Theory 2: MHD and multi-fluid dynamics

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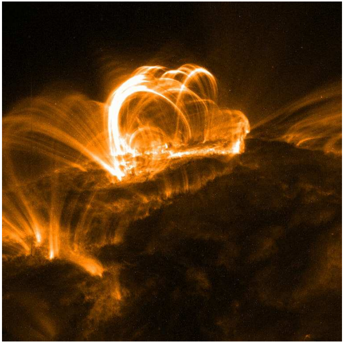
Overview

- Motivation
- Re-capitulate the Euler equations
- Write down ideal MHD equations by analogy
- Euler equations as basis for "deriving" multi-fluid equations
- The Generalised Ohm's Law
  - Weakly ionised approximation
  - Arbirarily ionised plasma
  - Fully ionised approximation

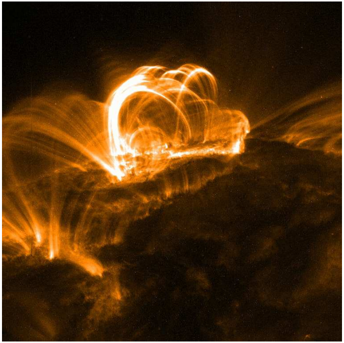
Motivation



Credit: NASA/LMSA

- Astrophysicists are used to thinking of ideal MHD
- Single fluid approximation
- Flux freezing (i.e. infinite conductivity)

Motivation



Credit: NASA/LMSA

Seems to make intuitive sense for fully ionized plasmas

Motivation

Credit: NASA, ESA, STScI, J. Hester and P. Scowen (Arizona State University)

- Arguably makes sense for weakly ionized plasmas (sometimes)

Motivation

The main assumptions of MHD are

- The plasma is locally electrically neutral
- The plasma acts as a single fluid (mean free path is "short")
- Ohm's law can be used (for ideal MHD assume infinite conductivity)
- The displacement current can be neglected (for non-relativistic MHD)

Awkward systems

Credit: NASA, ESA, STScI, J. Hester and P. Scowen (Arizona State University)

- Molecular clouds:
  - Low ionization fraction
  - Multiple species present in system
  - Turbulent

Awkward systems

Credit: J. Bally (University of Colorado) and H. Throop (SWRI)

- Proto-planetary (accretion) disks:
  - “Small”
  - Weakly ionized
  - Turbulent

Pulsars

Credit: X-ray Image: NASA/CXC/ASU/J. Hester et al. Optical Image: NASA/HST/ASU/J. Hester et al.

- Pulsars:
  - “Small”
  - Significant differences in velocities of charged species

The Euler equations

For inviscid fluids our conservation laws can be written:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P\hat{\mathbf{I}}) = 0, \tag{2}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot [(e + P)\mathbf{u}] = 0. \tag{3}$$

The MHD equations

For "ideal" MHD we assume $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{4}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P^* \hat{\mathbf{I}} - \mathbf{B}\mathbf{B}) = 0, \tag{5}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot [(e + P)\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}] = 0. \tag{6}$$

where

$$e = 0.5\rho u^2 + \frac{P}{\gamma - 1} + 0.5B^2,$$

$$P^* = P + 0.5B^2.$$

Flux freezing

The magnetic flux crossing a contour, S, is

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}, \tag{7}$$

the rate of change of which can be calculated by taking account of the change in **B** and in C with time:

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_S \mathbf{B} \cdot \mathbf{u} \times d\mathbf{l}, \\ &= -\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} + \int_S (\mathbf{u} \times B) \cdot d\mathbf{l}, \\ &= -\int_S \nabla \times \{\mathbf{E} + \mathbf{u} \times \mathbf{B}\} \cdot d\mathbf{S}. \end{aligned} \tag{8}$$

Multiple species

Now consider:

- Multiple species, labelled by a, I and possibly E
- Each species can be considered a fluid
- Species can be coupled via collisions
- Can write our conservation laws for each of these species individually

