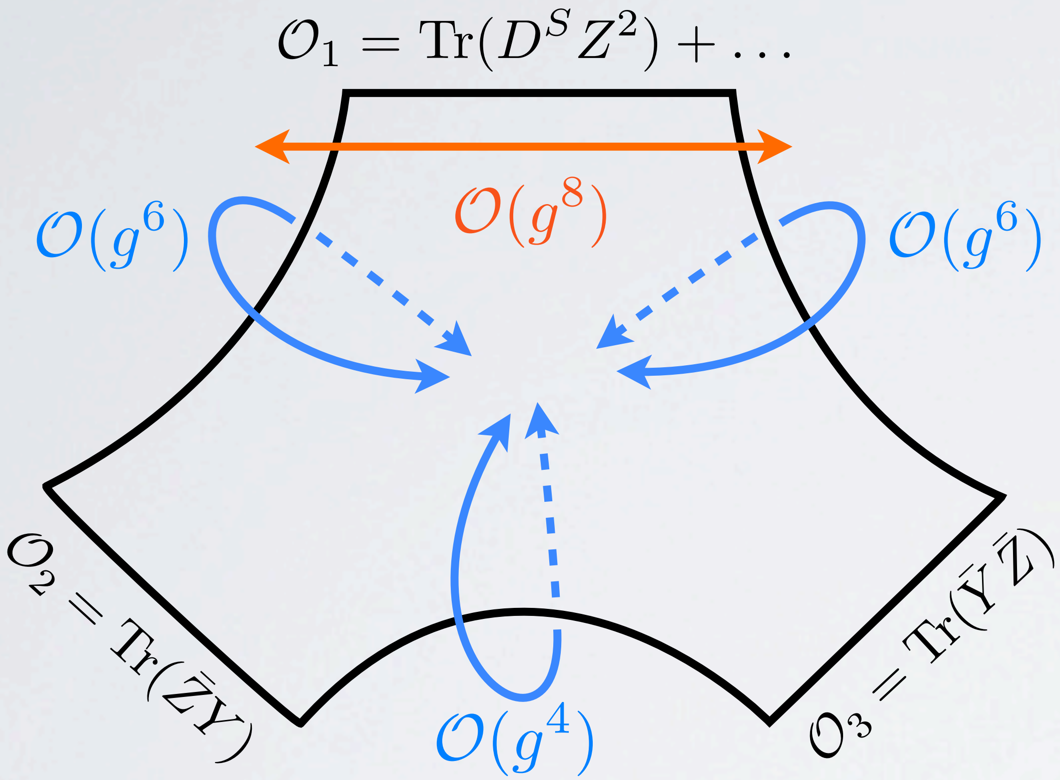
bridge length

$$\delta\mathcal{A} = \sum_{a \geqslant 1} \int \frac{du}{2\pi} \mu_a^{\gamma}(u) \times \left(\frac{1}{x^{[+a]} x^{[-a]}} \right)^{\ell} \times \mathrm{int}_a(u|\{u_i\})$$

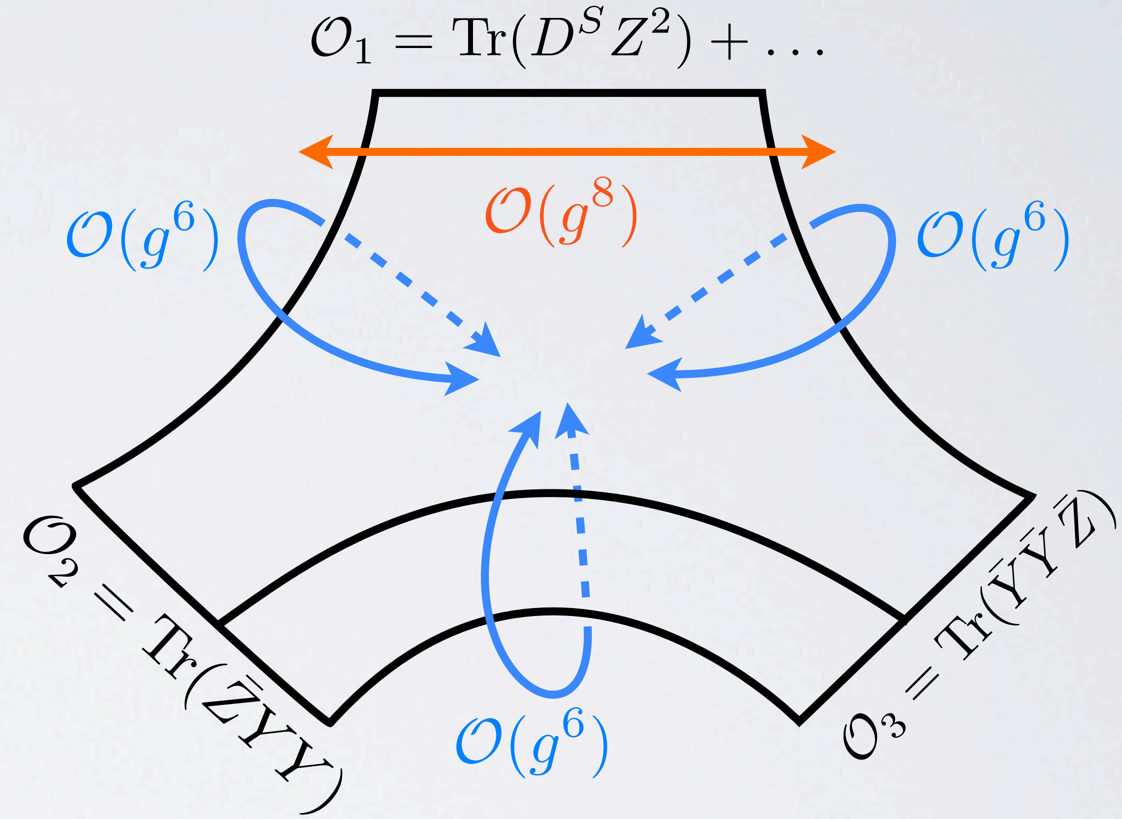
int includes hexagon interaction between exchanged mirror particle and magnons on spin chain

