WARPED CONFORMAL THEORIES

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DEFINITION

At the level of infinitesimal transformations, a WCFT is defined as a classical field theory invariant under

$$\begin{aligned} x^+ &\to x^+ + g(x^-) \\ x^- &\to f(x^-) \end{aligned}$$

Global symmetries

$SL(2,\mathbb{R}) \times U(1)$

This is not a CFT_2, where instead we have

$$\begin{aligned} x^+ &\to g(x^+) \\ x^- &\to f(x^-) \end{aligned}$$

Global symmetries

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

DEFINITION

At the level of infinitesimal transformations, a WCFT is defined as a classical field theory invariant under

$$
\begin{aligned}
x^+ &\rightarrow x^+ + g(x^-) \\
x^- &\rightarrow f(x^-)
\end{aligned}
$$

shift/translation

$SL(2,\mathbb{R}) \times U(1)$

scaling direction

MOTIVATION

CFT are the building blocks of relativistic QFTs.

What are the building blocks of non-relativistic QFTs?

- Very few examples.
- Exotic fixed points are easy to miss.

Can one build an interesting example of a WCFT?

YES

PLAN

Massive Weyl Fermion:

1. Spectrum
2. Modular invariance

[AC, D. Hofman, G. Sarosi, arxiv:1508.06302]

New geometric structures:

1. Thermodynamics
2. Entanglement entropy

[AC, D. Hofman, N. Iqbal, arxiv:1511.00707]

A NON TRIVIAL EXAMPLE

MASSIVE WEYL FERMION

1. SPECTRUM

ACTION

To start: a single complex anti-commuting field

[Hofman & Rollier 2014]

Not Lorentz invariant

$$I = \int dx^+ dx^- \left(i\bar{\Psi}\partial_+\Psi + m\bar{\Psi}\Psi\right)$$

Mass term: not allowed for CFT

ACTION

To start: a single complex anti-commuting field

[Hofman & Rollier 2014]

$$\underbrace{I = \int dx^+ dx^- \left(i\bar{\Psi}\partial_+\Psi + m\bar{\Psi}\Psi\right)}_{\substack{\text{Global symmetries} \\ SL(2,\mathbb{R})\times U(1)}}$$

Under scaling $x^- \to \lambda x^-$ fermion has weight ½.