Multiple species

Each species obeys:

$$\frac{\partial \rho_{al}}{\partial t} + \nabla \cdot (\rho_{al} \mathbf{u}_{al}) = S_{al}, \tag{9}$$

$$\frac{\partial \rho_{al} \mathbf{u}_{al}}{\partial t} + \nabla \cdot (\rho_{al} \mathbf{u}_{al} \mathbf{u}_{al} + \hat{\mathbf{p}}_{al}) = \mathbf{F}_{al} + \mathbf{R}_{al}, \tag{10}$$

$$\frac{\partial e_{al}}{\partial t} + \nabla \cdot [(e_{al} + \hat{\mathbf{p}}_{al}) \mathbf{u}_{al}] = \mathbf{F}_{al} \mathbf{u}_{al} + \frac{E_{al}}{m_a} S_{al} + M_{al}. \tag{11}$$

Multiple species

The term $\mathbf{F}_{al}$ includes all terms which change the momentum of the species. We'll only consider the electromagnetic forces:

$$\mathbf{F}_{al} = \rho_{al}\alpha_{al}\left(\mathbf{E} + \mathbf{u}_{al} \times \mathbf{B}\right) \tag{12}$$

Multiple species

Before digging deeper let's simplify our notation:

Note

By a species we mean a grouping of particles which can be described as a fluid and, in particular, in which all particles have the same charge-to-mass ratio and collision coefficients

Multiple species

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = S_i, \tag{13}$$

$$\frac{\partial \rho_i \mathbf{u}_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i \mathbf{u}_i + \hat{\mathbf{p}}_i) = \mathbf{F}_i + \mathbf{R}_i, \tag{14}$$

$$\frac{\partial e_i}{\partial t} + \nabla \cdot [(e_i + \hat{\mathbf{p}}_i)\mathbf{u}_i] = \mathbf{F}_i \mathbf{u}_i + \frac{E_i}{m_i} S_i + M_i. \tag{15}$$

Multiple species

And so

$$\mathbf{F}_i = \rho_i \alpha_i \left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B} \right), \tag{16}$$

and we have also defined

$$\hat{\mathbf{p}}_i \equiv \rho_i \left\langle \mathbf{c}_i \mathbf{c}_i \right\rangle, \tag{17}$$

is the tensor pressure. Note the definition of the tensor pressure indicates the reference frame is different for each component.

Collisions

The impact of collisions between species is contained in $\mathbf{R}_i$. In the case of elastic collisions

$$\mathbf{R}_i^{\mathrm{el}} = \sum_j \rho_i \rho_j \nu_{ij} (\mathbf{u}_i - \mathbf{u}_j), \tag{18}$$

where we require $\nu_{ij} = \nu_{ji}$ and in general $\nu_{ij} \equiv \nu_{ij}(\mathbf{u}_i - \mathbf{u}_j)$.
For inelastic collisions (chemical reactions, radiative collisions ...)

$$\mathbf{R}_i^{\mathrm{inel}} = \sum_j \rho_j P_j \mathbf{u}_j - \rho_i P_i \mathbf{u}_i, \tag{19}$$

where each term in the sum denotes momentum which moved from species j to species i as a result of the collision.

The induction equation

The induction equation is derived from

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{20}$$

and the next issue is to find **E**. For ideal MHD we assume

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}, \tag{21}$$

so if we're in the instantaneous rest frame of the fluid (i.e. $\mathbf{u} = 0$), there is no electric field.

The induction equation

Once we make this assumption we immediately find

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot [\mathbf{uB} - \mathbf{Bu}] = 0. \tag{22}$$

More realistically, the right hand side should be $\mathbf{E}'$, the electric field in the frame of the fluid.

Some questions

Questions

- What, exactly, is “the fluid”?
- How do we find $\mathbf{E}'$?

Answers

- There is no such thing in multi-fluid MHD, but we can define a **u**.
- First define **u** and then see the next slides ...

The Generalised Ohm's Law

In ideal MHD we have a single fluid system with the magnetic field frozen in:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot [\mathbf{uB} - \mathbf{Bu}] = 0,$$

and clearly if we have a multi-fluid system, with necessarily some averaged **u**, then we must have extra terms to account for the possibility that the field can no longer be transported by the vector field **u**.

