Thermoelectric Conductivity and Stokes Flows on Black Hole Horizons

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In holography some very special quantities of the dual CFT are captured, universally, by the black hole horizon:

- The temperature of the CFT is given by the surface gravity

$$T_H = \frac{\kappa}{2\pi}$$

- The entropy of the CFT is given by the area of the horizon

$$S_{BH} = \frac{A}{4G}$$

- Shear viscosity?

$$\cancel{\eta = \frac{s}{4\pi}}$$

DC thermoelectric conductivity matrix

$$\begin{pmatrix} J \\ Q \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \bar{\alpha} T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E \\ -(\nabla T)/T \end{pmatrix}$$

J^i	Electric current density	$\partial_i J^i = 0$
$Q^i = T^i{}_t - \mu J^i$	Heat current density	$\partial_i Q^i = 0$

DC conductivity matrix is universally obtained exactly by solving generalised Stokes equations for an incompressible charged fluid on the curved black hole horizon

[Donos,JPG]

Long history of connecting fluids to black holes

Starting with the membrane paradigm

[Damour][McDonald,Price,Thorne][…]

And more recently

[Bhattacharya,Hubeny,Minwalla,Rangamani]

[Eling,Fouxon,Oz]

[Bredberg,Keeler,Lysov,Strominger][…]

By contrast our result is an exact and universal statement about specific correlation functions in holography
(no hydrodynamic limit taken)

One early clue: [Iqbal,Liu]

General framework: Holographic Lattices

CFT with a deformation by an operator that explicitly breaks translation invariance.

Provides a mechanism for momentum dissipation.

[Horowitz, Santos, Tong] [Chesler, Lucas, Sachdev] [Donos, JPG][.........]

- For example, consider D=4 black holes with bulk scalar field

$$\phi(r, x^i) \to \frac{\phi_s(x^i)}{r} + \frac{v(x^i)}{r^2} + \dots \qquad r \to \infty$$

Corresponds to deformation

$$L_{CFT} \to L_{CFT} + \int dx \phi_s(x) \mathcal{O}(x)$$

Holographic lattices are interesting

The lattice deformation can lead to:

- Drude physics - "coherent metals"

  [Horowitz, Santos, Tong][…]

  These arise when momentum is nearly conserved

  The T=0 geometry recovers translation invariance in the IR

- Novel 'incoherent' metals

  [Donos,JPG][Gouteraux][…..]

- Insulators and M-I transitions

  [Donos,Hartnoll][Donos,JPG][……]

The T=0 geometry breaks translation invariance in the IR

DC Conductivity and Stokes Flows

Illustrate (for now) with D=4 Einstein-Maxwell Theory

$$S = \int d^4x\sqrt{-g}\Big[R + 6 - \frac{1}{4}F^2 + \dots\Big]$$

Also, assume for now:

- There is a single black hole horizon with planar topology

- That the holographic lattice does not break time reversal invariance

DC Conductivity and Stokes Flows

- Background lattice black holes

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^i dx^j$$
$$A = A_t dt$$

with $g_{\mu\nu}$ and A_t functions of (r, x^i) , periodic in the x^i

- Behaviour at AdS boundary $r \to \infty$

$$ds^2 \to r^{-2}dr^2 + r^2 \left[g_{tt}^{\infty}(x)dt^2 + g_{ij}^{\infty}(x)dx^i dx^j\right]$$

$$A \to A_t^{\infty}(x)dt$$

- Killing horizon at $r = 0$

- Step1: Perturb the black hole

$$\delta(ds^2) = \delta g_{\mu\nu} dx^\mu dx^\nu + 2t g_{tt} \zeta_i dt dx^i$$

$$\delta A = \delta a_\mu dx^\mu - tE_i dx^i + tA_t \zeta_i dx^i$$

- Behaviour at AdS boundary

The only sources are provided by the closed one-forms

$E = E_i dx^i \qquad \zeta = \zeta_i dx^i$

which source the electric and heat currents

- Regular behaviour at the horizon

Use Kruskal co-ordinates

- Step 2: Electric and heat currents

- Electric current

The gauge equations of motion are

$$\nabla_\mu F^{\mu\nu} = 0$$

Define the bulk electric current density as

$$J^i = \sqrt{-g} F^{ir}$$

which satisfy

$$\boxed{\partial_i J^i = 0\,, \qquad \partial_r J^i = \partial_j \left(\sqrt{-g} F^{ji}\right)}$$