ACTION

To start: a single complex anti-commuting field

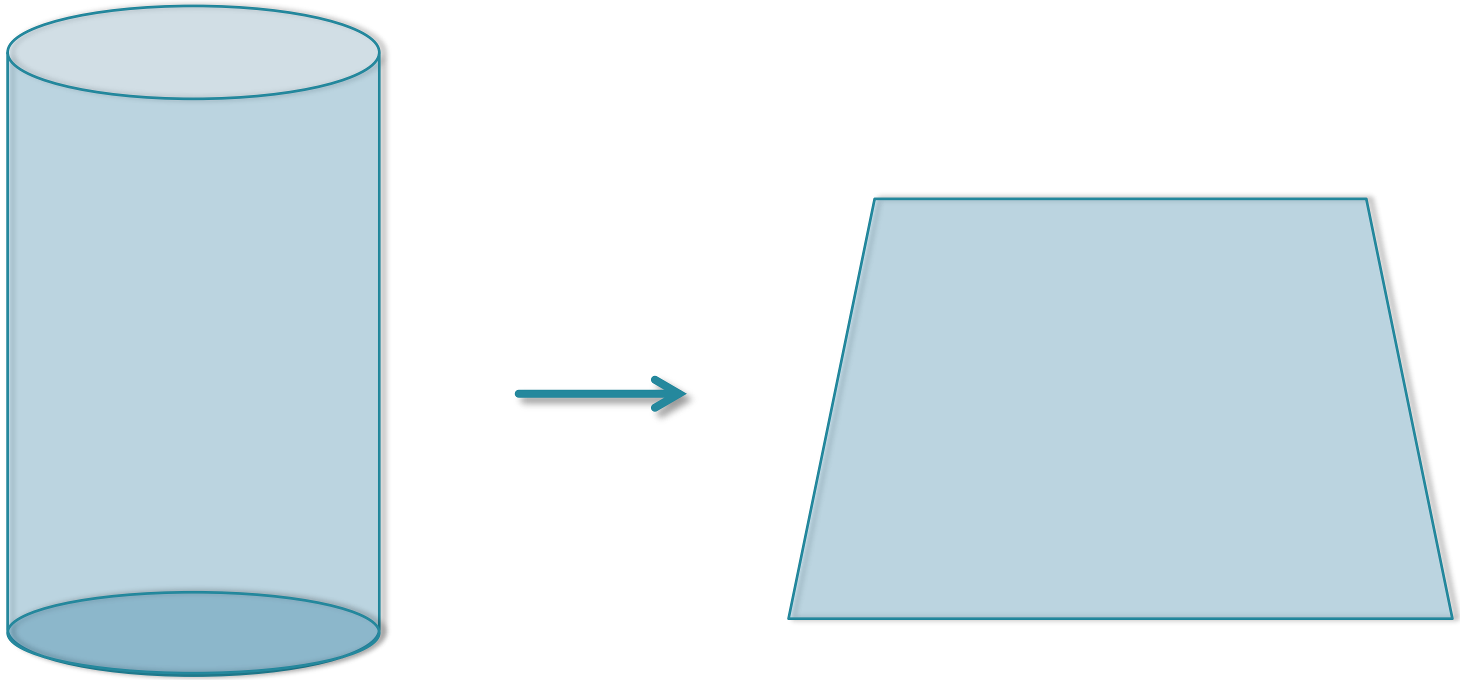
[Hofman & Rollier 2014]

$$I = \underbrace{\int dx^+ dx^- \left(i\bar{\Psi}\partial_+ \Psi + m\bar{\Psi}\Psi \right)}_{\text{Global symmetries } SL(2,\mathbb{R}) \times U(1)}$$

My favorite things about this example:

- Explicit
- Not a CFT
- Novel structure
- Unitary WCFT

CYLINDER TO PLANE



$$x^- + x^+ = 2t$$
$$x^+ - x^- = 2\varphi$$
$$\varphi \sim \varphi + 2\pi$$

$$z = e^{-ix^-} = e^{-i(t-\varphi)} = e^{t_E + i\varphi}$$
$$w = x^+ - x^- = 2\varphi$$

SYMMETRIES

Solutions to equations of motion

$$\Psi = e^{imw}\eta(z) \qquad \bar{\Psi} = e^{-imw}\bar{\eta}(z)$$

Noether currents

$$\begin{aligned} z \ &\rightarrow \ f(z): \qquad && T(z) = -\frac{1}{2}\left(\bar{\eta}\partial_z\eta + \eta\partial_z\bar{\eta}\right) \\ w \ &\rightarrow \ w + g(z): \qquad && P(z) = m\,\bar{\eta}\,\eta \end{aligned}$$

ALGEBRA

$$L_n = -\frac{i}{2\pi}\int dz\, \zeta_n(z) T(z) \qquad P_n = -\frac{1}{2\pi}\int dz\, \chi_n(z) P(z)$$

$$\begin{aligned}
[L_n, L_{n'}] &= (n-n')L_{n+n'} + \frac{c}{12} n(n^2-1)\delta_{n,-n'} \\
[L_n, P_{n'}] &= -n' P_{n'+n} \\
[P_n, P_{n'}] &= k\frac{n}{2}\delta_{n,-n'}
\end{aligned}$$

$$\boxed{c = 1 \qquad k = 2m^2}$$

Virasoro-Kac Moody

ALGEBRA

$$\begin{aligned}
[L_n, L_{n'}] &= (n-n')L_{n+n'} + \frac{c}{12}n(n^2-1)\delta_{n,-n'} \\
[L_n, P_{n'}] &= -n' P_{n'+n} \\
[P_n, P_{n'}] &= k\frac{n}{2}\delta_{n,-n'}
\end{aligned}$$