Examples

Short (length two or three) operators



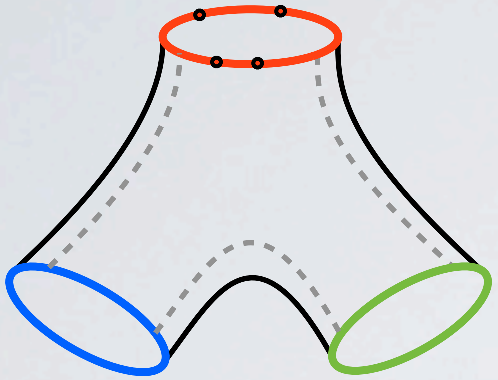
New 2-loop
virtual effect



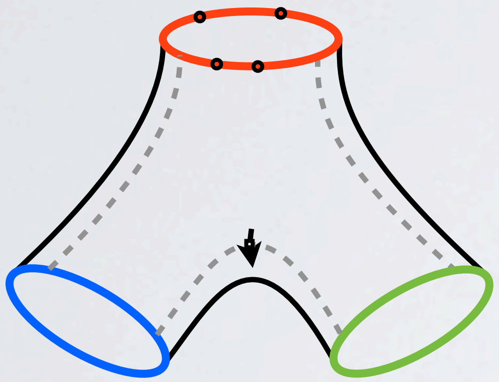
Similar configuration with a slightly bigger bridge delays the new virtual effect to 3-loops

Conclusion : the asymptotic result is the same for both
- but it is valid up to 1-loop on the left
- and up to 2-loop on the right

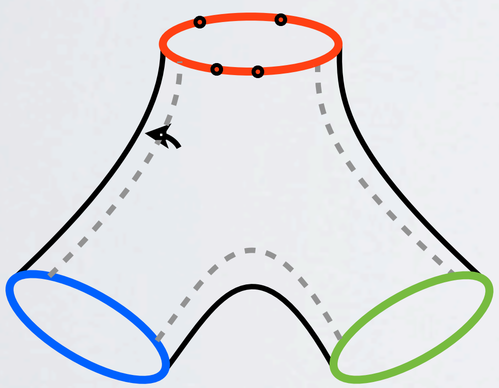
3 loop match



At ***tree level*** and ***one loop***, the asymptotic result is not corrected



At ***two loops*** the first single particle virtual corrections in the opposing channel kicks in



At ***three loops*** the first single particle virtual corrections in neighboring channels kicks in

with finite size effect included we get a perfect match with known results up to 3 loops!

[Eden,Sfondrini‘15],
[BB,Goncalves,Komatsu,Vieira’15]
[Eden’12],
[Chicherin,Drummond,Heslop,Sokatchev’15]

Conclusions

Integrability comes with powerful new strategies for computing quantities at any value of the coupling in planar N=4 SYM theory

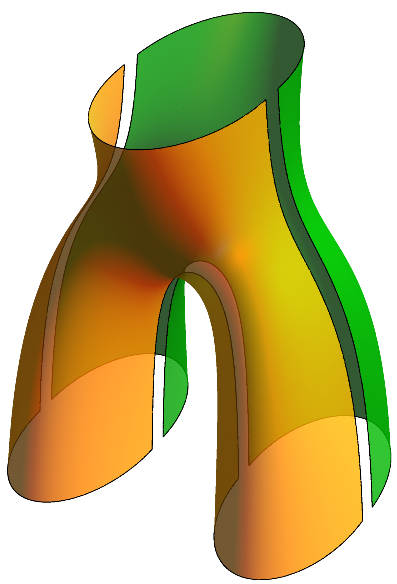
It allows us to attack increasingly complicated objects and find all-loop expressions (conjectures) for them,
like for amplitudes, structure constants, etc.