At AdS boundary, $J^i\big|_\infty$, is the electric current of the dual CFT

- Heat current

Want a suitable two-form. Let $k = \partial_t$ and define

$$G^{\mu\nu} \equiv -2\nabla^{[\mu}k^{\nu]} + \dots$$

(think of First Law or Kaluza-Klein reduction)

$$\nabla_\mu G^{\mu\nu} = \alpha k^\nu + \dots$$

Define the bulk heat current density

$$Q^i = \sqrt{-g}G^{ir}$$

which satisfy

$$\boxed{\partial_i Q^i = 0 \qquad\qquad \partial_r Q^i = \partial_j(\sqrt{-g}G^{ji})}$$

At AdS boundary, $Q^i|_\infty$, is the heat current of the dual CFT

• Step 3: Hamiltonian decomposition of equations of motion with respect to the radial coordinate

The Hamiltonian is a sum of constraints:

$$\mathcal{H} = N\,H + N_\mu\,H^\mu + \Phi\,C\,,$$

Evaluate the constraints in an expansion at the horizon

Find that there is a decoupled sector for a subset of the perturbation and forms a closed set of equations

Define

$$v_i \equiv \delta g_{it}^{(0)} \qquad w \equiv \delta a_t^{(0)} \qquad p \equiv p(\delta g_{rt}^{(0)}, \delta g_{it}^{(0)}, g_{tt}^{(0)})$$

Find

$$\nabla_i v^i = 0 \qquad \leftrightarrow \qquad \partial_i Q^i_{(0)} = 0$$

$$\nabla^2 w - v^i \nabla_i a_t^{(0)} = 0 \qquad \leftrightarrow \qquad \partial_i J^i_{(0)} = 0$$

$$\nabla^2 v_j + R_{ji} v^i + a_t^{(0)} \nabla_j w - \nabla_j p = -4\pi T \zeta_j - a_t^{(0)} E_j$$

Linear, time-independent, forced Navier-Stokes equations for a charged, incompressible fluid on the curved black hole horizon!

Also called Stokes equations

Equivalently:

$$Q^i_{(0)} = T s_H v^i \qquad \partial_i Q^i_{(0)} = 0$$

$$J^i_{(0)} = \rho_H v^i + \Sigma_H^{ij}(E_j + \nabla_j w) \qquad \partial_i J^i_{(0)} = 0$$

$$\eta_H \left(-2\nabla^j \nabla_{(i} v_{j)} + \nabla_i p\right) = \rho_H (E_i + \nabla_i w) + T s_H \zeta_i$$

$$\rho_H = \sqrt{g_{(0)}}\, a_t^{(0)} \qquad s_H = 4\pi \sqrt{g_{(0)}}$$

$$\eta_H = \frac{s_H}{4\pi} \qquad \Sigma_H^{ij} = \sqrt{g_{(0)}}\, g^{ij}_{(0)}$$

NOTE! In general $\sigma^{ij} \neq \Sigma_H^{ij}$ and $\eta \neq \eta_H$

- Step 4: have J^i, Q^i on the horizon as functions of E_i , ζ_i

what do we know about $J^i|_\infty, Q^i|_\infty$ (the holographic currents)?

Need to integrate:

$$\partial_r J^i = \partial_j(\sqrt{-g}F^{ji}) \qquad \partial_r Q^i = \partial_j(\sqrt{-g}G^{ji})$$

e.g. If the lattice deformation is periodic in x^i with period L_i

define the current flux densities:

$$\bar{J}^1 \equiv \frac{1}{L_2}\int J^1 dx^2 \qquad \bar{J}^2 \equiv \frac{1}{L_1}\int J^2 dx^1$$

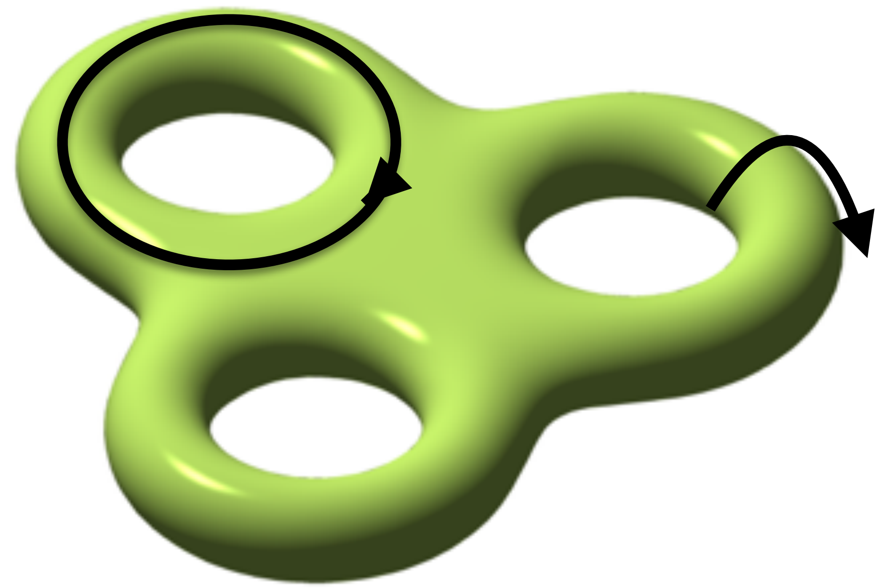
with $\partial_r \bar{J}^i = 0$. Thus we obtain $\bar{J}^i_\infty, \bar{Q}^i_\infty$ as functions of E_i , ζ_i

