NBIA PhD School: Neutrinos underground & in the heavens II (August 1–5, 2016). Problems

Neutrino Theory and Phenomenology

1. 3ν **oscillations in vacuum-general case**

   Given the Hamiltonian in vacuum $H_{\rm mass} = {\rm diag}(E_1, E_2, E_3) \simeq p\delta_{ij} + m_i^2/(2E)\delta_{ij}$ prove that:

   $$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j} {\rm Re}J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} {\rm Im}J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{4E}\right) \;, \quad (1)$$

   where $\Delta m^2_{ij} = m_i^2 - m_j^2$ and $J^{ij}_{\alpha\beta} = U_{\alpha i}U^\star_{\beta i}U^\star_{\alpha j}U_{\beta j}$ (Jarlskog invariant).
   Hints: Since the Hamiltonian is x-independent, the evolution operator is simply obtained by exponentiation $\widehat{S} = \exp(-i\widehat{H}x)$. In flavor basis: $S_{\beta\alpha} =< \nu_\beta|\widehat{S}|\nu_\alpha >= \sum_{ij} U_{\beta j}S_{ji}U^\star_{\alpha i} = \sum_i U^\star_{\alpha i}U_{\beta i}\exp(-\imath m_i^2 x/(2E))$. Flavor oscillation probability:
   $P(\nu_\alpha \to \nu_\beta) = |S_{\beta\alpha}|^2 = |\sum_i U^\star_{\alpha i}U_{\beta i}\exp-(\imath m_i^2 x)/(2E)|^2$.

   Prove that $P_{\alpha\beta}$ in vacuum for $(\Delta m^2 x)/(4E) \sim \mathcal{O}(1)$ and $(\delta m^2 x)/(4E) \ll 1$ (one-dominant mass-scale approximation) becomes:

   $$P_{\alpha\alpha} = 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)\sin\left(\frac{\Delta m^2 x}{4E}\right) \;, \quad (2)$$

   $$P_{\alpha\beta} = 4|U_{\alpha 3}|^2|U_{\beta 3}|^2\sin\left(\frac{\Delta m^2 x}{4E}\right) \text{ with } \alpha \neq \beta \;. \quad (3)$$

   Hints: $P_{\alpha\alpha} = 1 - 4{\rm Re}(J^{13}_{\alpha\alpha} + J^{23}_{\alpha\alpha})\sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 2{\rm Im}(J^{13}_{\alpha\alpha} + J^{23}_{\alpha\alpha})\sin\left(\frac{\Delta m^2 x}{2E}\right)$ and $P_{\alpha\beta} = 4|U_{\alpha 3}|^2|U_{\beta 3}|^2\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$. From here one can recover some probabilities for the PMNS matrix within the limit of $\theta_{13} \to 0$, which lead to the so-called *Pontecorvo formulae* for 2ν oscillations:

   $$P_{\mu\tau} \simeq \sin^2 2\theta_{23}\sin^2\left(\frac{\Delta m^2 x}{4E}\right) \;, \quad (4)$$

   $$P_{\mu\mu} \simeq 1 - \sin^2 2\theta_{23}\sin^2\left(\frac{\Delta m^2 x}{4E}\right) \;, \quad (5)$$

   and $P_{\mu e} = P_{ee} \simeq 0$.

2. 2ν **oscillations in matter with constant density**

   Let us consider the (ν_1, ν_2) states with oscillation parameters $(\delta m^2, \theta_{12}) \neq 0$ and

