Non-Relativistic Hydrodynamics and Lifshitz Black Branes

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Current Themes in Holography

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Introduction

- Question: Given a Lifshitz invariant field theory with dynamical exponent $z > 1$ what are the equations that govern the hydrodynamic approximation?
- Lifshitz: H (time transl.), P_i (d space transl.), J_{ij} (SO(d) rotations), D (dilatations: $t \to \lambda^z t$ and $x^i \to \lambda x^i$).
- For theories with only scalar fields, do we know all Lifshitz invariant actions?
- Is there a universal theory for Lifshitz hydrodynamics or are there different possibilities?
- Which kinds of fluids can be described using fluid/gravity for Lifshitz black branes?

Outline Talk

- New examples of Lifshitz scalar field theories
- Lifshitz fluids from:
  - Simple scalar models
  - Null reduction
  - Fluid/gravity
- Summary/Outlook

Lifshitz Scalar Models I

- Higher derivative single real scalar model

$$\mathcal{L} = \frac{1}{2}(\partial_t\theta)^2 - \frac{\lambda}{2}(\partial_i\partial_i)^n\theta(\partial_i\partial_i)^n\theta\,, \qquad z = 2n$$

- Shift symmetry: $\theta \to \theta + c$
  Scaling dimension: $[\theta] = (d-z)/2$
  Global symmetries: $\mathbb{R} \times$ Lif
  Hamiltonian is bounded from below
  Time reversal invariance

- For $z < d$ we can add a potential $V = V(\theta)$. For $z \geq d$ we keep the shift symmetry for otherwise we can add arbitrary high powers of θ.

Lifshitz Scalar Models II

- Lifshitz is a subgroup of Schrödinger so the Schrödinger model is a Lifshitz scalar theory

$$\mathcal{L} = i\phi^\star \partial_t \phi - i\phi \partial_t \phi^\star - \partial_i \phi \partial_i \phi^\star - V_0 \left(\phi\phi^\star\right)^{(d+2)/d}$$

$$\phi = \frac{1}{\sqrt{2}} \varphi e^{i\theta}\,, \qquad \mathcal{L} = -\varphi^2 \left[\partial_t \theta + \frac{1}{2}\partial_i \theta \partial_i \theta\right] - \frac{1}{2}\partial_i \varphi \partial_i \varphi - V_0 \varphi^{2(d+2)/d}$$

- Shift symmetry: $\theta \to \theta + c$
  Scaling dimension: $[\varphi] = d/2$, $[\theta] = 0$
  Global symmetries: Sch($z = 2$) $\supset U(1) \times$ Lif($z = 2$)
  Hamiltonian is bounded from below
  No time reversal invariance

Lifshitz Scalar Models III

$$\mathcal{L} = -\varphi^2 \left[\partial_t \theta + \frac{1}{2} \partial_i \theta \partial_i \theta \right] - \frac{1}{2} \partial_i \varphi \partial_i \varphi - V_0 \varphi^{2(d+2)/d}$$

- Galilean boosts (generator G_i):
  $x^i = x'^i - v^i t' \,, \qquad t = t' \,, \qquad \theta = \theta' - v^i x'^i + \frac{1}{2} v^i v^i t'$
  Invariant: $\partial_t \theta + \frac{1}{2} \partial_i \theta \partial_i \theta$
  $[P_i, G_j] = N \delta_{ij}$: mass N (shift θ)
- At an algebraic level we can break
  Sch $\rightarrow$ Lif by breaking N
- Or we can break
  Sch $\rightarrow U(1) \times$ Lif by breaking G_i

Lifshitz Scalar Models IV

$$\mathcal{L} = -\varphi^\alpha \partial_t \theta - \frac{1}{2}\varphi^2 \partial_i \theta \partial_i \theta - \frac{1}{2}\partial_i \varphi \partial_i \varphi - V_0 \varphi^{\frac{2(d+z)}{d+z-2}}\,, \qquad \alpha = \frac{2d}{d+z-2}$$

- Shift symmetry: $\theta \to \theta + c$

  Scaling dimensions: $[\varphi] = (d+z-2)/2$, $\quad [\theta] = 0$

  Global symmetries: $U(1) \times$ Lif(z) for $z \neq 2$

  General $z > 1$ scaling without higher spatial derivatives

  For $z \neq 2$ Galilean boosts are broken

  Hamiltonian is bounded from below

  No time reversal invariance

$$\mathcal{L} = (\phi\phi^\star)^{(\alpha-2)/2}\left[i\phi^\star \partial_t \phi - i\phi \partial_t \phi^\star\right] - \partial_i \phi \partial_i \phi^\star - V_0 \left(\phi\phi^\star\right)^{\frac{d+z}{d+z-2}}$$