Calculating $\mathbf{E}'$

Let us start by re-writing our equations of motion as

$$\alpha_i \rho_i (\mathbf{E}' + \mathbf{u}_i \times \mathbf{B}) = \nabla p_i + \rho_i \frac{D\mathbf{u}_i}{Dt} - \mathbf{R}_i. \tag{23}$$

where we have now assumed an inviscid flow with an isotropic pressure for all fluids. We can use these equations to determine $\mathbf{E}'$.

Notes

- We have not assumed much about the various fluids in the system

Weakly ionised plasma

System

- N fluids in our system labelled $i = 1, \ldots, n$, $i = 1$ is the neutral fluid
- Centre of mass velocity is the velocity of the neutrals (defines $\mathbf{u}$)
- Inertia of the charged species is negligible
- Collisions between different charged species are negligible

Weakly ionised plasmas

Our equations of motion

$$\alpha_i \rho_i (\mathbf{E}' + \mathbf{u}_i \times \mathbf{B}) = \nabla p_i + \rho_i \frac{D\mathbf{u}_i}{Dt} - \mathbf{R}_i, \tag{24}$$

reduce to

$$\alpha_i \rho_i (\mathbf{E}' + \mathbf{u}_i \times \mathbf{B}) + \mathbf{R}_i = 0, \tag{25}$$

where, now,

$$\mathbf{R}_i = \{\rho_i \rho_1 \nu_{i1} (\mathbf{u}_i - \mathbf{u}_1)\} + \{\rho_1 P_1 \mathbf{u}_1 - \rho_i P_i \mathbf{u}_i\}\,. \tag{26}$$

Weakly ionised plasmas

A useful quantity is the Hall parameter:

$$\beta_i \equiv \frac{\alpha_i B}{\nu_{1i}\rho_1}, \tag{27}$$

and is a measure of how well tied to the field a species is. Moving to the rest frame of the neutrals, our equations of motion become

$$\begin{aligned} 0 &= \alpha_i\rho_i(\mathbf{E}' + \mathbf{u}_i' \times \mathbf{B}) + \rho_i\rho_1\nu_{i1}(-\mathbf{u}') \\ &= \alpha_i\rho_i(\mathbf{E}' + \mathbf{u}_i' \times \mathbf{B}) - \frac{B}{\beta_i}(\alpha_i\rho_i\mathbf{u}_i'). \end{aligned} \tag{28}$$

This condition simplifies our search for $\mathbf{E}' \equiv \mathbf{E}'(\mathbf{J})$.

Weakly ionised plasmas

To proceed, define a coordinate system such that the z axis is parallel to **B**, the y axis is defined such that **B** lies entirely in the yz-plane, and the x axis is perpendicular to both y and z. Then our equations of motion become:

$$0 = \alpha_i \rho_i B(u_i')_y - \frac{B}{\beta_i}\alpha_i \rho_i (u_i')_x, \tag{29}$$

$$0 = \alpha_i \rho_i (E_\perp' - B(u_i')_x - \frac{B}{\beta_i}\alpha_i \rho_i (u_i')_y, \tag{30}$$

$$0 = \alpha_i \rho_i E_\parallel' - \frac{B}{\beta_i}\alpha_i \rho_i (u_i')_z, \tag{31}$$

Weakly ionised plasmas

or

$$\alpha_i \rho_i (u_i')x = \frac{1}{B} \frac{\alpha_i \rho_i \beta_i^2}{(1 + \beta_i^2)} E_\perp', \tag{32}$$

$$\alpha_i \rho_i (u_i')y = \frac{1}{B} \frac{\alpha_i \rho_i \beta_i}{(1 + \beta_i^2)} E_\perp', \tag{33}$$

$$\alpha_i \rho_i (u_i')z = \frac{1}{B} \alpha_i \rho_i \beta_i E_\parallel'. \tag{34}$$

