1 The four mesons

Explain why the following systems are irrelevant to flavor oscillations:

1. $B^+ - B^-$
2. $K - K^*$
3. $T - \overline{T}$ (a T is a meson made out of a t and a $\bar{u}$ quarks.)
4. $K^* - \overline{K}^*$ oscillation

2 Mixing formalism

In this question, you are asked to develop the general formalism of meson mixing.

1. Show that the mass and width differences are given by

$$4(\Delta m)^2 - (\Delta\Gamma)^2 = 4(4|M_{12}|^2 - |\Gamma_{12}|^2), \qquad \Delta m\Delta\Gamma = 4\Re(M_{12}\Gamma_{12}^*), \tag{1}$$

and that

$$\left|\frac{q}{p}\right| = \left|\frac{\Delta m - i\Delta\Gamma/2}{2M_{12} - i\Gamma_{12}}\right|. \tag{2}$$

Answer:
Define

$$H = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \tag{3}$$

From the eigenvalue equation we get

$$(a-\omega)^2 - bc = 0 \quad \Longrightarrow \quad \omega = a \pm \sqrt{bc} \quad \Longrightarrow \quad \Delta\omega = 2\sqrt{bc} \tag{4}$$

From one of the eigenfunction equations we get

$$(a-\omega)p + bq = 0 \quad \Longrightarrow \quad \frac{q}{p} = \pm\frac{\Delta\omega}{2b} \tag{5}$$

Which gives (2). Using

$$4bc = (2M_{12} - i\Gamma_{12})(2M_{12}^* - i\Gamma_{12}^*) = 4|M_{12}|^2 - |\Gamma_{12}|^2 - 4i\Re(M_{12}\Gamma_{12}^*) \tag{6}$$

and

$$(\Delta\omega)^2 = (\Delta m)^2 + \frac{(\Delta\Gamma)^2}{4} - i(\Delta m \Delta\Gamma) \tag{7}$$

Then from $(\Delta\omega)^2 = 4bc$ and comparing the real and imaginary parts we get (1).

2. When CP is a good symmetry all mass eigenstates must also be CP eigenstates. Show that CP invariance requires $|q/p| = 1$.

   Answer:

   under CP $|B\rangle \to |\bar{B}\rangle$ (up to an unphysical phase which we set to zero). Thus,

$$|B_1\rangle = \ p\,|B^0\rangle + q\,|\bar{B}^0\rangle \to q\,|B^0\rangle + p\,|\bar{B}^0\rangle \tag{8}$$

   and it is clear that in order for $|B_1\rangle$ to stay invariant under CP transformation we need $|q| = |p|$.

3. In the limit $\Gamma_{12} \ll M_{12}$ show that

$$\Delta m = 2|M_{12}|, \qquad \Delta\Gamma = 2|\Gamma_{12}|\cos\theta, \qquad \left|\frac{q}{p}\right| = 1\,, \tag{9}$$

   where $M_{12}\Gamma_{12}^* \equiv |M_{12}||\Gamma_{12}|\mathrm{e}^{i\theta}$.

   Answer:

   From (1) we can solve for Δm and $\Delta\Gamma$ and we get

$$(\Delta\Gamma)^2 = 2\left(\sqrt{a^2+b^2} - a\right), \qquad (\Delta m)^2 = \frac{1}{2}\left(\sqrt{a^2+b^2} + a\right), \tag{10}$$

   where

$$a = 4|M_{12}|^2 - |\Gamma_{12}|^2, \qquad b = 4\Re(M_{12}\Gamma_{12}^*) \tag{11}$$

   It is now clear how to take the required limit.

4. In class we derived the expressions for $\Gamma_{B^0\to f}(t)$ and $\Gamma_{\overline{B^0}\to f}(t)$ in the limit $y \ll 1, x$. Find the general formulae for these rates.
Answer:

$$\begin{aligned}
\Gamma(B^0 \to f)[t] = |A_f|^2 e^{-\tau}\Big\{&(\cosh y\tau + \cos x\tau) + |\lambda_f|^2(\cosh y\tau - \cos x\tau) \\
&-2\Re\left[\lambda_f(\sinh y\tau + i\sin x\tau)\right]\Big\}, \\
\Gamma(\bar{B} \to f)[t] = |\overline{A}_f|^2 e^{-\tau}\Big\{&(\cosh y\tau + \cos x\tau) + |\lambda_f|^{-2}(\cosh y\tau - \cos x\tau) \\
&-2\Re\left[\lambda_f^{-1}(\sinh y\tau + i\sin x\tau)\right]\Big\}.
\end{aligned} \tag{12}$$

5. Show that when $\Delta\Gamma = 0$ and $|q/p| = 1$

$$\begin{aligned}
\Gamma_{B\to X\ell^-\bar{\nu}}(t) =& \; e^{-\Gamma t}\sin^2(\Delta m t/2)\,, \\
\Gamma_{B\to X\ell^+\nu}(t) =& \; e^{-\Gamma t}\cos^2(\Delta m t/2)\,.
\end{aligned} \tag{13}$$

Answer

In the $|q/p| = 1$ case we have

$$|B\rangle = \frac{1}{\sqrt{2}}\left(|B_1\rangle + |B_2\rangle\right), \qquad |\bar{B}\rangle = \frac{1}{\sqrt{2}}\left(|B_1\rangle - |B_2\rangle\right). \tag{14}$$

The mass eigenstate evolve according to

$$a_i(t) = a_i(0)\exp[(\Gamma/2 + iM_i)t] \tag{15}$$

Since $a_i(0) = 1/\sqrt{2}$ we find

$$\begin{aligned}
P(B \to \bar{B}) = |a_1(t) - a_2(t)|^2 = e^{-\Gamma t}\sin^2(\Delta m t/2), \\
P(B \to B) = |a_1(t) + a_2(t)|^2 = e^{-\Gamma t}\cos^2(\Delta m t/2)
\end{aligned} \tag{16}$$

3 Kaon system

1. Explain why $y_K \approx 1$.

2. In a hypothetical world where we could change the mass of the kaon without changing any other masses, how would the value of y_K change if we made m_K smaller or larger.

4 Mixing beyond the SM

Consider a model without a top quark, in which the first two generations are as in the SM, while the left–handed bottom (b_L) and the right–handed bottom (b_R) are $SU(2)$ singlets.

1. Draw a tree-level diagram that contributes to $B - \bar{B}$ mixing in this model.
2. Is there a tree-level diagram that contributes to $K - \bar{K}$ mixing?
3. Is there a tree-level diagram that contributes to $D - \bar{D}$ mixing?