Waves, Instabilities, and Shocks

NBIA Summer School on Astrophysical Plasmas

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1 Sound Waves

Let us first derive the equations describing sound waves starting from the continuity and momentum equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{v}) = 0 \ , \tag{1}$$

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} \right] = -\boldsymbol{\nabla} P \, . \tag{2}$$

We consider a homogeneous, time-independent equilibrium background with density ρ_0, pressure P_0 and velocity $\boldsymbol{v}_0 = 0$ and assume small amplitude perturbations given by $\delta\rho$, δP and $\delta\boldsymbol{v}$, such that

$$\rho = \rho_0 + \delta\rho \ , \tag{3}$$

$$P = P_0 + \delta P \ , \tag{4}$$

$$\boldsymbol{v} = \delta\boldsymbol{v} \ . \tag{5}$$

Substituting these into the continuity and momentum equations, we find

$$\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \delta\rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho_0 \delta\boldsymbol{v} + \cancel{\delta\rho\delta\boldsymbol{v}}) = 0 \ , \tag{6}$$

$$(\rho_0 + \cancel{\delta\rho}) \left[\frac{\partial \delta\boldsymbol{v}}{\partial t} + \cancel{(\delta\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\boldsymbol{v})} \right] = \cancel{-\boldsymbol{\nabla} P_0} - \boldsymbol{\nabla}\delta P \ \ . \tag{7}$$

We have crossed out all the terms that can be ignored because they are either identically zero or negligible when the perturbations are small, i.e., $\delta P/P_0 \ll 1$, $\delta\rho/\rho_0 \ll 1$.

The *linearized* equations for the perturbations are then

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla}\boldsymbol{\cdot}\delta\boldsymbol{v} = 0 \, , \tag{8}$$

$$\rho_0 \frac{\partial \delta\boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\delta P \ . \tag{9}$$

Let us now differentiate the equation for $\delta\rho$ with respect to time. We find

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \boldsymbol{\nabla}\boldsymbol{\cdot}\frac{\partial \delta\boldsymbol{v}}{\partial t} = 0 \, . \tag{10}$$

We can then use the linearized momentum equation to substitute for $\partial\delta\boldsymbol{v}/\partial t$, yielding

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \boldsymbol{\nabla}^2 \delta P \ . \tag{11}$$

For an isothermal gas, the fluctuations in pressure and density are related by

$$\delta P = c_{\mathrm{s}}^2 \delta\rho \ , \tag{12}$$

[1]Adapted from NBI Theoretical Astrophysics course, with contributions from O. Gressel and G. Murphy.

where c_s has a constant value. This implies that

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_\mathrm{s}^2 \boldsymbol{\nabla}^2 \delta\rho \ . \tag{13}$$

This is a wave equation for the density perturbation. The solution of this equation is a sound wave that propagates with speed c_s. Let us take a perturbation of the form

$$\delta\rho = A\cos(kx - \omega t) \ , \tag{14}$$

where A is the amplitude. So that the one dimensional wave equation

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_\mathrm{s}^2 \frac{\partial^2 \delta\rho}{\partial x^2} \ , \tag{15}$$

gives

$$-A\omega^2 \cos(kx - \omega t) = -Ak^2 c_\mathrm{s}^2 \cos(kx - \omega t) \ . \tag{16}$$

Simplifying we find the dispersion relation for sound waves

$$\omega^2 = c_\mathrm{s}^2 k^2 \ . \tag{17}$$

The dispersion relation is a relation between the scale of the perturbation and its frequency.

2 Linear Mode Analysis for Sound Waves

Let us start with the linearized equations (8) and (9) but specializing them to one spatial dimension, x, and setting $\delta P = c_\mathrm{s}^2 \delta\rho$. In this case we have

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial \delta v_x}{\partial x} = 0 \ , \tag{18}$$

$$\rho_0 \frac{\partial \delta v_x}{\partial t} + c_\mathrm{s}^2 \frac{\partial \delta\rho}{\partial x} = 0 \ . \tag{19}$$

Before proceeding we make these equations dimensionless by using the background value, ρ_0, the characteristic scale L, the soundspeed c_s and the characteristic time $t_0 = L/c_\mathrm{s}$. In terms of the dimensionless parameters

$$\delta\rho' = \frac{\delta\rho}{\rho_0} \ , \qquad x' = \frac{x}{L} \ , \qquad \delta v_x' = \frac{\delta v_x}{c_\mathrm{s}} \ , \qquad t' = \frac{t}{L/c_\mathrm{s}} \ , \tag{20}$$

the equations for the perturbations then become

$$\frac{\partial \delta\rho'}{\partial t'} + \frac{\partial \delta v_x'}{\partial x'} = 0 \ , \tag{21}$$

$$\frac{\partial \delta v_x'}{\partial t'} + \frac{\partial \delta\rho'}{\partial x'} = 0 \ . \tag{22}$$

