Accretion disks

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Why disks?

Accretion disks

We get accretion disks because of conservation of angular momentum

- Any cloud of gas/dust generally has net angular momentum
- Gravitational collapse proceeds more rapidly parallel to rotation vector
- Naturally leads to "flattened" structure
- Disks are typically stable

Why disks?

This paradigm naturally leads to presuming that

$$\frac{GM}{r^2} = \frac{v^2}{r}, \tag{1}$$

or

$$v = \sqrt{\frac{GM}{r}}. \tag{2}$$

Which is, of course, Keplerian motion. Since $\Omega = v/r$ we have

$$\Omega_k = \sqrt{\frac{GM}{r^3}}. \tag{3}$$

Why disks?

Note

- We have assumed our disk is “thin”
- We have assumed its mass is “small”

Disk dynamics

We'll continue with these assumptions and follow the standard approach[1]. Suppose now our disk has some kinematic viscosity associated with it. Then the torque is

$$T(r) = 2\pi\nu\Sigma r^3 \frac{d\Omega}{dr}, \tag{4}$$

where Σ is surface density and $\Omega = (GM/R^3)^{1/2}$ is the Keplerian angular speed.

[1] e.g. Barbara Ryden's notes

Disk dynamics

Mass & angular momentum conservation yield

$$r\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma u_r) = 0, \tag{5}$$

$$r\frac{\partial \Sigma r^2 \Omega}{\partial t} + \frac{\partial}{\partial r}(\Sigma u_r r^3 \Omega) = \frac{1}{2\pi}\frac{\partial T}{\partial r}. \tag{6}$$

and so

$$r\frac{\partial \Sigma}{\partial t} = -\frac{1}{2\pi}\frac{\partial}{\partial r}\left\{\frac{1}{(r^2\Omega)'}\frac{\partial T}{\partial r}\right\}. \tag{7}$$

Disk dynamics

Using the Keplerian angular speed and $T = 2\pi\nu\Sigma r^3\Omega'$

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left\{r^{1/2}\frac{\partial}{\partial r}(\nu\Sigma r^{1/2})\right\} \tag{8}$$

This is a diffusion equation for Σ. If the disk begins as a thin ring it will spread out (in r) over time (Pringle 1981 for solution).
The radial drift (accretion speed) is then

$$u_r(r,t) = -\frac{3}{\Sigma r^{1/2}}\frac{\partial}{\partial r}(\nu\Sigma r^{1/2}). \tag{9}$$

Meaning of “thin”

In the vertical direction we have hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{r^3} \tag{10}$$

or, using the isothermal sound speed,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{GMz}{c_s^2 r^3} = -\frac{u_k^2 z}{c_s^2 r^2}, \tag{11}$$

and so

$$\rho = \rho_0 e^{-z^2/(2H^2)}, \tag{12}$$

where $H = r/M_k$, with M_k being the keplerian Mach number. So, a thin disk with $H << r$ requires $M_k >> 1$.

What accretion rates do we expect?

- The mean free path very close to the star is about 10^{-3} cm
- This yields a kinematic viscosity of $10^{3}\,\mathrm{cm}^2\,\mathrm{s}^{-1}$
- Hence $u_r \sim -5\,\mathrm{cm\ yr}^{-1}$.

Some disk instabilities

Stable disks are unwelcome

- If disks are stable no accretion occurs
- Observations suggest disks are not “very” unstable

Sufficient condition for stability

The Solberg-Hoiland criterion:

$$\frac{1}{r^3}\frac{\partial j^2}{\partial r} - \frac{1}{c_p \rho}\nabla P \cdot \nabla S > 0, \qquad (13)$$

$$\frac{\partial P}{\partial z}\left\{\frac{\partial j^2}{\partial r}\frac{\partial S}{\partial z} - \frac{\partial j^2}{\partial z}\frac{\partial S}{\partial r}\right\} < 0, \qquad (14)$$

j is specific angular momentum; S is the entropy; other symbols standard.