Here we presented a strategy for structure constants :
- cut open pair of pants into hexagons
- glue hexagons back together in the end

Summary hexagon picture

3-pt function = finite volume correlator of two hexagons

$C_{123} =$

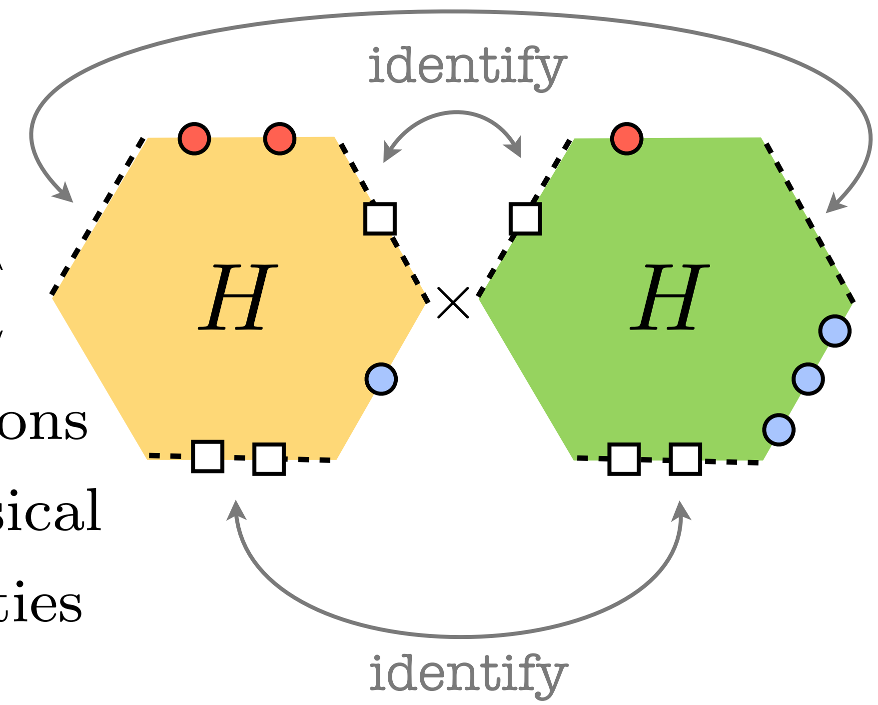


$\sim \int$

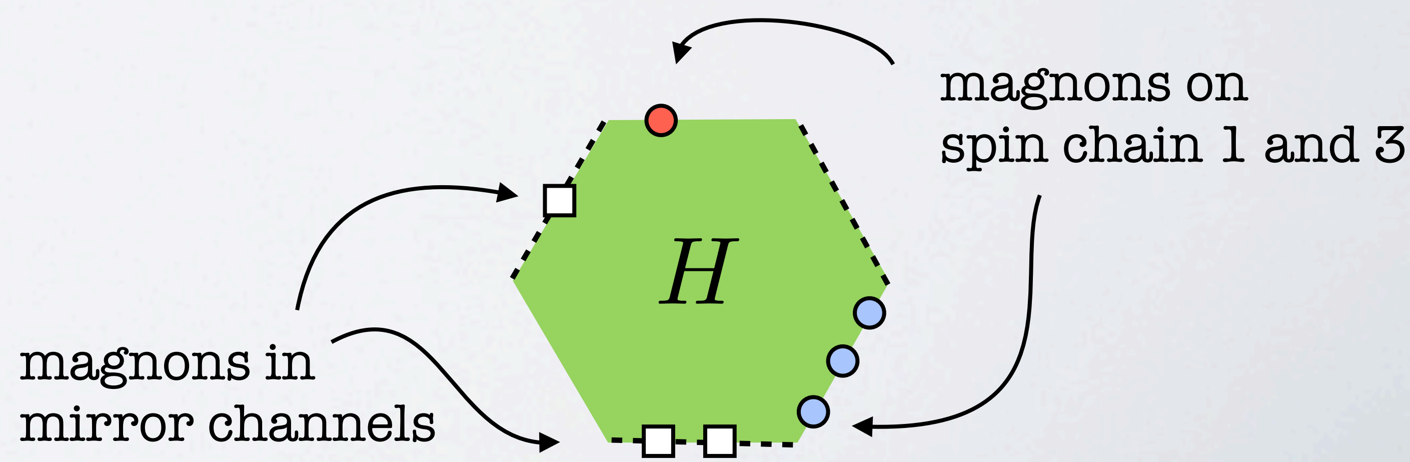
(momentum of) mirror particles where we glue □

$\sum$

partitions of physical rapidities ○



Elementary patch =
hexagon form factor
(can be found using an integrable bootstrap)



THANK YOU!