We drop the primes on the dimensionless variables in the following. The main idea of a linear mode analysis is to now assume that the perturbations depend on space and time as

$$\delta\rho(x,\, t) = \sum_k \delta\rho_k(t) e^{ikx} \qquad \text{and} \qquad \delta v_x(x,\, t) = \sum_k \delta v_k(t) e^{ikx} \,. \tag{23}$$

where k is the wavenumber. Substituting this into our linearized equations we obtain

$$\sum_k \left[\frac{\partial \delta\rho_k(t)}{\partial t} + ik\delta v_k(t)\right] e^{ikx} = 0 \qquad \text{and} \qquad \sum_k \left[\frac{\partial \delta v_k(t)}{\partial t} + ik\delta\rho_k(t)\right] e^{ikx} = 0 \,. \tag{24}$$

Because the complex exponentials are orthogonal functions, each of the terms in the previous sums must vanish individually. This can be written in matrix form as

$$\frac{\partial}{\partial t}\begin{pmatrix} \delta\rho_k(t) \\ \delta v_k(t) \end{pmatrix} = \begin{pmatrix} 0 & -ik \\ -ik & 0 \end{pmatrix}\begin{pmatrix} \delta\rho_k(t) \\ \delta v_k(t) \end{pmatrix}. \tag{25}$$

This matrix equation can be written as

$$\frac{\partial}{\partial t}\boldsymbol{a}_k(t) = M\boldsymbol{a}_k(t)\,. \tag{26}$$

In order to solve this matrix equation, it is convenient to work in the basis of eigenvectors in which the matrix is diagonal. In this basis, the action of M over the set of eigenvectors is equivalent to a scalar multiplication, i.e.,

$$M_{\rm diag}\boldsymbol{e}_{k,j}(t) = \sigma_j\,\boldsymbol{e}_{k,j}(t)\,, \qquad \text{for} \quad j = 1, 2, \tag{27}$$

where $\boldsymbol{e}_{k,j}(t)$ are the eigenvectors associated with the matrix M and σ_j are the roots of the characteristic polynomial associated with the matrix M and are in general complex scalars.

The advantage of working in this basis is that the equations are decoupled, i.e.,

$$\frac{\partial\delta\rho_k(t)}{\partial t} = \sigma_j\delta\rho_k(t)\,, \qquad \text{and} \qquad \frac{\partial\delta v_k(t)}{\partial t} = \sigma_j\delta v_k(t)\,, \tag{28}$$

and the solutions are exponentials

$$\delta\rho_k(t) = \delta\rho_k(0)\,e^{\sigma_j t}\,, \qquad \text{and} \qquad \delta v_k(t) = \delta v_k(0)\,e^{\sigma_j t}\,. \tag{29}$$

In order to find the eigenvalues σ_j, we need to find σ_j such that

$$\begin{pmatrix} 0 & -ik \\ -ik & 0 \end{pmatrix}\begin{pmatrix} \delta\rho_k(t) \\ \delta v_k(t) \end{pmatrix} = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}\begin{pmatrix} \delta\rho_k(t) \\ \delta v_k(t) \end{pmatrix}, \tag{30}$$

or, moving everything to the LHS,

$$\begin{pmatrix} \sigma_j & ik \\ ik & \sigma_j \end{pmatrix}\begin{pmatrix} \delta\rho_k(t) \\ \delta v_k(t) \end{pmatrix} = 0\ . \tag{31}$$

Recall from linear algebra that such a linear, homogeneous system only has non trivial solutions (that is a solution that is not just $\delta v_k = \delta\rho_k = 0$) if the determinant of the matrix is zero. This is equivalent to finding the roots of the characteristic polynomial, in this case,

$$\sigma_j^2 + k^2 = 0\ , \tag{32}$$

which has solutions

$$\sigma_j = \pm ik\,. \tag{33}$$

If only one mode k is excited, and we restore the physical dimensions, e.g. $\sigma = kc_{\rm s}$, then the perturbations in real space become

$$\delta\rho(x,\,t) = \delta\rho_{-k}(0)e^{-(ikx\pm i\sigma t)} + \delta\rho_k(0)e^{(ikx\pm i\sigma t)}\,. \tag{34}$$

Since the density is a real function, we must have $\rho_{-k} = \rho_k^\star$, where $\star$ stands for complex conjugate, and

$$\cos\theta \equiv \frac{e^{-i\theta} + e^{-i\theta}}{2}\,, \tag{35}$$

we find the same result that we have obtained previously when working directly with the differential equations, i.e.,

$$\delta\rho(x,\,t) \sim \cos(kx - \omega t)\,. \tag{36}$$

2.1 Kelvin-Helmholtz Instability

The Kelvin-Helmholtz Instability can arise when there is a velocity gradient in a fluid. This may occur within a single continuous fluid but it can also take place at the interface between two different fluids, for example air and water, see Figure 1. For the sake of simplicity we will assume that the two fluids are incompressible with constant background densities ρ_1 and ρ_2 and we will work in the frame of reference in which the fluid velocity is zero on one side of the interface and $\boldsymbol{v}$ on the other. We are going to look for solutions that are of the form $\exp[i(kx - \omega t)]$. If the dispersion relation between ω and the wavenumber k leads to an ω with a positive imaginary part, then the small amplitude perturbations will go unstable.