From Scalars to Fluids I

- A real relativistic scalar is a perfect fluid

$$\mathcal{L} = \frac{1}{2}\left(\partial_t \phi\right)^2 - \frac{1}{2}\left(\partial_i \phi\right)^2 - V(\phi)$$

- The EMT can be written in perfect fluid form

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \eta_{\mu\nu}\mathcal{L} = (E+P)U_\mu U_\nu + \eta_{\mu\nu} P$$

$$U_\mu = \frac{\partial_\mu \phi}{\sqrt{-(\partial\phi)^2}}\,, \qquad P = \mathcal{L}\,, \qquad E + P = -(\partial\phi)^2\,.$$

- The thermodynamic properties of the perfect fluid are not described by the scalar model.

From Scalars to Fluids II

$$\mathcal{L} = -\varphi^2 \left[\partial_t\theta + \frac{1}{2}\partial_i\theta\partial_i\theta\right] - \frac{1}{2}\partial_i\varphi\partial_i\varphi - V(\varphi)$$

- Fluid variables: mass density $\rho = \varphi^2$, velocity $V^i = \partial_i\theta$, pressure $P = \frac{1}{2}\varphi V' - V$, energy $\mathcal{E} = V$
- Eom of Sch model can be written as: mass continuity, modified Euler equation, energy conservation

$$\partial_t\rho + \partial_i\left(\rho V^i\right) = 0$$

$$\rho\left(\partial_t V^i + V^j\partial_j V^i\right) = -\partial_i P + \rho\partial_i I\,, \qquad I = \frac{1}{4\rho}\partial_j\partial_j\rho - \frac{1}{8\rho^2}\partial_j\rho\partial_j\rho$$

$$\left(\partial_t + V^i\partial_i\right)\mathcal{E} + (\mathcal{E} + P)\,\partial_i V^i = 0$$

Speed of Sound

- Fluctuate around a constant background (perform G-boost: $\partial_{t'} = \partial_t + V_0^i \partial_i$ and $\partial'_i = \partial_i$)

$$\partial_{t'}^2 \delta\mathcal{E} - \frac{\mathcal{E}_0 + P_0}{\rho_0}\left(\partial'_i\partial'_i \delta P - \frac{1}{4}(\partial'_i\partial'_i)^2 \delta\rho\right) = 0$$

- Assume equation of state $2\mathcal{E} = dP$
- 1st law: $\delta\mathcal{E} = T\delta s = \frac{\mathcal{E}+P}{\rho}\delta\rho + T\rho\delta\left(\frac{s}{\rho}\right)$
- Flow is isentropic: entropy per unit mass s/ρ is cst

$$\partial_{t'}^2 \delta\mathcal{E} - \frac{2}{d}\frac{\mathcal{E}_0 + P_0}{\rho_0}\partial'_i\partial'_i \delta\mathcal{E} + \frac{1}{4}(\partial'_i\partial'_i)^2 \delta\mathcal{E} = 0$$

- Speed of sound $c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_{s/\rho}$

Null Reduction

- Minkowski in null coordinates: $ds^2 = 2dtdu + dx^i dx^i$
- Null reduction of EMT: $t^{\mu}{}_{u} = T^{\mu}$, $t^{u}{}_{u} = T^{t}{}_{t}$, $t^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}$
- Perfect fluid: $t^{A}{}_{B} = (E+P)U^A U_B + P\delta^A_B$, $U^2 = -1$
- Reduction of fluid:

$$U_u^2 = \frac{\rho}{E+P}\,, \qquad U_t = -\frac{1}{2}U_u\left(V^i V^i + U_u^{-2}\right)$$
$$U_i = U_u V^i\,, \qquad E = 2\mathcal{E} + P\,,$$

- Lower-dimensional EMT and mass current:

$$T^{\mu}{}_{\nu} = \left(\mathcal{E} + P + \frac{1}{2}\rho V^i V^i\right) u^{\mu}\tau_{\nu} + P\delta^{\mu}_{\nu} + \rho u^{\mu} h_{\nu\rho} u^{\rho}\,, \quad T^{\mu} = -\rho u^{\mu}$$
$$\tau_{\mu} u^{\mu} = -1\,, \quad u^i = -V^i\,, \qquad \tau_{\mu} = \delta^t_{\mu}\,, \quad h_{t\mu} = 0\,, \quad h_{ij} = \delta_{ij}$$

