1 FCNC in the SM

Three unique features suppress flavor changing neutral currents in the SM: loop suppression, CKM suppression, and GIM suppression. This exercise summarizes these three ingredients.

The local symmetry group of the SM is

$$G_{\text{SM}} = SU(3)_C \times SU(2)_W \times U(1)_Y \xrightarrow{\text{EWSB}} SU(3)_C \times U(1)_{\text{EM}} \tag{1}$$

Its matter content includes three fermion generations of the following representations:

$$\begin{aligned} Q_{L_i} &= \begin{pmatrix} U_{L_i} \\ D_{L_i} \end{pmatrix} \sim (3,2)_{+1/6}\,, \quad U_{R_i} \sim (3,1)_{+2/3}\,, \quad D_{R_i} \sim (3,1)_{-1/3} \\ L_{L_i} &= \begin{pmatrix} \nu_{L_i} \\ E_{L_i} \end{pmatrix} \sim (1,2)_{-1/2}\,, \quad E_{R_i} \sim (1,1)_{-1}\,. \end{aligned} \tag{2}$$

The covariant derivative terms for each of the left and right handed fields yield interactions with the EW gauge bosons, which read:

$$\mathcal{L}_{\text{GK}} = e\bar{\psi}_a e Q \not{A} \psi_a + \frac{e}{s_W c_W} \bar{\psi}_a \left(T^3 - Q s_W^2\right) \not{Z} \psi_a + \frac{e}{\sqrt{2} s_W} \bar{\psi}_a \left(T^+ \not{W}^+ + T^- \not{W}^-\right) \psi_a\,. \tag{3}$$

Here the various T operator vanish when acting on $SU(2)_W$ singlets, and, for $SU(2)_W$ doublets, are given by

$$T^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

1. Write down the Yukawa Lagrangian of the SM and the resulting fermion mass terms after EWSB. Rotate the fermion fields to their physical mass basis and write down the interaction terms between the gauge bosons and the quarks in this basis.

2. **Loop suppression:** Explain why there are no photon mediated FCNC tree-level diagrams. Explain why there are no Z-boson mediated FCNC tree-level diagrams. Explain why there are no W-boson mediated FCNC tree-level diagrams.

3. **CKM suppression:** The W boson can mediate FCNC process at one-loop level. As an example, the $B^0 - \overline{B}^0$ mixing amplitude comes from box diagrams, dominated by intermediate top quarks. Draw these diagrams and estimate the CKM suppression they exhibit.

4. **GIM suppression:** Draw the one-loop box diagrams for $K^0 - \overline{K}^0$ mixing and show why, in the limit of $m_u = m_c = m_t$ their sum vanish.

2 SM1.5

At the late sixties the quark model of hadrons explained the spectrum of the observed mesons and baryons as $q\bar{q}$ and qqq bound states of $q = (u, d, s)$. Consider a toy model with three quark flavors, u, d and s, in the following $SU(3) \times SU(2) \times U(1)_Y$ representations:

$$\begin{aligned} &Q_L = \begin{pmatrix} U_L \\ D_L \end{pmatrix} \sim (3,2)_{+1/6}\,, \quad S_L \sim (3,1)_{-1/3}\,, \\ &U_R \sim (3,1)_{+2/3}\,, \quad D_R \sim (3,1)_{-1/3}\,, \quad S_R \sim (3,1)_{-1/3}. \end{aligned} \tag{5}$$

1. Write down the $SU(2) \times U(1)$ parts of the covariant derivatives for the different fermions in the physical basis of the gauge bosons.

2. Let the Higgs field be $\phi \sim (1,2)_{+1/2}$. Write down the bare mass terms $\mathcal{L}_\psi$ and the Yukawa terms $\mathcal{L}_{Yuk}$. Derive the mass matrix of the down-type quarks after EWSB. Rotate the fermions to the physical mass basis.

3. Write down the charged current interactions. What is the form of the "CKM" matrix that controls the flavor changing W couplings? How many physical angles and phases the model exhibits?

4. Write down the neutral current interactions. Are there flavor changing Z couplings (FCNC) in the model? Deduce the FCNC coupling if it exists in terms of θ_C and θ_W.

5. Roughly estimate the ratio

$$\frac{\Gamma(K^0 \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu_\mu)} \tag{6}$$

in this model.

6. Explain why the observation of a tiny BR($K^0 \to \mu^+\mu^-$) requires the introduction of a fourth quark, the charm. Find this BR in the PDG.