Some disk instabilities

Sufficient condition for stability

$$\frac{1}{r^3}\frac{\partial j^2}{\partial r} - \frac{1}{c_p \rho}\nabla P \cdot \nabla S > 0, \quad (15)$$

$$\frac{\partial P}{\partial z}\left\{\frac{\partial j^2}{\partial r}\frac{\partial S}{\partial z} - \frac{\partial j^2}{\partial z}\frac{\partial S}{\partial r}\right\} < 0, \quad (16)$$

Assumptions

Perturbations are infinitesimal, axisymmetric and adiabatic.

Some disk instabilities

For Keplerian disks[2] the condition becomes

$$\Omega_{\rm K}^2 + N_r^2 > 0, \quad (17)$$
$$N_z^2 > 0, \quad (18)$$

where N_r and N_z are the radial and vertical parts of the Brunt-Väisälä frequency.

[2] following, e.g., Fromang & Lesur (2017)

Magnetic fields in disks

- It is hard to imagine any way a disk could form without magnetic fields
- Fields exert forces

What impact do magnetic fields have on disks?

Magnetic fields in disks

Let's assume ideal MHD:

$$\frac{D \ln \rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \tag{19}$$

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla \cdot \left[P + \frac{B^2}{8\pi}\right] - \frac{1}{4\pi\rho}(\mathbf{B} \cdot \nabla)\mathbf{B} + \nabla\Phi = 0, \tag{20}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times \mathbf{u} \times \mathbf{B} = 0. \tag{21}$$

The Magnetorotational Instability

Now linearise our equations:

$$\mathbf{u}_1 \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{u}_1 = 0, \quad (22)$$

$$\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_1 + \frac{1}{\rho_0}\nabla P_1 + \frac{\rho_1}{\rho_0^2}\nabla P_0 + \frac{1}{4\pi\rho_0}\nabla(2\mathbf{B}_0\mathbf{B}_1) - \frac{\rho_1}{8\pi\rho_0}\nabla \mathbf{B}_0^2 - \frac{1}{4\pi\rho_0}(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 - \frac{1}{4\pi\rho_0}(\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0 = 0, \quad (23)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times \mathbf{u}_1 \times \mathbf{B}_0 - \nabla \times \mathbf{u}_0 \times \mathbf{B}_1 = 0. \quad (24)$$

The Magnetorotational Instability

Assuming all our perturbed quantities can be represented by some amplitude times $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ in the usual way:

$$\frac{\partial}{\partial t} \rightarrow -i\omega,$$
$$\nabla \rightarrow i\mathbf{k}.$$

After some algebra (e.g. Balbus & Hawley 1991):

$$\frac{k^2}{k_z^2}\tilde{\omega}^4 - \left\{\kappa^2 + \left[\frac{k_z}{k}N_z - N_r\right]^2\right\}\tilde{\omega}^2 - 4\Omega^2 k_z^2 v_{Az}^2 = 0, \qquad (25)$$

where N_z and N_r are the vertical and radial parts of the Brunt-Väisälä frequency, κ is the epicyclic frequency and

$$\tilde{\omega}^2 \equiv \omega^2 - k_z^2 v_{Az}^2 \qquad (26)$$

For instability, need $\omega^2 < 0$.

The Magnetorotational Instability

Clearly it is possible for $\omega^2 > 0$ (exercise) since equation 25 is a quadratic for $\tilde{\omega}^2$ yielding

$$\omega^2 = k_z^2 v_{Az}^2 + k^2 \frac{A \pm \sqrt{A^2 + 16\Omega^2 k_z^2 v_{Az}^2}}{2k_z^2} \qquad (27)$$

where for convenience A is the coefficient of $\tilde{\omega}^2$ in equation 25.

A necessary condition for instability is that $\omega = 0$ for some k.

The Magnetorotational Instability

Requiring $\omega = 0$ and rearranging yields

$$k_r^2(k_z^2 v_{Az}^2 + N_z^2) - 2k_r k_z N_z N_r + k_z^2 \left(\frac{d\Omega^2}{d\ln r} + N_r^2 + k_z^2 v_{Az}^2 \right) = 0. \quad (28)$$

Viewing this as a quadratic in k_r, and recalling that we require there to be no real solution, our necessary condition for instability becomes

$$k_z^4 v_{Az}^4 + k_z^2 v_{Az}^2 \left(N^2 + \frac{d\Omega^2}{d\ln r} \right) + N_z^2 \frac{d\Omega^2}{d\ln r} > 0. \quad (29)$$