and hence the DC conductivity

Comments

- We implicitly assumed that the spatial topology of the space which the CFT lives, Σ_d was a torus.

We can extend to arbitrary topology by defining current fluxes through a basis of $d-1$ cycles



The conductivity matrix is then a matrix of size $b_1(\Sigma_d)^2$

- We can also consider the horizon being disconnected

Analyse using language of differential forms

Comments

- The formalism can be generalised from Einstein-Maxwell to any theory of gravity e.g. add scalar field ϕ

$$S=\int d^Dx\sqrt{-g}\left(R-V(\phi)-\frac{Z(\phi)}{4}F^2-\frac{1}{2}(\partial\phi)^2\right)$$

Extra viscous terms appear in the Stokes equations

$$\partial_i Q^i_{(0)}=0$$

[Banks,Donos,JPG]

$$\partial_i J^i_{(0)}=0$$

$$\nabla^2 v_j+R_{ji}v^i-\nabla_j\phi^{(0)}\nabla_i\phi^{(0)}v^i+Z^{(0)}a_t^{(0)}\nabla_j w-\nabla_j p$$
$$=-4\pi T\zeta_j-Z^{(0)}a_t^{(0)}E_j$$

Comments

- We can generalise to break T: e.g. include magnetic fields

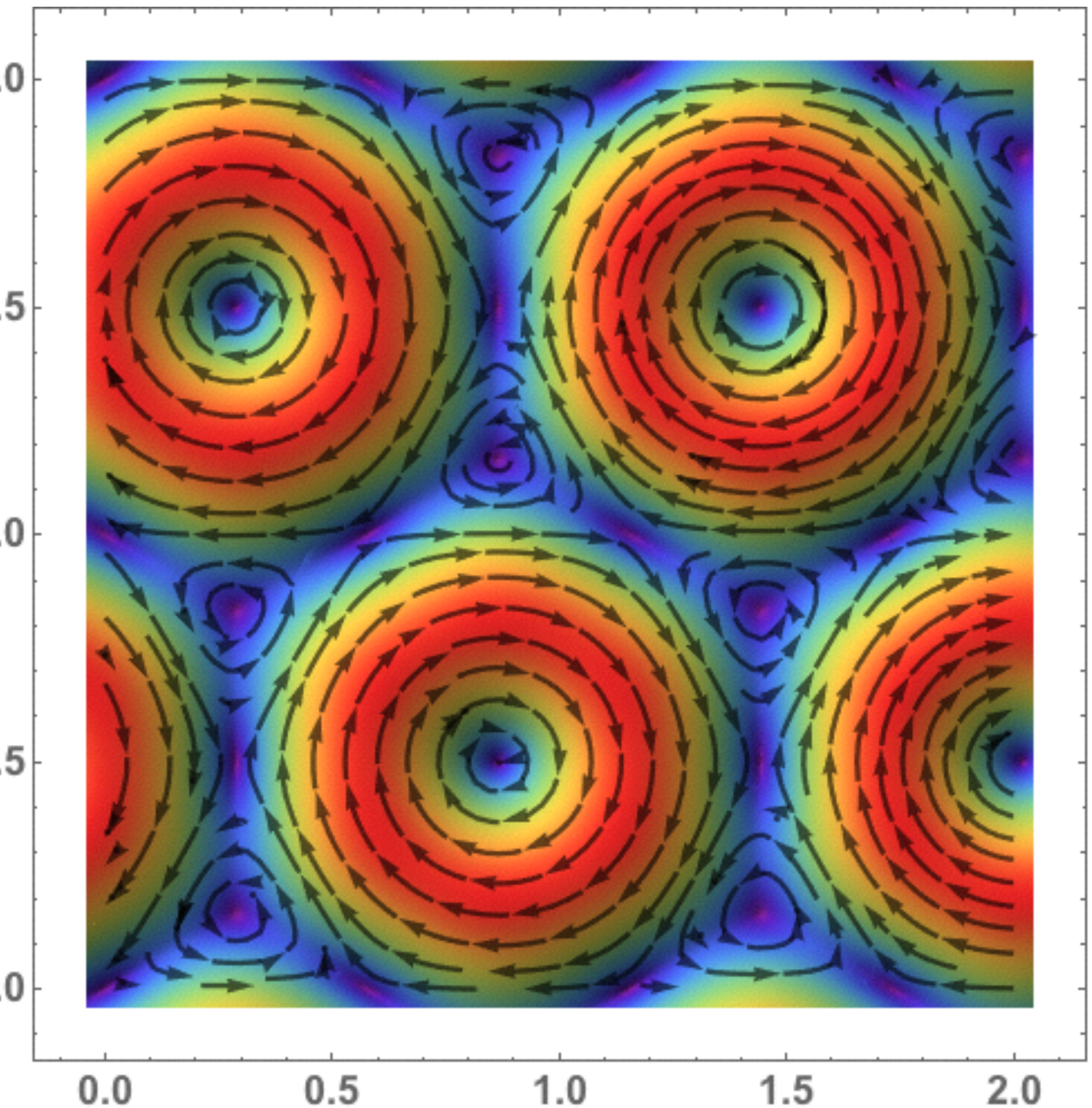
[Donos,JPG,Griffin,Melgar]