The linearized equations for an incompressible ideal fluid are $\boldsymbol{\nabla \cdot} \delta \boldsymbol{v} = 0$ and

$$\frac{\partial \delta \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\delta \boldsymbol{v} = -\frac{\nabla \delta P}{\rho} \,. \tag{37}$$

Taking the divergence of this equation we find that the pressure perturbations must satisfy Laplace's equation, i.e.,

$$\nabla^2 \delta P \equiv 0 \,. \tag{38}$$

Thus assuming that the pressure perturbations are of the form

$$\delta P = f(z) e^{i(kx - \omega t)} \,, \tag{39}$$

Laplace's equation leads to

$$\frac{d^2 f}{dz^2} = k^2 f \,, \tag{40}$$

which has solutions $f(z) \propto \exp[\pm kz]$.

With a little bit of algebra using the previous equations, it can be seen that demanding that the pressure is continuous at the interface, i.e., $\delta P_1 \equiv \delta P_2$, leads to

$$\rho_1 (kv - \omega)^2 \equiv -\rho_2 \omega^2 \,, \tag{41}$$

This is no other than the dispersion relation associated with the Kelvin-Helmholtz Instability, which has solutions

$$\omega = kv \left(\frac{\rho_1 \pm i\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2} \right) . \tag{42}$$

If the two fluids have the same density, i.e., $\rho_1 \equiv \rho_2$, then the imaginary part of ω reads

$$\Im(\omega) = \sigma(k) = \frac{kv}{2} \,. \tag{43}$$

The growth rate σ depends linearly on the wavenumber k. Obviously, this growth can not be infinite for large k, or small physical scales, and in reality viscous dissipation prevents the small scales from going unstable.

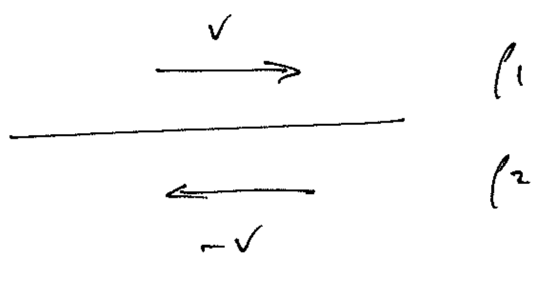

Figure 1: Initial configuration for the Kelvin-Helmholtz instability.

2.2 Rayleigh-Taylor Instability

Let us consider two fluids with different densities separated by an interface which is slightly distorted according to $\xi(x)$ in the presence of a gravitational field as shown in Figure 2. This setup is unstable in case of a heavier fluid on top of a lighter fluid. In order to simplify this problem, let us assume that we have an irrotational, $\nabla \times \boldsymbol{v} = 0$, and incompressible, $\nabla \cdot \boldsymbol{v} = 0$. flow. In this case the velocity can be obtained as the gradient of a potential scalar function Ψ, which satisfies Laplace's equation, i.e.,

$$\boldsymbol{v} \equiv \nabla \Psi \,, \qquad \text{with} \qquad \nabla^2 \Psi \equiv 0 \,. \tag{44}$$

Using the Ansatz for the potential function

$$\Psi(x, z) = f(z) \cos(kx - \omega t) \,, \tag{45}$$

where again $f(z) = e^{\pm kz}$ and requesting that the velocity perturbations vanish at the top and at the bottom

$$v(h') = v(-h) \equiv 0 \,, \tag{46}$$

it can be seen that the solutions for Ψ_1 and Ψ_2 are:

$$\Psi_1 = A \cosh[k(z + h)] \cos(kx - \omega t), \tag{47}$$

$$\Psi_2 = B \cosh[k(z - h')] \cos(kx - \omega t). \tag{48}$$

After some algebra involving the equations of motion for Ψ, it can be seen that ω must satisfy

$$\omega^2 = \frac{kg(\rho - \rho')}{\rho \coth(kh) + \rho' \coth(kh')} \,. \tag{49}$$

This dispersion relation for $\omega(k)$ governs the stability properties for the small amplitude perturbations. In particular when $\rho < \rho'$, then

$$\omega^2 < 0 \,, \tag{50}$$

and the system is Rayleigh-Taylor unstable. Note that for $\rho > \rho'$ there are various types of waves, which are very interesting.