Lifshitz Fluids from Null Reduction

- Add a scalar with shift symmetry and consider the Ward identities: $\partial_A t^A{}_B = -O_\psi \partial_B \psi$ and $t^A{}_A = 0$
- Twisted null reduction: $\psi = -u$
- Lower-dimensional Ward identities: $\partial_\mu T^\mu{}_\nu = 0$, $\partial_\mu T^\mu = O_\psi$ (mass no longer conserved), $2T^t{}_t + T^i{}_i = 0$
- Fluid equations imply a conserved entropy current if we take

$$\mathcal{E} + P - \frac{1}{2}\rho V^2 = Ts\,, \qquad \delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho$$

- Velocity V^i becomes a chemical potential
- Lifshitz PF from Schrödinger PF by breaking N

Holographic Setup

- SS reduction of $\mathcal{L}_{(5)} = \sqrt{-\gamma}\left(R + 12 - \frac{1}{2}(\partial\psi)^2\right) + \mathcal{L}_{\text{ct}}$

$$\mathcal{L}_{(4)} = \sqrt{-g}\left(R - \frac{1}{4}e^{3\Phi}F^2 - 2B^2 - \frac{3}{2}(\partial\Phi)^2 - 2e^{-3\Phi} + 12e^{-\Phi}\right) + \mathcal{L}_{\text{ct}}$$

- Admits $z = 2$ Lifshitz solutions
- Near boundary ($r = 0$) exp.: $ds_4^2 = e^{\Phi}\frac{dr^2}{r^2} + h_{\mu\nu}dx^\mu dx^\nu$

$$\begin{aligned}
\Phi &= -\frac{1}{8}r^2\rho - r^4\left(\frac{1}{6}T^t{}_t + \frac{1}{64}\rho^2\right) + \mathcal{O}(r^6) \\
B_t &= r^{-2} + \frac{1}{4}\rho + r^2\left(\frac{1}{12}T^t{}_t + \frac{1}{16}\rho^2\right) + \mathcal{O}(r^4) \\
B_i &= -\frac{1}{4}r^2 T^t{}_i + \mathcal{O}(r^4) \qquad \text{likewise for } h_{\mu\nu}
\end{aligned}$$

Moving Lifshitz Black Branes

- Ansatz for full solution (all functions depend on r only):

$$ds_4^2 = -r^{-4}F_1 dt^2 + \frac{dr^2}{r^2 F_2} + \frac{F_3}{r^2}dx^2 + \frac{F_4}{r^2}(dy + N^y dt)^2$$

$$B = r^{-2}G dt + A_y (dy + N^y dt) \,, \qquad \Phi = \Phi(r)$$

- Effective action for this ansatz has two scale symmetries:

$$1). \quad [G] = 1\,, \quad [F_1] = 2\,, \quad [F_{3,4}] = 1\,, \quad [A_y] = -1/2\,, \quad [N^y] = 3/2$$

$$2). \quad [F_3] = 2\,, \quad [F_4] = -2\,, \quad [A_y] = -1\,, \quad [N^y] = 1$$

Associated Noether charges Q_1 and Q_2 are first integrals of motion along r

Thermodynamics

- Near horizon regularity: G, F_1, F_2 first order zeros at $r = r_h$, the rest are nonzero
- Horizon generator $X = \partial_t - N^y(r_h)\partial_y$ gives temperature and chemical potential ($N^y(r_h) = -V$)
- Conserved Noether charges $Q_1 - \frac{3}{2}Q_2 = Ts$ at the horizon and $\mathcal{E} + P - \frac{1}{2}\rho V^2$ at the boundary
- We can also derive the first law by computing the on-shell action which gives the grand potential as a function of temperature T and chemical potential V

$$\mathcal{E} + P - \frac{1}{2}\rho V^2 = Ts\,, \qquad \delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho$$

Summary

- Many ways to realize Lifshitz invariance in field theory. New models with general z scaling without higher spatial derivatives.
- Simple scalar models behave similar to fluids
- From Schrödinger to Lifshitz: break Galilean boosts or mass
- When mass is broken we have found the PF form of the Lifshitz fluid by null reduction and from holography
- Case with breaking Gal. boosts is under construction. This gives $U(1) \times$ Lif

Outlook

- Consider other holographic setups: with massless vector and running dilaton [Kiritsis, Matsuo, 2015]
- Compare with approach by [Hoyos, Kim, Oz, 2013]
- Recently found a Chern–Simons version of 3D Horava–Lifshitz gravity with $z = 2$ Lifshitz solutions [JH, Lei, Obers, to appear]. Develop fluid/gravity for HL gravity theories with Lifshitz solutions.
- Coupling to NR electrodynamics [Festuccia, Hansen, JH, Obers, to appear]