Hence, stability is only guaranteed if

$$\frac{d\Omega^2}{dr} \geq 0. \quad (30)$$

The Magnetorotational Instability

For a disk in Keplerian rotation

$$\frac{d\Omega^2}{dr} < 0, \qquad (31)$$

and so it is potentially unstable.
Assuming the pressure gradient in the radial direction is much less than that in the vertical direction, so that $N_z^2 \approx N^2$, the disk will become unstable to waves with

$$k_z < \frac{1}{v_{Az}} \left| \frac{d\Omega^2}{d \ln r} \right|^{1/2} \qquad (32)$$

The wavelength corresponding to this value of k_z must fit in the vertical extent of the disk for the disk to be unstable to the MRI.

Why care about the MRI?

Importance of the MRI

- The MRI is clearly viable in real astrophysical disks
- Simulations show that it grows strongly
- It does not saturate (except by other instabilities)
- It can drive disk turbulence

The MRI is good news

- Stable accretion disks are not welcome.

The MRI - caveats

Care is required

- We have assumed ideal MHD
- We have assumed infinite extent in the vertical direction

- Proto-planetary disks are problematic
- Disks around (other) compact objects seem susceptible to the MRI

The MRI in PPDs

The equations for our (isothermal) weakly ionized system are

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{v}_i) = 0, \quad (33)$$

$$\frac{\partial \rho_1 \mathbf{v}_1}{\partial t} + \nabla \cdot \left(\rho_1 \mathbf{v}_1 \mathbf{v}_1 + a^2 \rho_1 \mathbf{I}\right) = \mathbf{J} \times \mathbf{B}, \quad (34)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}_1 \mathbf{B} - \mathbf{B} \mathbf{v}_1) = -\nabla \times \left(r_0 \frac{(\mathbf{J} \cdot \mathbf{B})\mathbf{B}}{B^2} + r_1 \frac{\mathbf{J} \times \mathbf{B}}{B} \right. \quad (35)$$

$$\left. + r_2 \frac{\mathbf{B} \times (\mathbf{J} \times \mathbf{B})}{B^2} \right), \quad (36)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (37)$$

$$\nabla \times \mathbf{B} = \mathbf{J}. \quad (38)$$

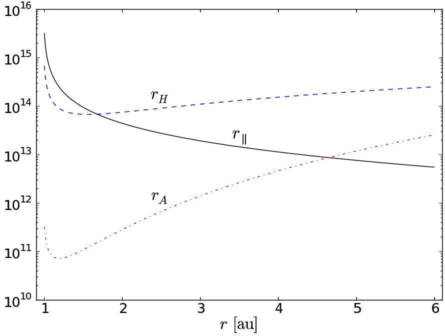
Numerical Approach

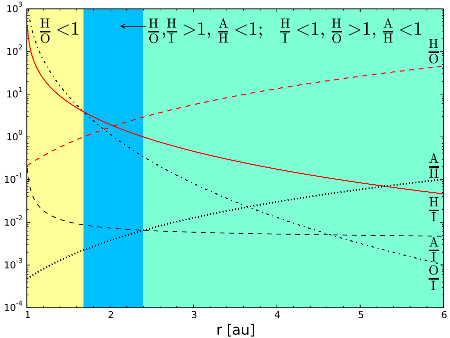
- Use the HYDRA code (Hall Diffusion Scheme for Hall term, super-time-stepping for ambipolar term)
- Written to simulate weakly ionised system with N fluids
- Parallelised: scales from 8k cores to 400k cores with circa 70% efficiency (strong scaling)

Numerical set-up

- (Quasi-)Global simulations
- Cartesian grid
- Weakly ionised multifluid approximation (3 fluids incl neutrals)
- Radially stratified ionisation and density (following Salmeron & Wardle 2003)
- Not vertically stratified (i.e. cylindrical “disk”)