$\theta_{13} = 0$, prove that the survival probability in matter for N_e =const. has the same vacuum-like structure, i.e.:

$$P_{ee}^{2\nu} = 1 - \sin^2 2\tilde{\theta}_{12} \sin^2 \left(\frac{\delta \tilde{m}^2 x}{4E} \right) \tag{6}$$

with

$$\sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}} \quad \text{and} \quad \delta \tilde{m}^2 = \delta m^2 \frac{\sin 2\theta_{12}}{\sin 2\tilde{\theta}_{12}} \tag{7}$$

with $A = \pm 2\sqrt{2} G_F N_e E$ ($+$ for ν's and $-$ for $\bar{\nu}$'s).
Hints: In the 2ν limit governed by the oscillation parameters $(\delta m^2_{12}, \theta_{12})$, the Hamiltonian of ν propagation in matter is $\tilde{H} = H_{\rm vac} + {\rm diag}(V, 0)$ with $H_{\rm vac} = 1/(2E)\ U(\theta_{12}){\rm diag}(m_1^2, m_2^2)U^\star(\theta_{12})$. Extracting the part proportional to the trace of the matrix and making $\tilde{H}$ traceless, $\tilde{H}$ can be diagonalized with the following rotation $\tilde{H} = 1/(4E)\ U(\tilde{\theta}_{12}){\rm diag}(-\delta m^2, +\delta m^2)U(\tilde{\theta}_{12})^T$.
The corresponding evolution operator is

$$\tilde{S} = \exp -\imath \tilde{H} x = \tilde{U} {\rm diag}(\exp(\imath \frac{\delta \tilde{m}^2 x}{4E}), \exp(-\imath \frac{\delta \tilde{m}^2 x}{4E}))\tilde{U}^T \ . \tag{8}$$

By squaring the diagonal element of $\tilde{S}$, one gets the survival probability in matter

$$\tilde{P}_{ee}^{2\nu} = 1 - \sin^2 2\tilde{\theta}_{12} \sin \left(\frac{\delta \tilde{m}^2 x}{4E} \right) \ . \tag{9}$$

The qualitative behavior of $\tilde{m}^2_{1,2}$ and $\tilde{\theta}_{12}$ is shown in Fig. 1.

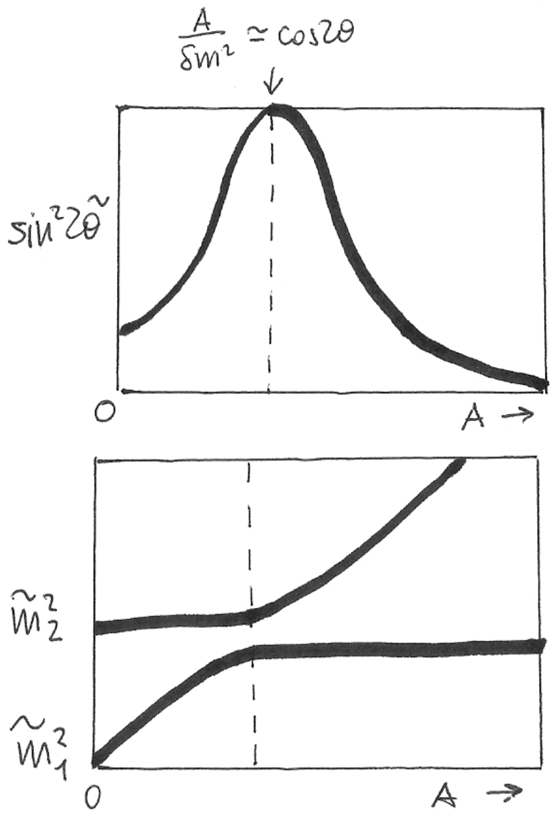

Figure 1: The *Mykheev-Smirov-Wolfenstein (MSW) resonance* occurs for $A/\delta m^2 \sim \mathcal{O}(1)$. For $A/\delta m^2 > 0$, the effective parameters have a resonant behavior around $A/\delta m^2 \simeq \cos 2\theta$ (only for ν's). For antineutrinos $A < 0$ and no resonance occurs. For $A/\delta m^2 \ll 1$ ($A/\delta m^2 \gg 1$) a vacuum-like (matter-dominated) behavior is expected.