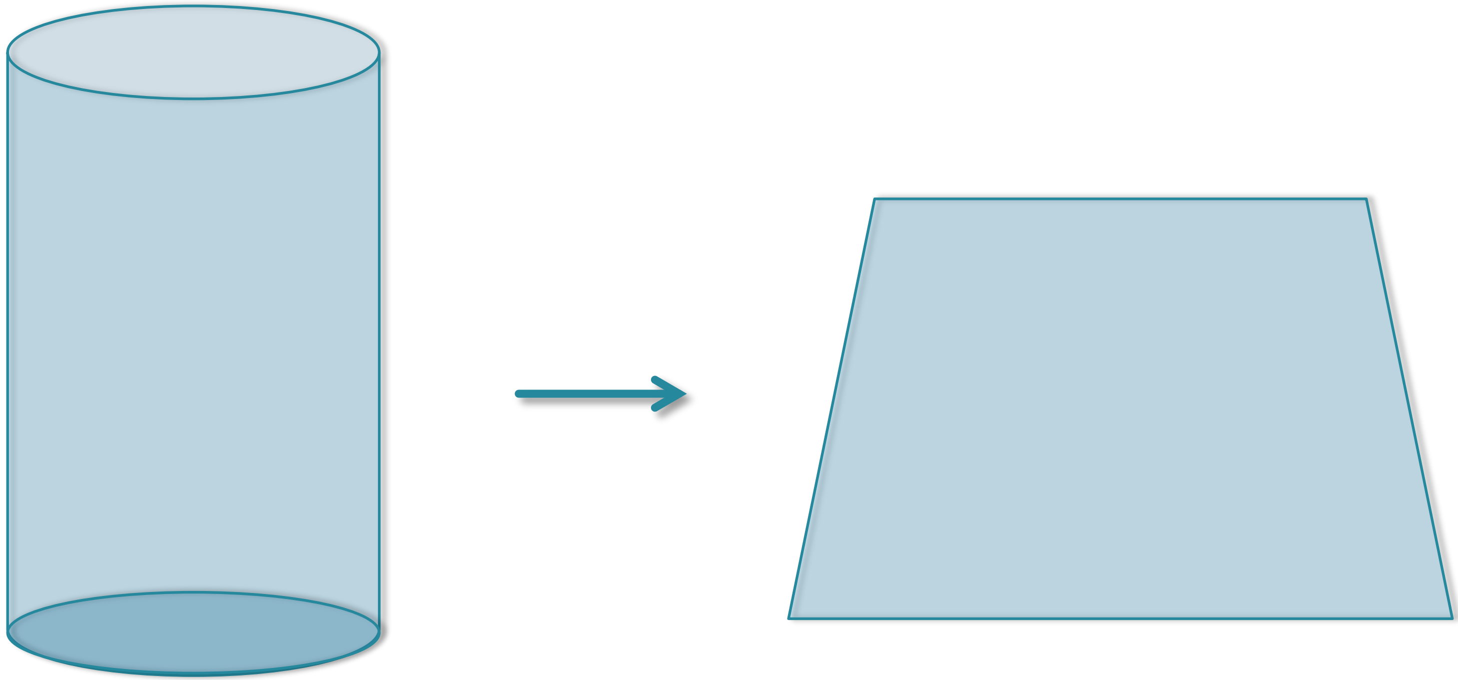
Under a tilt of the axis

$$z = z' \qquad w = w' + 2\gamma z'$$

$$\begin{aligned}
L_n &\to L_n^{(\gamma)} = L_n + 2\gamma\, P_n + \gamma^2\, k\, \delta_{n,0} \\
P_n &\to P_n^{(\gamma)} = P_n + \gamma\, k\, \delta_{n,0}
\end{aligned}$$

