

The Vlasov Equation and the Distribution Function

- distrib. function
- Vlasov/Boltzmann Eq
- Collisions
- Coll. Invariants
- Moments of B.E
- Maxwell-Boltzmann distribution + Euler Eq's
- Chapman-Enskog Theory
- viscosity

$f(x, u, t)$

$dN = \int d^3x d^3u$

$P = m \int u \text{ phase space}$

Vlasov Equation

$\begin{cases} x = x_0 + u_0 \Delta t \\ u = u_0 + a_0 \Delta t \end{cases}$

$|J| = \begin{vmatrix} 1 & \Delta t \\ \frac{\partial u}{\partial x} \Delta t & 1 \end{vmatrix}$

$|J| = 1 - \frac{\partial a}{\partial x} \Delta t^2$

$|J| \approx 1$

$dN = \int d^3x d^3u$

$d^3x_0 d^3u_0 = d^3x d^3u$

$\rightarrow \int \frac{d^3x}{\Omega} = \frac{d^3u}{\Omega}$

$f(x(u), u(t), t) = f_0$

chain rule

$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \dot{x} + \frac{\partial f}{\partial u} \cdot \dot{u} = 0$

$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + a \frac{\partial f}{\partial u} = 0$

RHS = $\left(\frac{Df}{Dt} \right)_{\text{coll}} - \left(\frac{Df}{Dt} \right)_{\text{ext}} \neq 0$

→ Boltzmann Equation collisions

Binary Elastic Collisions

Low density plasma

$M = m_1 + m_2$

$m_1 u_1 + m_2 u_2 = m_1 u_1' + m_2 u_2'$

$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$

$M \underline{V} = m_1 \underline{u}_1 + m_2 \underline{u}_2$

$\underline{a} = \underline{u}_1 - \underline{u}_2$

\underline{a}, b, ψ

$\rightarrow \underline{a}' = \gamma \underline{a} = \underline{a}, \theta$

Cross Section

$\langle Df \rangle = \dots$

Collisions

$M \underline{V} = m_1 \underline{u}_1 + m_2 \underline{u}_2$
 $\underline{v} = \underline{u}_1 - \underline{u}_2$
 $\underline{v}' = \underline{u}'_1 - \underline{u}'_2$
 $\tilde{m} = \frac{m_1 m_2}{m_1 + m_2}$
 $E = \frac{1}{2} M V^2 + \frac{1}{2} \tilde{m} v^2$
 $\|\underline{v}\| = \|\underline{v}'\|$

$\theta = \text{deflection angle}$
 $b = \text{impact parameter}$
 $\phi = \text{collision plane}$

\underline{v}, b, ϕ
 $\rightarrow \underline{v}' = \{r, \theta' = r, \phi\}$
 θ depends only on b

Cross Section
 $\left(\frac{Df}{Dt}\right)_{\text{out}} = \text{collision rate}$
 collision cylinder

 $\Delta V = b db d\phi v dt$
 $\Delta N_2 = f_2 \Delta V d^3 u_2$
 $\frac{\Delta N_2}{dt} = f_2 b db d\phi v d^3 u_2$

$\left(\frac{Df}{Dt}\right)_{\text{out}} = f_1 f_2 b db d\phi v d^3 u_2$
 Definition: $b db = \sigma \sin \theta d\theta$
 cross section
 $\left(\frac{Df}{Dt}\right)_{\text{out}} = f_1 f_2 \sigma \sin \theta d\theta v d^3 u_2$
 $\left(\frac{Df}{Dt}\right)_{\text{in}} = \int_{\mathbb{R}^3} f_2 \sigma v d^3 u_2$
 $\sigma = \sigma$
 $N' = N$
 $\int \delta^3(\underline{u}'_1 - \underline{u}'_2) d^3 u_2 = 1$

Collision Invariants

Collision Invariants

Moments of f

$$\rho(\underline{x}, t) = \int_{\mathbb{R}^3} m f d^3u$$

$$\rho \underline{v}(\underline{x}, t) = \int_{\mathbb{R}^3} m \underline{u} f d^3u$$

$$E = \int_{\mathbb{R}^3} \frac{1}{2} m u^2 f d^3u$$

$u^2 = u_x^2 + u_y^2 + u_z^2$

$\underline{v} = \underline{u}_1 - \underline{u}_2$

$$I(\underline{x}, t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Q(\underline{u}_1, \underline{u}_2) (f_1' f_2' - f_1 f_2) d^3u_1 d^3u_2$$