Neutrino Cosmology

We shall need the expressions for the number density, energy density, and pressure of relativistic species in the early Universe.

$$\rho(\text{bosons}) = \frac{\pi^2 g T^4}{30} \text{ and } \rho(\text{fermions}) = \frac{7\pi^2 g T^4}{240} , \tag{10}$$

$$n(\text{bosons}) = \frac{\xi(3) g T^3}{\pi^2} \text{ and } n(\text{fermions}) = \frac{3\xi(3) g T^3}{4\pi^2} , \tag{11}$$

$$P = \rho/3 , \tag{12}$$

where $\xi(3) = 1.20206...$ is the Riemann zeta-function. g is the number of internal degrees of freedom (spin states) for the species. $g_\gamma = 2$, $g_\nu = g_{\bar{\nu}} = 1$, $g^{e^-} = g^{e^+} = 2$. NOTE: We are using natural units where $\hbar = c = k_B = 1$.

Example: The energy density and number density of a boson species in SI or cgs units (so that $[\rho] = g/cm^3$ and $[n] = cm^{-3}$) is

$$\rho = \frac{\pi^2 g (k_B T)^4}{30 c^5 \hbar^3} \text{ and } n = \frac{\xi(3) g (k_B T)^3}{\pi^2 c^3 \hbar^3} . \tag{13}$$

1. **Expansion rate in the early universe**
   The present density of non-relativistic matter ($P = 0$) in the Universe is $\Omega_M h^2 = \rho_M/\rho_c h^2 = 0.12$. The critical density is $\rho_c = 1.88 \times 10^{-29} h^2 \text{g}/\text{cm}^3$. The present photon temperature is 2.726 K. Use the equation of energy conservation

   $$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{14}$$

   which applies separately for matter and radiation to show that $\rho_M \propto a^{-3}$ and $\rho_R \propto a^{-4}$.

   Calculate the present value of ρ_γ/ρ_M.

   At which value of a did the Universe become matter dominated ($\rho_M > \rho_\gamma$), assuming that $a_0 = 1$?

   Assume that in the early Universe at $T > 1$ MeV, the energy density is completely dominated by neutrinos, photons, electrons, and positrons. There are 3 different neutrino species (electron, muon and tau neutrinos).

   Calculate $H(T)$ in the early Universe at $T = 1$ MeV (assuming that it is flat).

   Show that in a radiation dominated Universe $a \propto t^{1/2}$ and $H = 1/2t^{-1}$.

   Assuming that $a(t = 0) = 0$, calculate the age of the Universe at $T = 1$ MeV.

   Hints: See book by Dodelson and equations above. For radiation $P = \rho/3$, for matter $P = 0$. $H(1\text{MeV}) \simeq 0.6$ s^{-1}. Age of the Universe at $T = 1$ MeV: $t \simeq 0.85$ s.

2. **Neutrino decoupling**

In the early Universe, neutrinos can be created and destroyed by the process

$$\nu\bar{\nu} \rightleftharpoons e^+e^- \tag{15}$$

The thermally averaged cross section for this process is given by $<\sigma|v|>= KG_F^2T^2$, where K is a constant of order unity. Assume that $K=1$. Use the condition $\Gamma \equiv n <\sigma|v|>= H$ to calculate the decoupling temperature of neutrinos.

Hints: Use $H(1\mathrm{MeV})$ from previous exercise, $T_{\rm dec} \sim 1$ MeV.

3. **Entropy conservation in the expanding universe**
Show that in a comoving volume of the universe, the total entropy is conserved:

$$\frac{d}{dt}(a^3 s) = 0 \;\; \text{where} \;\; s = \frac{\rho + p}{T} \; . \tag{16}$$

Hints: Use the results

$$\frac{dp}{dT} = \frac{\rho + p}{T} \;\; \text{and} \;\; \frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}a^3 \; , \tag{17}$$

where the former follows from thermodynamic reasoning and the latter is obtained when deriving the second Friedmann equation.