Spectral flow transformation

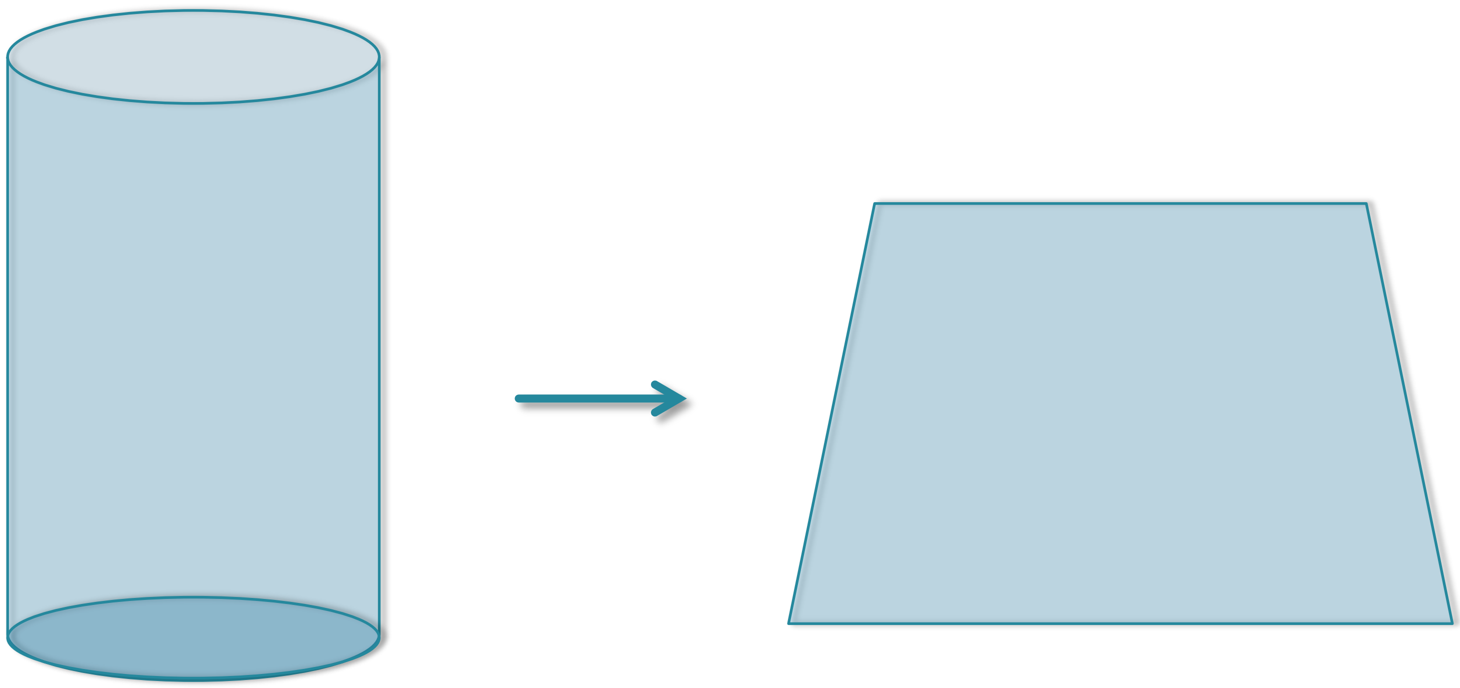
SPECTRUM



$$H = -i\partial_t = -i(\partial_+ + \partial_-)$$
$$J = -i\partial_\varphi = -i(\partial_+ - \partial_-)$$

$$H = L_0 - \frac{c}{24} - \frac{k}{4}$$
$$J = 2P_0 - L_0 + \frac{c}{24} - \frac{3}{4}k$$

SPECTRUM



$$\varphi \sim \varphi + 2\pi \quad \begin{cases} \text{R : periodic fields} \\ \text{NS: anti-periodic fields} \end{cases}$$

CONSTRAINTS

Quantization of Angular Momentum J:

$$\boxed{6k + c = 12p \qquad\qquad p \in \mathbb{Z}}$$

Restrictions on central charge.