In general this instability occurs in a stratified medium if

$$\frac{d\rho}{dz} < 0 \,. \tag{51}$$

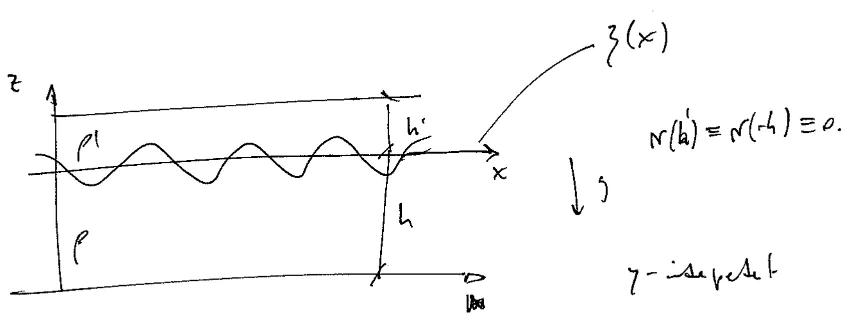

Figure 2: Setup for the Rayleigh-Taylor instability.

3 Ideal MHD Waves

We begin by recapitulating the equations for ideal MHD

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\rho &= -\rho\,\nabla\cdot\boldsymbol{v}\,, && (52)\\ \rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right] &= -\nabla P \;+\; \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\,, && (53)\\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla\times(\boldsymbol{v}\times\boldsymbol{B})\,, && (54) \end{aligned}$$

where μ_0 is the permeability of the vacuum. Using vector identities, the term associated with the Lorentz force in the momentum equation can be written as

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\nabla\left(\frac{B^2}{2}\right) + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}\,. \tag{55}$$

The first term on the right hand side is associated with magnetic pressure whereas the second one is associated with magnetic tension. The force associated with field line tension provides a restoring force (in the transverse direction) introducing an “elastic” property of the field-line “fabric”. Together with the flux freezing property implied by the ideal induction equation and the plasma's inertia, this allows the propagation of wave-like disturbances along field lines.

3.1 Linear Mode Analysis for MHD Waves

We can derive the dispersion relation for MHD waves by following the procedure that we have already seen for acoustic waves. We begin by writing down the linearised MHD equations, where the subscript “0” refers to the homogeneous background state, and the subscript “1” indicates perturbations, for instance, $\rho = \rho_0 + \rho_1$. As previously, we ignore any terms quadratic in fluctuations, and thus obtain:

$$\begin{aligned} \partial_t\rho_1 + \rho_0\nabla\cdot\boldsymbol{v}_1 &= 0\,, && (56)\\ \rho_0\partial_t\boldsymbol{v}_1 + \nabla P - \mu_0^{-1}(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0 &= 0\,, && (57)\\ -\partial_t\boldsymbol{B}_1 + \nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0) &= 0\,, && (58)\\ \partial_t\left(\frac{P_1}{P_0} - \gamma\frac{\rho_1}{\rho_0}\right) &= 0\,, && (59) \end{aligned}$$

with $\gamma \equiv C_\mathrm{p}/C_\mathrm{V}$ the ratio of specific heats, and where the last equation is derived from the isentropic equation of state P/ρ^γ =const. We now insert an Ansatz of the form

$$\rho_1 = \tilde{\rho_1}\,\exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right], \tag{60}$$

and accordingly for the other perturbed variables. With this ansatz, spatial derivatives (curl, gradient, divergence) will morph into factors of the wavevector $\boldsymbol{k}$, preserving the respective vector operation associated, for instance $\nabla\times\cdots \to i\boldsymbol{k}\times\ldots$; similarly, the partial time derivative will translate into a factor $-i\omega$. In the following we will moreover drop the subscript “1” to ease the notation. Dividing by common factors of the imaginary unit, i, we obtain the following algebraic system of equations:

$$\begin{aligned} -\omega\rho + \rho_0\boldsymbol{k}\cdot\boldsymbol{v} &= 0\,, && (61)\\ -\rho_0\omega\boldsymbol{v} + \boldsymbol{k}P - \mu_0^{-1}(\boldsymbol{k}\times\boldsymbol{B})\times\boldsymbol{B}_0 &= 0\,, && (62)\\ \omega\boldsymbol{B} + \boldsymbol{k}\times(\boldsymbol{v}\times\boldsymbol{B}_0) &= 0\,, && (63)\\ -\omega\left(\frac{P}{P_0} - \gamma\frac{\rho}{\rho_0}\right) &= 0\,. && (64) \end{aligned}$$