Partial answer:
Let us write the NC interactions in the gauge boson mass basis:

$$\begin{aligned}
\mathcal{L}_{NC} &= g Z_\mu J_Z^\mu + e A_\mu J_{EM}^\mu \,, \\
J_Z^\mu &= \frac{1}{\cos\theta_W} \bar{f}_i \gamma^\mu \left(T_3 - Q\sin^2\theta_W\right) f_i \,, \\
J_{EM}^\mu &= Q \bar{f}_i \gamma^\mu f_i \,,
\end{aligned} \tag{7}$$

where, for the question in hand,

f_i	T_3	Q
u_L	$1/2$	$2/3$
d_L	$-1/2$	$-1/3$
s_L	0	$-1/3$
u_R	0	$2/3$
d_R	0	$-1/3$
s_R	0	$-1/3$

Now, being agnostic about the exact form of the mass matrix, one can write the rotation of the down quarks to the mass basis, as written above:

$$\begin{pmatrix} d_R \\ s_R \end{pmatrix} = V_R^\dagger \begin{pmatrix} d'_R \\ s'_R \end{pmatrix} \,, \quad \begin{pmatrix} d_L \\ s_L \end{pmatrix} = V_L^\dagger \begin{pmatrix} d'_L \\ s'_L \end{pmatrix} \,, \tag{8}$$

where the primes indicate the fermions in the mass basis. The Z current of the down type quarks then takes the form

$$\begin{aligned}
J_Z^\mu &= \frac{1}{\cos\theta_W}\left[\begin{pmatrix}\bar{d}'_R & \bar{s}'_R\end{pmatrix} V_R \mathcal{J}_R V_R^\dagger \begin{pmatrix} d'_R \\ s'_R\end{pmatrix} + \begin{pmatrix}\bar{d}'_L & \bar{s}'_L\end{pmatrix} V_L \mathcal{J}_L V_L^\dagger \begin{pmatrix} d'_L \\ s'_L\end{pmatrix}\right] , \\
\mathcal{J}_R &= \frac{\sin^2\theta_W}{3}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} , \\
\mathcal{J}_L &= \begin{pmatrix} -\frac{1}{2}+\frac{\sin^2\theta_W}{3} & 0 \\ 0 & \frac{\sin^2\theta_W}{3}\end{pmatrix} .
\end{aligned} \tag{9}$$

Clearly,

$$V_R \mathcal{J}_R V_R^\dagger = \frac{\sin^2\theta_W}{3}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{10}$$

hence no Z mediated FCNC arise for the right handed quarks. On the other hand,

$$\begin{aligned}
V_L \mathcal{J}_L V_L^\dagger &= V_L \begin{pmatrix} -\frac{1}{2}+\frac{\sin^2\theta_W}{3} & 0 \\ 0 & \frac{\sin^2\theta_W}{3}\end{pmatrix} V_L^\dagger \\
\textit{subsection} &= V_L \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0\end{pmatrix} V_L^\dagger + \frac{\sin^2\theta_W}{3}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{\sin(2\theta_C)}{4} \\ -\frac{\sin(2\theta_C)}{4} & 0\end{pmatrix} + \begin{pmatrix} \frac{\sin^2\theta_W}{3}-\frac{\cos^2\theta_C}{2} & 0 \\ 0 & \frac{\sin^2\theta_W}{3}-\frac{\sin^2\theta_C}{2}\end{pmatrix}
\end{aligned} \tag{11}$$

where in the last equality we use

$$V_L = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C\end{pmatrix} . \tag{12}$$

We see that Z mediated FCNC arises for the left handed quarks. It's strength is $g\sin(2\theta_C)/(4\cos\theta_W)$. Now let us look at the EM interactions of the down quarks. Since the EM charge of those is $-1/3$, for both the left and right handed ones, this is given by

$$V_{L,R}\mathcal{J}_{EM}V_{L,R}^\dagger = -\frac{1}{3}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{13}$$

which of course do not raise photon mediated FCNC. This is a consequence $U(1)_{EM}$. Since it is conserved we get no mixing between different representations of this group.

3 2HDM

1. Write down the general Yukawa Lagrangian with two Higgs doublets.

2. Assuming the scalar potential respect CP, rotate the Higgs fields to the mass basis, both for the CP-odd and the CP-even particles.

3. Show that tree-level FCNC is controlled by the Yukawa couplings of the CP-odd scalar.

4 CP violation

Consider a two generation Standard Model. Write down the most general 2×2 unitary matrix and show how, by basis and field redefinitions, it contains only one real physical parameter.