Exploit spectral flow: $\quad z = z' \qquad w = w' + 2\gamma z'$

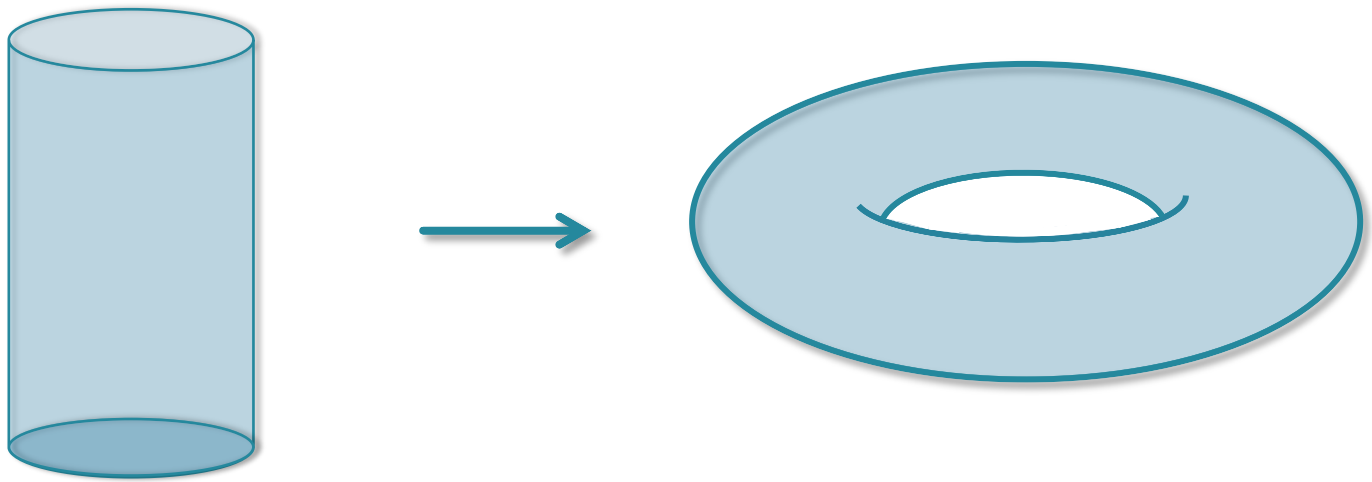
$$\begin{aligned} H &= L_0^{(\gamma)} + 2\gamma P_0^{(\gamma)} - \frac{c}{24} - \frac{k}{4}\left(1 - 4\gamma^2\right) \\ J &= 2(1-\gamma)P_0^{(\gamma)} - L_0^{(\gamma)} + \frac{c}{24} - \frac{3}{4}k + k\gamma(2-\gamma) \end{aligned}$$

$$\boxed{J \leq J_{\text{max}} \equiv \frac{c}{24} + \frac{k}{4}}$$

Angular momentum is bounded.

2. MODULAR INVARIANCE

ON THE TORUS



$$(x^-, x^+) \sim \overbrace{(x^- - 2\pi, x^+ + 2\pi)}^{\text{Spatial cycle}} \sim \overbrace{(x^- - 2\pi\tau, x^+ + 2\pi\bar{\tau})}^{\text{Thermal cycle}}$$

$$x^+ = t + \varphi \qquad x^- = t - \varphi$$

$$2\pi\tau = \mu - i\beta \qquad 2\pi\bar{\tau} = \mu + i\beta$$

ON THE TORUS

$$(u, v) \sim (u - 2\pi\ell, v + 2\pi\bar{\ell}) \sim (u - 2\pi\tau, v + 2\pi\bar{\tau})$$

$$Z_{\bar{\ell}|\ell}(\bar{\tau}|\tau) = \mathrm{Tr}_{\bar{\ell},\ell}\left(e^{2\pi i\bar{\tau}P_0^{\mathrm{cyl}}} e^{-2\pi i\tau L_0^{\mathrm{cyl}}}\right)$$