Assuming $\omega \neq 0$, we can solve the resulting system of equations for three of the dependent variables

$$\begin{aligned}
\rho &= \rho_0 \boldsymbol{k}\cdot\boldsymbol{v}/\omega\,, && (65)\\
P &= \gamma P_0 \boldsymbol{k}\cdot\boldsymbol{v}/\omega\,, && (66)\\
\boldsymbol{B} &= [(\boldsymbol{k}\cdot\boldsymbol{v})\,\boldsymbol{B}_0 - (\boldsymbol{k}\cdot\boldsymbol{B}_0)\,\boldsymbol{v}]/\omega\,, && (67)
\end{aligned}$$

already, leaving only the velocity $\boldsymbol{v}$ undetermined. By substituting these relations into the momentum equation (62), we obtain the following vector equation for $\boldsymbol{v}$,

$$\left[\omega^2 - \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{\rho_0\mu_0}\right]\boldsymbol{v} \;=\; -\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)\,(\boldsymbol{v}\cdot\boldsymbol{B}_0)}{\rho_0\mu_0}\,\boldsymbol{k} + \left[(c_{\mathrm{s}}^2 + c_{\mathrm{A}}^2)\boldsymbol{k} - \frac{\boldsymbol{k}\cdot\boldsymbol{B}_0}{\rho_0\mu_0}\,\boldsymbol{B}_0\right](\boldsymbol{k}\cdot\boldsymbol{v})\,, \qquad (68)$$

where we identify the adiabatic sound speed and the Alfvén speed, which corresponds to the speed at which transverse perturbations propagate along the field lines,

$$c_{\mathrm{s}} \equiv \sqrt{\gamma\frac{P_0}{\rho_0}}\,, \qquad (69)$$

$$c_{\mathrm{A}} \equiv \sqrt{\frac{B_0^2}{\rho_0\mu_0}}\,. \qquad (70)$$

Without loss of generality, we can orient the z-axis so that $\boldsymbol{B}_0 \parallel \hat{z}$. Furthermore, we can orient the y-axis so that it is perpendicular to $\boldsymbol{k}$, i.e., $\boldsymbol{k}\cdot\hat{y} = 0$. In this way $\boldsymbol{k}$ lies on the x-z-plane. Denoting the angle θ between $\boldsymbol{B}_0$ and $\boldsymbol{k}$, we can rewrite equation (68) in matrix form as

$$\begin{pmatrix} \omega^2 - k^2(c_{\mathrm{A}}^2 + c_{\mathrm{s}}^2\sin^2\theta) & 0 & -k^2 c_{\mathrm{s}}^2 \sin\theta\cos\theta \\ 0 & \omega^2 - k^2 c_{\mathrm{A}}^2\cos^2\theta & 0 \\ -k^2 c_{\mathrm{s}}^2\sin\theta\cos\theta & 0 & \omega^2 - k^2 c_{\mathrm{s}}^2 cos^2\theta \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = 0\,. \qquad (71)$$

The non-trivial solution of this linear system can be found by requiring that the determinant of the matrix vanishes. This yields the dispersion relation

$$\left(\omega^2 - k^2 c_{\mathrm{A}}^2\cos^2\theta\right)\left[\omega^4 - \omega^2 k^2(c_{\mathrm{A}}^2 + c_{\mathrm{s}}^2) + k^4 c_{\mathrm{A}}^2 c_{\mathrm{s}}^2\cos^2\theta\right] = 0\,, \qquad (72)$$

which is a third-order polynomial in ω^2. The three independent roots for ω^2 correspond to 3 distinct wave types.

3.1.1 Alfvén Waves

The obvious pair of roots in the dispersion relation (72) for ideal-MHD waves can be obtained by setting $(\omega^2 - k^2 c_{\mathrm{A}}^2\cos^2\theta) = 0$, which yields

$$\omega = \pm k c_{\mathrm{A}}\cos\theta\,,$$

where the plus and minus sign correspond to the left- and right-travelling waves, respectively. The group speed, $c_\phi \equiv \omega/k$, of the perturbations is simply the "projected" Alfvén speed $c_{\mathrm{A}}\cos\theta$, that is, the maximum propagation speed is obtained for $\boldsymbol{k} \parallel \boldsymbol{B}_0$, and waves that are perpendicular to the field lines do not propagate at all. The associated eigenvector of this particular solution of (72) is $\boldsymbol{v}_{\mathrm{A}} = (0, v_y, 0)$ satisfying both $\boldsymbol{k}\cdot\boldsymbol{v} = 0$ (incompressible perturbation), and $\boldsymbol{v}\cdot\boldsymbol{B}_0 = 0$ (transverse polarisation of the wave).