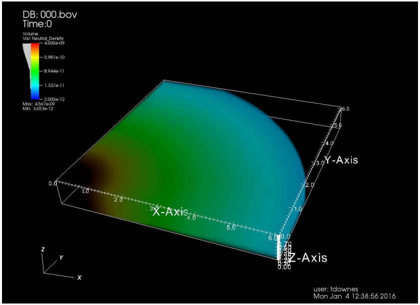
Multifluid effects

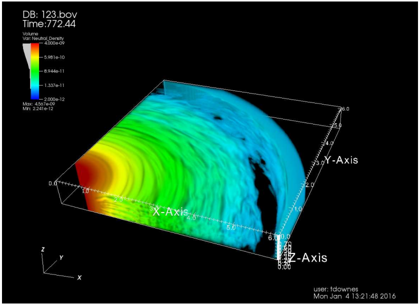




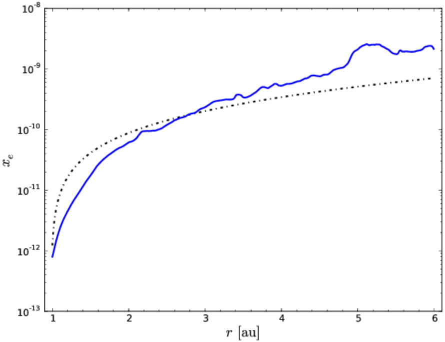
Results

Overview

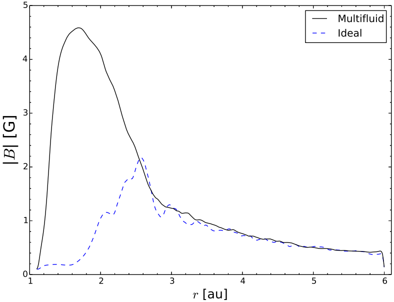




Ionisation fraction evolution

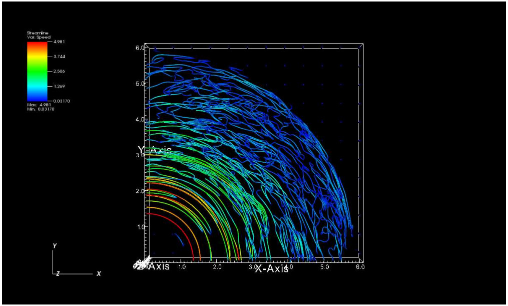


Magnetic field



Significant field amplification at low *r*.

Magnetic field



Magnetic field becomes “ordered” at low r.

Jets and Outflows

Credit: NASA, ESA, and M. Livio and the Hubble 20th Anniversary Team (STScI)

- Outflows are seen in many accreting objects
- Magnetically launched outflows can carry significant angular momentum
- Outflows are becoming more favoured, given issues with MRI

Some hydro disk instabilities

- Subcritical transition to turbulence
- Baroclinic instability
- Convective over-stability
- Vertical shear instability

We'll discuss these in a *very* hand-waving way.

Baroclinic instability

This relies on the thermal structure of the disk

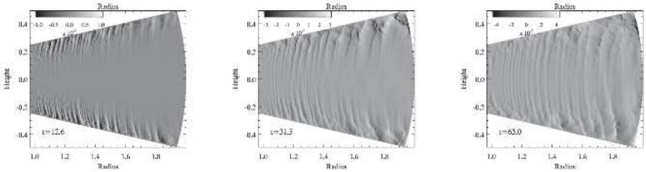
We require:

- A finite cooling time (i.e. non-adiabatic)
- $N_r^2 < 0$ - i.e. radial convection is a possibility

Convective over-stability

- Driven in the same fashion as the Baroclinic Instability
- Epicyclic orbits become amplified
- Turbulence results (or seeds the baroclinic instability)

Vertical Shear Instability



Credit: Nelson (2013)

This relies on the vertical structure of the disk

- Perturb a particle from (e.g.) the mid-plane *vertically*
- It then has “too much” angular momentum

Gravitational Instability

If the disk is sufficiently massive we can have

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma} < 1. \tag{39}$$

and the disk can be unstable to gravitational collapse.

- Require relatively cool, dense disks
- Possible route to planet formation
- Spiral waves emitted from collapsing regions transport angular momentum

Overview

- We expect disks around all accreting objects (angular momentum)
- Tricky to then allow accretion (hydro disks rather stable)
- We seem to need turbulence (or something)
- Instability can arise if magnetic fields are present
- Fields may not be coupled to the gas in PPDs
- We can appeal to outflows to transport momentum
- We can appeal to certain hydro instabilities, depending on our disk