Is all this notation needed?
Tampering with periodicities introduces anomalies. For example,

$$\hat{u} = \frac{u}{\ell} \qquad \hat{v} = v + \frac{\bar{\ell}}{\ell}u$$

$$Z_{\bar{\ell}|\ell}(\bar{\tau}|\tau) = e^{\pi i k\bar{\ell}\left(\bar{\tau} - \frac{\tau\bar{\ell}}{2\ell}\right)} Z_{0|1}(\bar{\tau} - \frac{\bar{\ell}\tau}{\ell}|\frac{\tau}{\ell}) \equiv e^{\pi i k\bar{\ell}\left(\bar{\tau} - \frac{\tau\bar{\ell}}{2\ell}\right)} \hat{Z}(\bar{\tau} - \frac{\bar{\ell}\tau}{\ell}|\frac{\tau}{\ell})$$

MODULAR INVARIANCE

Invariance under S: exchange of thermal and spatial cycle.

$$\boxed{Z_{0|1}(\bar{\tau}|\tau) = Z_{\bar{\tau}|\tau}(0|-1)}$$

Invariance under T: add the spatial cycle to the thermal.

$$\boxed{Z_{1|1}(\bar{\tau}+1|\tau+1) = Z_{1|1}(\bar{\tau}|\tau)}$$

MODULAR INVARIANCE

Invariance under S: exchange of thermal and spatial cycle.

$$\boxed{\hat{Z}(z|\tau) = e^{\pi i k \frac{z^2}{2\tau}} \hat{Z}(\frac{z}{\tau}| - \frac{1}{\tau})}$$

Invariance under T: add the spatial cycle to the thermal.

$$\boxed{\hat{Z}(z|\tau) = e^{\pi i \frac{k}{2}} \hat{Z}(z|\tau + 1)}$$

IMPLEMENTATION

Can we build a modular invariant partition function for the fermion?
Start by building the characters for each sector

$$\hat{Z}_{\mathrm{RR}} = q^{-\frac{1}{24}} \mathrm{Tr}_{\mathrm{R}} \left((-1)^F y^{\frac{\hat{P}_0}{m}} q^{\hat{L}_0} \right) \qquad \hat{Z}_{\mathrm{RNS}} = q^{-\frac{1}{24}} \mathrm{Tr}_{\mathrm{R}} \left(y^{\frac{\hat{P}_0}{m}} q^{\hat{L}_0} \right)$$

$$\hat{Z}_{\mathrm{NSNS}} = q^{-\frac{1}{24}} \mathrm{Tr}_{\mathrm{NS}} \left(y^{\frac{\hat{P}_0}{m}} q^{\hat{L}_0} \right) \qquad \hat{Z}_{\mathrm{NSR}} = q^{-\frac{1}{24}} \mathrm{Tr}_{\mathrm{NS}} \left((-1)^F y^{\frac{\hat{P}_0}{m}} q^{\hat{L}_0} \right)$$

Each character involves a specific theta function; for example

$$\hat{Z}_{\mathrm{NSR}}(z|\tau) \;=\; q^{-\frac{1}{24}} \prod_{n\geq 0} \left(1 - q^{n+\frac{1}{2}} y\right) \left(1 - q^{n+\frac{1}{2}} y^{-1}\right) = \frac{\theta_4(mz|-\tau)}{\eta(-\tau)}$$

INVARIANT COMBINATIONS

Impose three conditions:

- Invariance under S
- Invariance under T
- Positive integer coefficients

INVARIANT COMBINATIONS

$$Z_0 = \left(\hat{Z}_{\mathrm{NSNS}}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n} + \left(\hat{Z}_{\mathrm{RNS}}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n} + \left(\hat{Z}_{\mathrm{NSR}}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n}$$

No surprises: very similar to 8n chiral fermions.