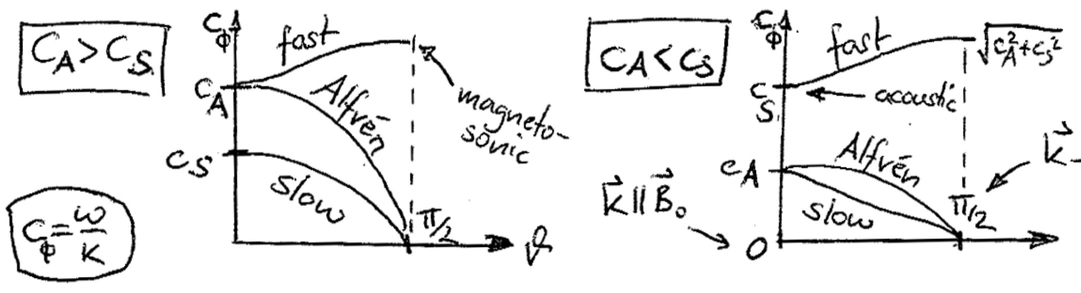

Figure 3: MHD wave speeds, $c_\phi \equiv \omega/k$, as a function of the angle θ between the background magnetic field and the wave vector, $\boldsymbol{k}$. *Left*: magnetically dominated case, $c_{\rm A} > c_{\rm s}$. *Right*: gas-pressure dominated case, $c_{\rm A} < c_{\rm s}$.

3.1.2 Slow and Fast Magnetosonic Waves

The two remaining roots of (72) follow from setting the term in square brackets to zero. Solving this simple quadratic equation in ω^2, we obtain

$$\omega = \pm k\, c_+\,, \qquad \text{and} \qquad \omega = \pm k\, c_-\,, \tag{73}$$

with the fast (c_+) and slow (c_-) magnetosonic velocities defined by

$$c_\pm \equiv \sqrt{\frac{1}{2}\left[c_{\rm A}^2 + c_{\rm s}^2 \pm \sqrt{\left(c_{\rm A}^2 + c_{\rm s}^2\right)^2 - 4c_{\rm A}^2 c_{\rm s}^2 \cos^2\theta}\,\right]}\,. \tag{74}$$

Note that $c_\pm \to c_{\rm s}$ for the unmagnetised case $B_0 \to 0$ (i.e., for $c_{\rm A} \to 0$), illustrating the relation of these waves with regular sound waves. Because all terms appearing in (74) are positive definite, we easily see that $c_+ > c_-$, explaining the names of the two wave branches.

In contrast to the Alfvén wave, the eigenvectors for the magnetosonic waves have the form $\boldsymbol{v}_{\rm MS} = (v_x, 0, v_z)$ implying compressional waves ($\boldsymbol{k}\cdot\boldsymbol{v} \neq 0$) with longitudinal polarisation ($\boldsymbol{v}\cdot\boldsymbol{B}_0 \neq 0$). The different propagation speed of the two wave branches can be understood in terms of the pressure perturbations being in phase or out of phase with the (linear) magnetic pressure fluctuation $\boldsymbol{B}_0\cdot\boldsymbol{B}/\mu_0$. More specifically

$$\frac{\boldsymbol{B}_0\cdot\boldsymbol{B}}{\mu_0} = \frac{\boldsymbol{k}\cdot\boldsymbol{v}\,B_0^2 - (\boldsymbol{k}\cdot\boldsymbol{B}_0)\,(\boldsymbol{v}\cdot\boldsymbol{B}_0)}{\mu_0\omega} = \frac{c_{\rm A}^2}{c_{\rm s}^2}\left(1 - \frac{k^2 c_{\rm s}^2\cos^2\theta}{\omega^2}\right). \tag{75}$$

From this equation, we can see that both perturbations have the same sign if $c_\phi > c_{\rm s}\cos\theta$, and have opposite signs if $c_\phi < c_{\rm s}\cos\theta$. As a simple exercise, check that this is indeed the case for $c_\phi = c_+$ and $c_\phi = c_-$, as stated above.

4 Hydrodynamic Shocks

When we studied sounds waves, we assumed that the amplitude of the perturbations was small. This conditions does not always hold. In particular because the sound speed c_s is proportional to (some power of) the density ρ (for an adiabatic, monoatomic gas $c_s \sim \rho^{1/3}$), regions of the fluid with higher densities will be moving faster, distorting the shape of the wave and causing it to steepen. This leads to the conclusion that after a finite time the density is bound to become a multiple-valued function of position. Because this is physically impossible, a sharp discontinuity, known as a shock, forms in the fluid. The processes involved cannot be understood in the framework of linear theory. Even though the detailed properties of the shock front could involve complicated micro-physics, a lot of progress can be made under a set of reasonable assumptions and by considering the conservation of mass, momentum, and energy on both sides of the shock.