4. **Upper bound on neutrino masses**
If neutrinos are non-relativistic today (and have masses $\ll 1$ MeV), then show that the contribution of the neutrinos to the Ω_0 (present-day density parameter) is

$$\Omega_{0,\nu} = \frac{\sum_i m_i}{50 \text{ eV}} \; , \tag{18}$$

Hints: Since we are assuming that all three neutrino species are non relativistic, the CνB density today is given by $\rho_{0,\nu} = n_{0,\nu}\sum_i m_i/3 \simeq 200\mathrm{g/cm}^3{\times}10^{-33}\sum_i(m_i/\mathrm{eV})$. The critical density today is $\rho_0 = 1.878{\times}10^{-29}\,\mathrm{h}^2$ g/cm^3 with $h = H_0/100\,\mathrm{km/sMpc} \simeq 0.73$. If we impose that the current matter in the universe is not all made by neutrinos: $\Omega_{0,\nu} \leq \Omega_{0,m} \simeq 0.25$ from which we deduce $\sum_i m_i \leq 13$ eV, i.e., any of the three neutrinos must have a mass $m_i \leq 4$ eV.

Neutrino Astronomy

Neutrino event rate.

1. Consider one of the TeV point-like sources in http://tevcat.uchicago.edu/ that you think could be a good candidate neutrino emitter. Describe the kind of source you picked up and its main features.
2. Describe relevant observations and measurements by gamma-ray experiments, for instance the spectrum and its energy range. Collect the useful documentation on the main features of the gamma emission from the source and prepare a nice documentation (e.g., a web page with the useful links to references).
3. Explain the reasons why the source can be a potential candidate neutrino emitter and why you chose it. Explain what kind of mechanism could be responsible for acceleration of cosmic rays.
4. According to the source you selected, pick up the neutrino effective area of a neutrino telescope that could see it, if you are performing a muon neutrino point-source analysis. Derive the neutrino event rate per year for the neutrino flux from that source. In order to do so, assume that all energy in γ's goes into hadronic processes and not into electromagnetic ones. Chose the neutrino production mechanism that can produce neutrinos in the source: proton-proton interaction or proton-gamma interactions and derive the neutrino flux from the measured gamma one, using average numbers as during the lectures and assuming that in the source environment gamma absorption is negligible. Also consider that the differential energy spectra are of the form $dN/dE = A_{p,\nu,\gamma}(E^{-\Gamma}/1\mathrm{TeV})\exp(-\sqrt{E/\epsilon})$ with $\epsilon =$ cut-off energy and $\Gamma_\nu \sim \Gamma_\gamma \sim \Gamma_p - 0.1$. Assume that all efficiencies (including the losses due to the pointing accuracy of the detector) are included un the effective area. Necessary inputs are provided in Figs. 2–4.

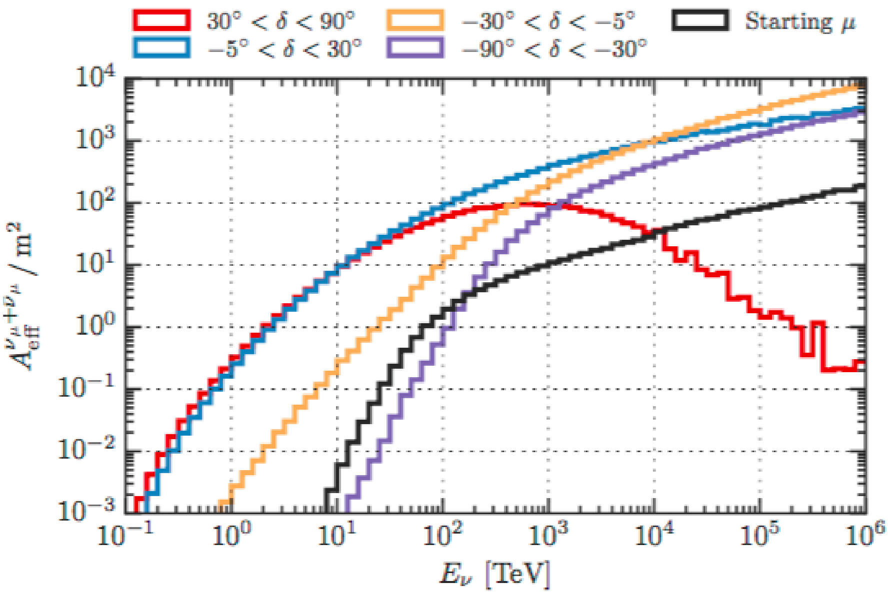

Figure 2: IceCube effective area vs neutrino energy after all cuts for point-like source searches. Ignore the starting muon area and consider the proper declination range of your selected source.