INVARIANT COMBINATIONS

$$Z_0 = \left(\hat{Z}_{\rm NSNS}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n} + \left(\hat{Z}_{\rm RNS}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n} + \left(\hat{Z}_{\rm NSR}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n}$$

$$\left.\begin{aligned} Z_+ &= Z_0 + \left(\hat{Z}_{\rm RR}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n} \\ Z_- &= Z_0 - \left(\hat{Z}_{\rm RR}(\frac{z}{\sqrt{8n}}|\tau)\right)^{8n} \end{aligned}\right\}$$

No CFT counterpart!

And each of these cases can be casted in terms of Vir-KM primaries

$$\hat{Z}(z|\tau) = q^{-c/24} \sum_p e^{2\pi i z p} \frac{q^{\frac{p^2}{k}}}{\phi(q)} \sum_{\hat{h}} \chi_{\hat{h}}(q) q^{\hat{h}}$$

CONSTRAINTS

S^2: $$\hat{Z}(z|\tau) = \hat{Z}(-z|\tau) \longrightarrow$$ P_0 has a reflection symmetry.

T:
$$Z_{\bar{\ell}|\ell}(\bar{\tau} + \bar{\ell}|\tau + \ell) = Z_{\bar{\ell}|\ell}(\bar{\tau}|\tau)$$
$$\hat{Z}(z|\tau) = e^{\pi i k \frac{\vec{\ell}^2}{2}} \hat{Z}(z|\tau + 1) \longrightarrow 6\vec{\ell}^2 k + c = 24p$$

$(ST)^3$: $$\hat{Z}(z|\tau) = e^{i\pi \frac{3k}{2}} \hat{Z}(-z|\tau) \longrightarrow k = \frac{4}{3}n$$

$$\downarrow$$

$$c = 8n$$

NEW UNIVERSAL CONTRIBUTION

GEOMETRIC STRUCTURE

ENTROPY

$$S_{\bar{\ell}|\ell}(\bar{\tau}|\tau) \equiv (1 - \bar{\tau}\partial_{\bar{\tau}} - \tau\partial_\tau) \log Z_{\bar{\ell}|\ell}(\bar{\tau}|\tau)$$

High temperature regime $\tau \to -i0$

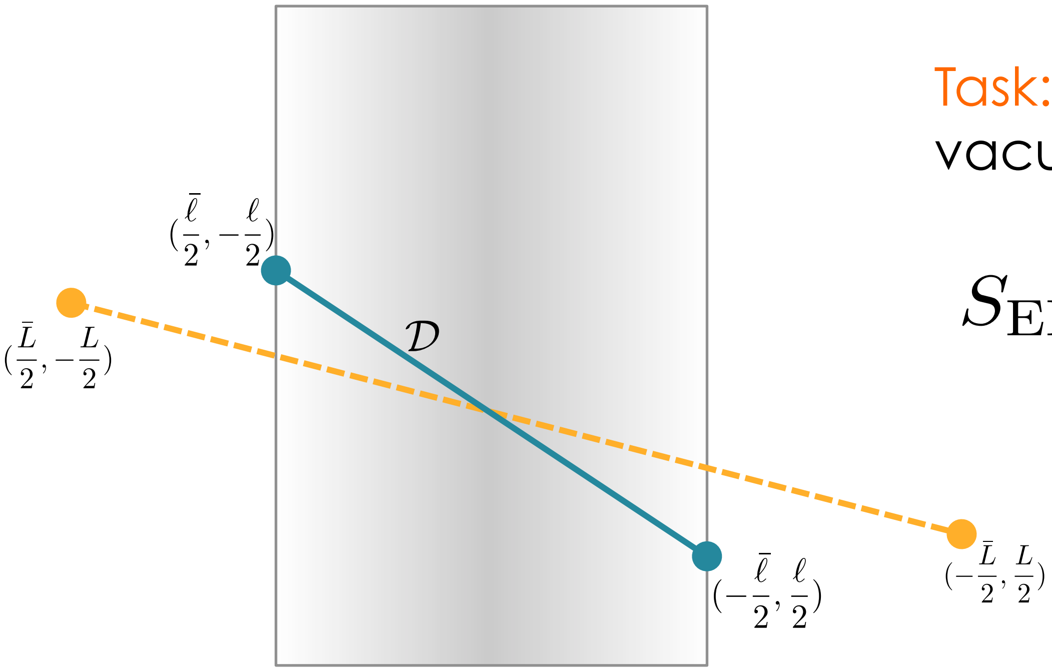
[Hartman,Hofman & Detournay, 2012]