When black holes have $A_i \neq 0$, $g_{ti} \neq 0$ they can have

local U(1) and heat magnetisation currents in the lattice deformation:

$$J_\infty^{(B)i} = \partial_j M^{(B)ij} \qquad Q_\infty^{(B)i} = \partial_j M_T^{(B)ij}$$

Models for magnetic impurities



Aside: these currents can also be generated spontaneously within holography

[Donos,JPG]

Extra Lorentz and Coriolis terms appear in the Stokes equations

$$\partial_i Q^i_{(0)} = 0 \qquad \partial_i J^i_{(0)} = 0$$

$$\nabla^2 v_j + R_{ji} v^i - \left[\nabla_j \phi^{(0)} \nabla_i \phi^{(0)} + d\chi^{(0)}_{ji}\right] v^i + Z^{(0)} a_t^{(0)} \nabla_j w - \nabla_j p$$
$$= -4\pi T \, \zeta_j - Z^{(0)} a_t^{(0)} E_j - F^{(0)}_{ij} \frac{J^j_{(0)}}{\sqrt{h^{(0)}}}$$

where $\chi^{(0)}_i = g^{(0)}_{ti}$

Subtlety: necessary to subtract off magnetisation currents

[Cooper,Halperin,Ruzin][Hartnoll,Kovtun,Muller,Sachdev]

- For some black holes can solve Stokes equations explicitly

i) For Q-lattices [Donos,JPG]

Holographic lattice black holes that exploit a global symmetry in the bulk

ii) For one-dim holographic lattices

Recover the exact results found earlier by [Donos,JPG]

[Blake,Donos,Lohitsiri]

[Kim,Kim,Seo,Sin]

(iii) Perturbative, periodic lattice about translationally invariant black branes - coherent metals

Let λ be the expansion parameter

The black hole horizon is a small expansion about flat space

$$g_{(0)ij} = g\,\delta_{ij} + \lambda\, h^{(1)}_{ij} + \cdots$$

$$a^{(0)}_t = a + \lambda\, a_{(1)} + \cdots$$

Solve Navier-Stokes perturbatively in λ

- At leading order in λ we find

$$\alpha_{ij} = \bar{\alpha}_{ij} = \frac{L^{-1}_{ij}}{\lambda^2} 4\pi\rho + \dots \qquad \bar{\kappa}_{ij} = \frac{L^{-1}_{ij}}{\lambda^2} 4\pi s T + \dots$$

$$\sigma_{ij} = \frac{L^{-1}_{ij}}{\lambda^2} \frac{4\pi\rho^2}{s} + \dots$$

where $L_{ij} = \int_H l_{ij}(h^{(1)}_{kl}, a^{(1)})$

- Note $\bar{\kappa}(\sigma T)^{-1} = \dfrac{s^2}{\rho^2}$ - generalised Wiedemann-Franz law

$$\bar{\kappa}\alpha^{-1} = \frac{Ts}{\rho}$$

Consistent with memory matrix formalism

[Barkeshli,Hartnol,Mahajan] [Lucas,Sachdev]

Final Comments

- New appearance of Navier-Stokes on black hole horizons

  Gives, universally, DC thermoelectric conductivities for all T

- Some exact results, including generalised W-F law

- General constraints on conductivity using new results

  [Grozdanov,Lucas,Sachdev,Schalm]

- Hydrodynamics for large T with lattices

  [Hartnoll,Kovtun,Muller,Sachdev][Blake]
  [Davison,Gouteraux][Lucas,Sachdev][Lucas]

- Holographic lattices and new ground states