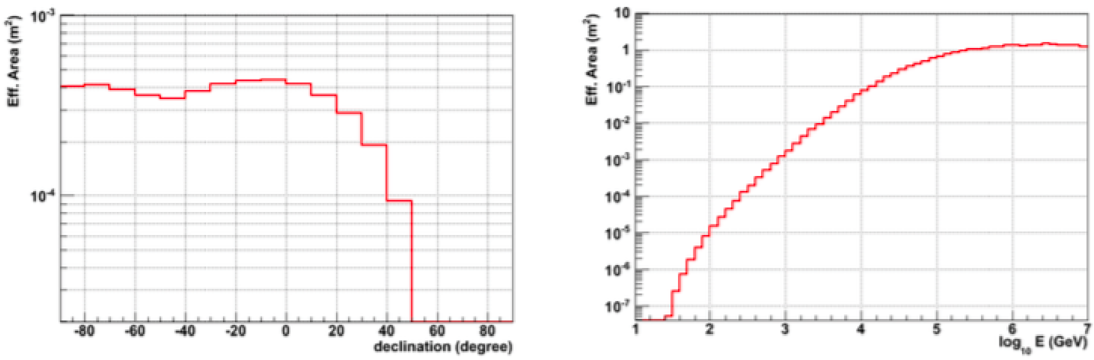

Figure 3: ANTARES effective area for E^{-2} vs declination and vs neutrino energy after all cuts for point-like source searches. For other spectra ANTARES area vs declination has quite the same shape so use the plot on the left to calculate an eventual global reduction factor to apply to the effective area vs energy when convolving with the spectrum depending on the declination of the source.

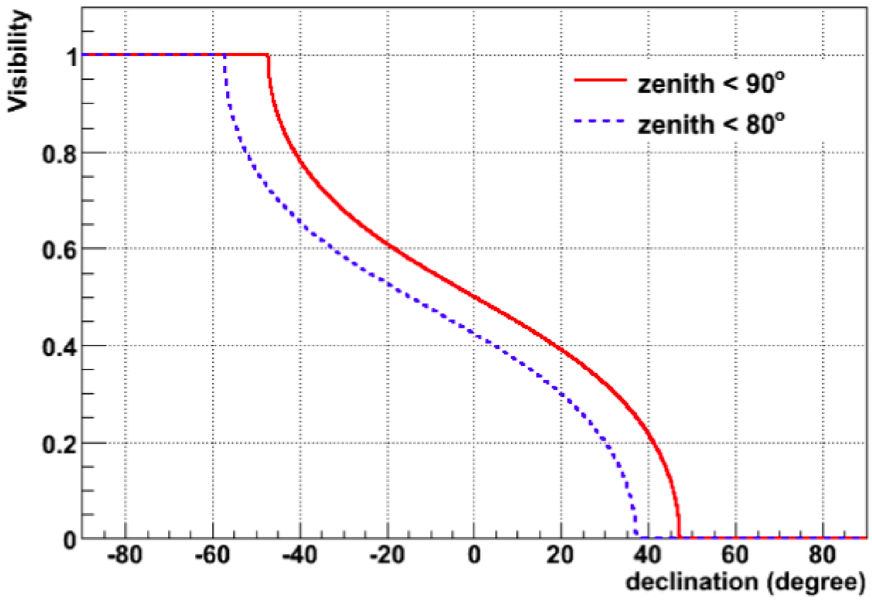

Figure 4: ANTARES visibility of a source for upgoing event analysis.