$$S_{\bar{\ell}|\ell} = 4\pi i \frac{\hat{P}_0^* \langle \hat{P}_0 \rangle}{k} + 2\pi \sqrt{\frac{c}{6}\left(\langle L_0 \rangle - \frac{\langle P_0 \rangle^2}{k}\right)}$$

Is this piece counting states?

No contribution for the Weyl Fermion

ENTANGLEMENT ENTROPY



Task: evaluate EE for the vacuum state on the cylinder.

$$S_{\mathrm{EE}} = -\mathrm{Tr}\left(\rho_{\mathcal{D}} \log \rho_{\mathcal{D}}\right)$$

$$\boxed{S_{\mathrm{EE}} = iP_0^{\mathrm{vac}}\ell\left(\frac{\bar L}{L} - \frac{\bar\ell}{\ell}\right) - 4L_0^{\mathrm{vac}}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right)}$$

SUMMARY

- We found interesting constraints:
  1. Bound from above of angular momentum.
  2. Quantization of central charge and U(1) level.

- Implementation of modular invariance:

  Preferred frame to quantize.

- New nontrivial fermionic sector:
  1. New modular invariant combinations.
  2. What is bosonization in WCFTs?

WHAT I HAVE NOT TOLD YOU

SECRET AGENDA

HOLOGRAPHY

$$\downarrow$$

EXTREMAL (ZERO TEMPERATURE) BLACK HOLES

Global symmetries

$$SL(2,\mathbb{R}) \times U(1)$$

More concretely, WCFTs appear in holography

- Warped AdS_3 : Topological massive gravity
  [Compere & Detournay, 2008]
- Warped AdS_3 : Massive vector fields
  [Guica et al, …]
- AdS_3 Gravity: Mixed boundary conditions
  [Troessaert, 2012; Compere, Strominger & Song, 2012]
- Lower spin gravity: Newton-Cartan formulation
  [Hofman & Rollier, 2014]

CURIOSITIES

$$S_{\bar{\ell}|\ell} = 4\pi i \frac{\hat{P}_0^* \langle \hat{P}_0 \rangle}{k} + 2\pi \sqrt{\frac{c}{6}\left(\langle L_0 \rangle - \frac{\langle P_0 \rangle^2}{k} \right)}$$

These terms are generic in holographic duals to WCFTs. And imaginary!

$$S_{\mathrm{EE}} = i P_0^{\mathrm{vac}} \ell \left(\frac{\bar{L}}{L} - \frac{\bar{\ell}}{\ell} \right) - 4 L_0^{\mathrm{vac}} \log \left(\frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right)$$

Are the holographic duals intrinsically not unitary?
Is it a bug or a feature?

OPEN HOLOGRAPHIC QUESTIONS

ENTROPY: all the terms in

$$S_{\bar{\ell}|\ell} = 4\pi i \frac{\hat{P}_0^* \langle \hat{P}_0 \rangle}{k} + 2\pi \sqrt{\frac{c}{6}\left(\langle L_0 \rangle - \frac{\langle P_0 \rangle^2}{k} \right)}$$

are need to reproduce the Wald entropy of a black hole.
Is the imaginary piece counting states?

OPEN HOLOGRAPHIC QUESTIONS

Entanglement: Lower spin gravity reproduces correctly

$$S_{\mathrm{EE}} = iP_0^{\mathrm{vac}}\ell\left(\frac{\bar{L}}{L} - \frac{\bar{\ell}}{\ell}\right) - 4L_0^{\mathrm{vac}}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right)$$

But in metric formulation of WAdS_3 this remains unknown.
What are the modifications of R-T and L-M when the spacetime is not asymptotically AdS?

THANK YOU