(1) and (2) identical rank 1 & 2

$$I(\underline{x}, t) = \frac{1}{2} \iiint (\varphi(\underline{u}_1) + \varphi(\underline{u}_2)) [\dots] d(\dots)$$

$$I(\underline{x}, t) = \frac{1}{4} \iiint [\varphi(\underline{u}_1) + \varphi(\underline{u}_2) - \varphi(\underline{u}_1') - \varphi(\underline{u}_2')] (\dots)$$

$\varphi = m$
 $\varphi = m \underline{u}$
 $\varphi = \frac{1}{2} m u^2$

$\rightarrow I(\underline{x}, t) = 0$

Taking Moments to get the Fluid Equations

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Moments of the Boltzmann Equation

$\underline{Q} = m$ $\underline{a} \cdot \frac{\partial f}{\partial \underline{x}} = \underline{\nabla} \cdot (\underline{f} \underline{u})$

$$\int_{\mathbb{R}^3} m \frac{\partial f}{\partial t} d^3 u + \int_{\mathbb{R}^3} m \underline{u} \frac{\partial f}{\partial \underline{x}} d^3 u + \int_{\mathbb{R}^3} m \underline{a} \frac{\partial f}{\partial \underline{v}} d^3 u = 0$$

($\underline{x}, \underline{u}, t$) independent variables

(1) $\frac{\partial \rho}{\partial t}$ (2) $\underline{\nabla} \cdot \left(\int_{\mathbb{R}^3} m \underline{u} f d^3 u \right) = \underline{\nabla} \cdot (\underline{p} \underline{u})$

(3) $\underline{a} \cdot \int_{\mathbb{R}^3} m \frac{\partial f}{\partial \underline{v}} d^3 u = 0$ if $f \rightarrow 0$ as $|\underline{v}| \rightarrow \infty$

(1/2) Consider

Moments through Partial Integrals

$$Q(u_x) = m u_x$$

$$\int m u_x \frac{\partial f}{\partial t} d^3u + \int m u_x \underline{u} \cdot \frac{\partial f}{\partial \underline{x}} d^3u + \int m u_x \underline{u} \cdot \frac{\partial f}{\partial \underline{u}} d^3u = 0$$

①
②
③

① $\frac{\partial}{\partial t} (\rho u_x)$ $\underline{u} =$ thermal velocity

② $u_x = v_x + w_x$ $\int m w_x f d^3u = 0$

$$\underline{u} \times \underline{u} \cdot \frac{\partial f}{\partial \underline{x}} = \underline{\nabla} \cdot (\underline{u} \times f \underline{u})$$

$$\underline{\nabla} \cdot \left(\int m u_x f \underline{u} \cdot d^3u \right)$$

Moments through Partial Integrals

$$\underline{M} + \underline{M} = \underline{v} \times \underline{v} + \underline{w} \times \underline{v} + \underline{v} \times \underline{w} + \underline{w} \times \underline{w}$$

$$= 0$$

$$\underline{\nabla} \cdot (\rho \underline{v} \otimes \underline{v} + 0 + 0 + \underline{P}) \quad (3) \quad \text{integration by parts}$$

$$P_{ij} = \int m w_i w_j d^3 u$$

Pressure tensor

$$\frac{\partial}{\partial t} (\rho \underline{v}) + \underline{\nabla} \cdot (\rho \underline{v} \otimes \underline{v} + \underline{P}) = \rho \underline{a}$$

Moments through Partial Integrals

$$E = \int \frac{1}{2} m u^2 f d^3 u = \frac{1}{2} \rho v^2 + e$$

$$e = \int \frac{1}{2} m w^2 f d^3 u$$

internal energy

$w^2 = w_x^2 + w_y^2 + w_z^2$

$$e = \frac{3}{2} \text{Tr } \underline{\underline{P}}$$

$$\underline{\underline{Q}} = \int \frac{m w w^2}{2} f d^3 u$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\underline{\underline{E}} \underline{\underline{v}} + \underline{\underline{P}} \cdot \underline{\underline{v}} + \underline{\underline{Q}}) = \underline{\underline{\rho}} \cdot \underline{\underline{a}}$$

The Maxwell-Boltzmann Distribution

parts

$$f_0 = \frac{\rho/m}{(\pi)^{3/2}} e^{-\frac{1}{2} \frac{(u-u_0)^2}{\sigma^2}}$$

$$\underline{P} = \underline{P \underline{1}}$$

$$\underline{Q} = 0$$

↳ Euler Equ.

$$\underline{P} = \underline{P a}$$