4.1 Conservative Form of Equations for Hydrodynamics

We have already seen the equations describing an ideal fluid. In order to understand the dynamical properties of shocks, it is useful to write this equations in conservative form,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0\,, \tag{76}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{v}\,\boldsymbol{v} + P\boldsymbol{I}) = 0\,, \tag{77}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\left[\rho \boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + \epsilon + \frac{P}{\rho}\right)\right] = 0\,. \tag{78}$$

In these equations, $\boldsymbol{I}$ is the 3×3 identity matrix, $E = (\rho \boldsymbol{v}^2)/2 + \rho\epsilon$ is the total energy per unit volume, i.e., the sum of the kinetic energy and internal energy, both per unit volume. Recall that for an ideal gas under isentropic conditions, P/ρ^γ=const., the pressure is related to the internal energy per unit volume according to $P = (\gamma - 1)\rho\epsilon$. Here, ϵ is the energy per unit mass, i.e., $\epsilon = k_B T/(\gamma - 1)\mu m_H$, where k_B is the Boltzmann constant, T is the temperature, μ is the mean molecular weight, and m_H is the mass of the hydrogen atom. The adiabatic index/ratio of specific heats is defined as $\gamma = C_p/C_v$. Typical values are $\gamma = 5/3$ for an ideal, monatomic gas and $\gamma = 7/5$ for an ideal diatomic gas.

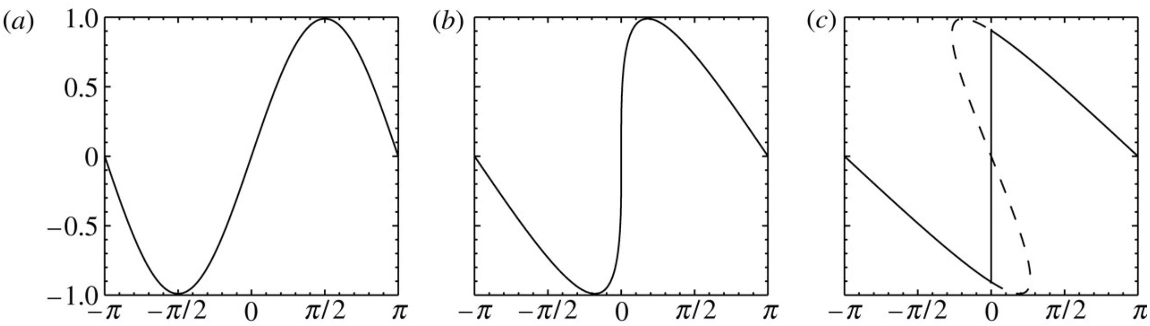

Figure 4: Sinusoidal velocity oscillation propagating in a gas. The "crests" have higher velocity than the "troughs", and outrun (b) and eventually overtake them (c). The system does not become multivalued and a shock forms (c, dashed line). M. Ruderman, 2006.

4.2 Rankine-Hugoniot Jump Conditions

We will consider the shock as a region of small thickness over which the dynamical variables change abruptly. We are interested in understanding the relationship between the fluid properties on the two sides of the shock, which we label 1 (pre-shock region) and 2 (post-shock region). In order to do this, it is useful to solve the problem in the reference frame in which the shock is at rest. Under steady conditions, the density, pressure, and velocity on the two sides of the shock are related by the Rankine-Hugoniot jump conditions,

$$\rho_1 v_1 = \rho_2 v_2\,, \tag{79}$$

$$\rho_1 v_1^2 + P_1 = \rho_2 v_2^2 + P_2\,, \tag{80}$$

$$\frac{1}{2}v_1^2 + h_1 = \frac{1}{2}v_2^2 + h_2\,, \tag{81}$$

where we have defined the enthalpy $h = \epsilon + P/\rho$. Note that in the frame of the undisturbed medium, the shock is propagating with velocity $v_{\rm s} = -v_1$, which is expected to be larger than the speed of sound, since otherwise the shock would produce ordinary sound waves that would propagate faster then the shock thereby weakening it.

4.3 Relationship Between Pre-shock and Post-shock Properties

The Rankine-Hugoniot jump conditions are three equations in six variables. It is convenient to write their solutions in terms of the ratios of the variables in the pre- and post-shock regions in terms of the mach number $\mathcal{M} = v_1/c_{\rm s}$, such that $\mathcal{M} > 1$, as mentioned above. After some algebraic manipulations we find

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,\mathcal{M}^2}{(\gamma+1)+(\gamma-1)(\mathcal{M}^2-1)} = \frac{v_1}{v_2}\,, \tag{82}$$

$$\frac{p_2}{p_1} = \frac{(\gamma+1)+2\gamma(\mathcal{M}^2-1)}{(\gamma+1)}\,, \tag{83}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma+1)+2\gamma(\mathcal{M}^2-1)][(\gamma+1)+(\gamma-1)(\mathcal{M}^2-1)]}{(\gamma+1)^2\,\mathcal{M}^2}\,. \tag{84}$$

It is clear from the above equations that $p_2 > p_1$, $\rho_2 > \rho_1$, $v_2 < v_1$, and $T_2 > T_1$. The strongest possible shock corresponds to $\mathcal{M} \to \infty$. In this limit the pressure and temperature ratios diverge but $\rho_2/\rho_1 = (\gamma+1)/(\gamma-1)$. This corresponds to a factor of $\rho_2/\rho_1 = 4$ for monoatomic gas and and $\rho_2/\rho_1 = 7$ for a diatomic gas.