Weakly ionised plasmas

Summing over i:

$$J_x = \sum_{i=1}^{N} \alpha_i \rho_i (u_i')x = \frac{1}{B} \sum_{i=1}^{N} \frac{\alpha_i \rho_i \beta_i^2}{(1+\beta_i^2)} E_\perp' = \sigma_{\mathrm{H}} E_\perp', \tag{35}$$

$$J_y = \sum_{i=1}^{N} \alpha_i \rho_i (u_i')y = \frac{1}{B} \sum_{i=1}^{N} \frac{\alpha_i \rho_i \beta_i}{(1+\beta_i^2)} E_\perp' = \sigma_\perp E_\perp', \tag{36}$$

$$J_z = \alpha_i \rho_i (u_i')z = \frac{1}{B} \sum_{i=1}^{N} \alpha_i \rho_i \beta_i E_\parallel' = \sigma_\parallel E_\parallel', \tag{37}$$

and so

$$J_\alpha = \sigma_{\alpha\beta} E_\beta' = \begin{pmatrix} \sigma_\perp & \sigma_{\mathrm{H}} & 0 \\ -\sigma_{\mathrm{H}} & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \mathbf{E}'. \tag{38}$$

Weakly ionised plasmas

Inverting this relation gives our generalised Ohm's law for weakly ionised plasmas:

$$\mathbf{E}' = r_0 \frac{(\mathbf{J} \cdot \mathbf{B})\mathbf{B}}{B^2} + r_1 \frac{\mathbf{J} \times \mathbf{B}}{B} - r_2 \frac{(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}}{B^2}, \quad (39)$$

where

$$r_0 = \frac{1}{\sigma_\parallel}, \quad (40)$$

$$r_1 = \frac{\sigma_{\rm H}}{\sigma_\perp^2 + \sigma_{\rm H}^2}, \quad (41)$$

$$r_2 = \frac{\sigma_\perp}{\sigma_\perp^2 + \sigma_{\rm H}^2}. \quad (42)$$

The induction equation ...

So our induction equation for a weakly ionised plasma is

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot [\mathbf{uB} - \mathbf{Bu}] = -\nabla \times \left\{ r_0 \frac{(\mathbf{J} \cdot \mathbf{B})\mathbf{B}}{B^2} + r_1 \frac{\mathbf{J} \times \mathbf{B}}{B} - r_2 \frac{(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}}{B^2} \right\}. \tag{43}$$

Arbitrarily ionised plasma

Proceeding as before yields

$$\begin{aligned}\mathbf{E}' &= r_0\frac{(\mathbf{J}\cdot\mathbf{B})\mathbf{B}}{B^2} - r_1\frac{\mathbf{B}\times\mathbf{J}}{B} + r_2\frac{\mathbf{B}\times(\mathbf{J}\times\mathbf{B})}{B^2} \\ &\quad + r_0'\frac{(\nabla p\cdot\mathbf{B})\mathbf{B}}{B^2} - r_1'\frac{\mathbf{B}\times\nabla p}{B} + r_2'\frac{\mathbf{B}\times(\nabla p\times\mathbf{B})}{B^2} \\ &\quad + \frac{(\vec{r_0''}\cdot\mathbf{B})\mathbf{B}}{B^2} - \frac{\mathbf{B}\times\vec{r_1''}}{B} + \frac{\mathbf{B}\times(\vec{r_2''}\times\mathbf{B})}{B^2} \qquad (44)\end{aligned}$$

where

$$r_0 = \frac{1}{\sigma_\parallel}; \quad r_1 = \frac{\sigma_\wedge}{(\sigma_\perp^2+\sigma_\wedge^2)}; \quad r_2 = \frac{\sigma_\perp}{(\sigma_\perp^2+\sigma_\wedge^2)} \qquad (45)$$

$$\begin{aligned} r_0' = \frac{\sigma_\parallel'}{\sigma_\parallel} = r_0\sigma_\parallel'; \quad r_1' &= \frac{\sigma_\wedge\sigma_\perp' - \sigma_\perp\sigma_\wedge'}{(\sigma_\perp^2+\sigma_\wedge^2)} = r_1\sigma_\perp' - r_2\sigma_\wedge'; \\ r_2' &= \frac{\sigma_\perp\sigma_\perp' + \sigma_\wedge\sigma_\wedge'}{(\sigma_\perp^2+\sigma_\wedge^2)} = r_1\sigma_\wedge' + r_2\sigma_\perp'. \qquad (46)\end{aligned}$$