A few remarks concerning the approximations involved are in order. We derived the junction conditions using the conservation of mass, momentum, and energy. The first two are usually very good approximations, however, the equation that we used for energy conservation is incomplete if energy can be gained or lost at the shock front. The former could take place if chemical or nuclear reactions can happen at the shock front. The latter could take place if the gas heated by the shock reaches a high enough temperature that radiative losses become important. In our derivation, we assumed that the shock itself can be thought of as a contact discontinuity. However, because the gradients in these regions are large dissipative processes are expected to be important in this region and they in fact determine the actual shock thickness. Further thickening of the shock front could arise do to energetic particles that can propagate faster than the sound speed and travel to the pre-shock region pre-heating the incoming gas. When none of these effects are important, mass, momentum, and energy are conserved at the shock discontinuity we discussed above and the shock is called adiabatic.

5 Phases of Supernova Remnant Evolution

Supernovae occur when massive stars implode and eject their outer layers. 99% of the energy released is carried away by neutrinos. The remainder drives a supernova remnant into the ISM. The material ejected expands into the interstellar medium, and shocks it. This process can be roughly divided into four phases: 1) free expansion/constant velocity phase; 2) adiabatic, energy-conserving/ Sedov phase; 3) radiative/momentum conserving/snowplow phase; and 4) merging of shock with interstellar medium. In what follows, we will summarise some of the salient characteristics of these phases.

- **Free expansion phase (constant velocity)** – Early on after the explosion, the shell of swept-up material in front of shock does not represent a significant mass compared to the mass ejected $M_{\rm ej}$. As long as the swept-up mass is much smaller that the mass ejected, momentum conservation implies that the velocity of the shock front remains approximately constant and $R_{\rm s}(t) \sim V_0\, t$. This phase ends when the swept-up mass becomes comparable to the mass of ejecta, i.e., $4\pi R_{\rm s}^3(t)\rho_0/3 = 4\pi V_0^3 t_{SW}^3 \rho_0/3 = M_{\rm ej}$.

- **Sedov phase (adiabatic/energy-conserving)** – The Sedov phase begins after the sweep-up time, at about 10^2 years, when enough mass has been accumulated to decelerate the remnant from a constant velocity. During 1941-1945, J. von Neumann, L. Sedov and G. I. Taylor independently studied the instantaneous input of fixed amounts of energy E_0 into systems with uniform density, ρ_0. They found that the evolution of the shock in this phase was characterised by only three parameters: the initial energy (at the end of the free expansion phase), the initial ambient density, and the radius of the blast wave. Dimensional analysis shows that the evolution of the shock front is $R_{\rm s}(t) \sim E_0^{1/5} \rho_0^{1/5} t^{2/5}$.

- **Snowplow phase (radiative/momentum-conserving)** – The Sedov phase ends after a few $\sim 10^3$ years, when the temperature has dropped to $T \sim 10^6$K. C,N,O ions now recombine and cool the remnant efficiently. During this phase, momentum is conserved but the energy is gradually lost. In order to calculate the expansion radius $R_{\rm s}(t)$ we consider that the momentum is conserved $4\pi R_{\rm s}^3 \rho_0 V_{\rm s}/3 =$ const. and the shock speed is just $V_{\rm s}(t) = dR_{\rm S}/dt$. This equation can be integrated to obtain $R_{\rm s}(t) \sim t^{1/4}$. During the final stages of the snow-plow phase, the swept-up mass gradually slows down the front to subsonic speeds and the remnant merges with the ISM.

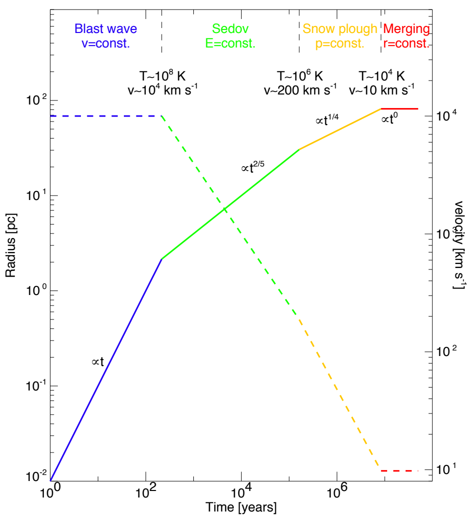

Figure 5: Radius and velocity evolution of supernova shell during the different phases.