Arbitrarily ionized Ohm's law

And

$$\vec{r_0''} = (r_0'')_x\hat{\imath} + (r_0'')_y\hat{\jmath} + (r_0'')_z\hat{k} \tag{47}$$

$$\text{where } (r_0'')_{x,y,z} = r_0 \sum_j \sigma''_{j\|} \left(\frac{d_j \mathbf{v}_j}{dt}\right)_{x,y,z}$$

$$\vec{r_1''} = (r_1'')_x\hat{\imath} + (r_1'')_y\hat{\jmath} + (r_1'')_z\hat{k} \tag{48}$$

$$\text{where } (r_1'')_{x,y,z} = r_1 \sum_j \sigma''_{j\perp} \left(\frac{d_j \mathbf{v}_j}{dt}\right)_{x,y,z} - r_2 \sum_j \sigma''_{j\wedge} \left(\frac{d_j \mathbf{v}_j}{dt}\right)_{x,y,z}$$

$$\vec{r_2''} = (r_2'')_x\hat{\imath} + (r_2'')_y\hat{\jmath} + (r_2'')_z\hat{k} \tag{49}$$

$$\text{where } (r_2'')_{x,y,z} = r_1 \sum_j \sigma''_{j\wedge} \left(\frac{d_j \mathbf{v}_j}{dt}\right)_{x,y,z} + r_2 \sum_j \sigma''_{j\perp} \left(\frac{d_j \mathbf{v}_j}{dt}\right)_{x,y,z}$$

Fully ionized Ohm's Law

If we assume full ionization, and two fluids, then we find

$$\begin{aligned}\mathbf{E} &= -\frac{1}{c}\mathbf{v}\times\mathbf{B} + r_0\frac{(\mathbf{J}\cdot\mathbf{B})\mathbf{B}}{B^2} - r_1\frac{\mathbf{J}\times\mathbf{B}}{B} + r_2\frac{\mathbf{B}\times(\mathbf{J}\times\mathbf{B})}{B^2} \\ &+ s_0\frac{(\nabla p\cdot\mathbf{B})\mathbf{B}}{B^2} - s_1\frac{\nabla p\times\mathbf{B}}{B} + s_2\frac{\mathbf{B}\times(\nabla p\times\mathbf{B})}{B^2} \\ &+ \frac{(\vec{t_0}\cdot\mathbf{B})\mathbf{B}}{B^2} - \frac{\vec{t_1}\times\mathbf{B}}{B} + \frac{\mathbf{B}\times(\vec{t_2}\times\mathbf{B})}{B^2} \qquad (50)\end{aligned}$$

Fully ionized Ohm's Law

where

$$r_0 = \frac{1}{\sigma_{\|}} \, , \, r_1 = \frac{\sigma_H}{\sigma_\perp^2 + \sigma_H^2} \, , \text{and } r_2 = \frac{\sigma_\perp}{\sigma_\perp^2 + \sigma_H^2} \tag{51}$$

and

$$s_0 = r_0 a_{\|} \, , \, s_1 = r_1 a_\perp - r_2 a_H \, , \text{and } s_2 = r_1 a_H + r_2 a_\perp \tag{52}$$

and

$$t_0 = r_0 (b_{\|})_z \, , \, (t_1)_x = r_1 (b_\perp)_x - r_2 (b_H)_x \, , \, (t_1)_y = r_1 (b_\perp)_y - r_2 (b_H)_y$$

$$(t_2)_x = r_1 (b_H)_x + r_2 (b_\perp)_x \, , \, (t_2)_y = r_1 (b_H)_y + r_2 (b_\perp)_y \tag{53}$$

where a's are similar to the σ's and the b's involve accelerations of charged species (neglected if plasma made up of ions and electrons).

Overview

Multi-fluid MHD

- Require each species to be a fluid
- Require a (useful) generalised Ohm's law
- Ohm's law derived from equations of motion of charged fluids
- Usually need some approximations to make progress

Overview

Single fluid MHD

- Can (arbitrarily) include terms in induction equation
- No need for generalised Ohm's law
- Limited applicability if equilibrium conditions not present

Suggested further reading

- Pelletier (2007) - nice derivation of MHD
- Cowling (1957) - Generalised Ohm's Law
- Jones (2011, PhD thesis) - derivation of arbitrarily & fully ionised Ohm's Law
- Ballester et al (2017) - review of partially ionised plasmas in astro

There are many more works, but this represents a good start!