Soft Graviton Theorem in Generic Quantum Theory of Gravity

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What is soft graviton theorem?

Take a general coordinate invariant quantum theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number N of external particles of finite momentum $p_1, \cdots p_N$

– M external gravitons carrying small momentum $k_1, \cdots k_M$.

Soft graviton theorem: Expansion of this amplitude in power series in $k_1, \cdots k_M$ in terms of the amplitude without the gravitons.

There are many explicit results.

1. General results at leading order in k   Weinberg; . . .

2. For <u>one</u> soft graviton, there are general subleading results in D=4 via BMS

Strominger; Strominger, Zhiboedov; Campiglia, Laddha; . . .

3. Results in specific theories in general dimensions

White; Cachazo, Strominger; Bern, Davies, Di Vecchia, Nohle; Elvang, Jones, Naculich; . . .
Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha
Bianchi, Guerrieri; Di Vecchia, Marotta, Mojaza; . . .

Our goal: Study soft graviton amplitudes in generic UV finite theories, in generic number of dimensions, for arbitrary mass and spin of external states

– including string theory

Validity

1. For tree amplitudes our analysis is valid in all dimensions

2. For loop amplitudes the results are valid if we assume that 1PI vertices do not generate soft factors in the denominator

True by power counting for

– subleading amplitudes for D > 5

– subsubleading amplitudes for D > 6

D: number of non-compact space-time dimensions 4

For single soft gravitons we can argue that the unwanted terms cancel in the sum over graphs and the results are also valid for D=5,6

We expect a similar result to hold for multiple soft gravitons, but this has not been proved.

In D=4 the S-matrix elements themselves are infrared divergent, introducing additional subtleties.

Bern, Davies, Nohle

In D $\leq$ 4, our analysis will apply to only tree amplitudes.

A.S. arXiv:1702.03934, 1703.00024: Subleading single soft
A. Laddha, A.S., arXiv:1706.00759: Sub-subleading single soft
Subhroneel Chakrabarti, Sitender Kashyap, Biswajit Sahoo, A.S., Mritunjay Verma, arXiv:1707.06803; subleading multiple soft

Single soft graviton
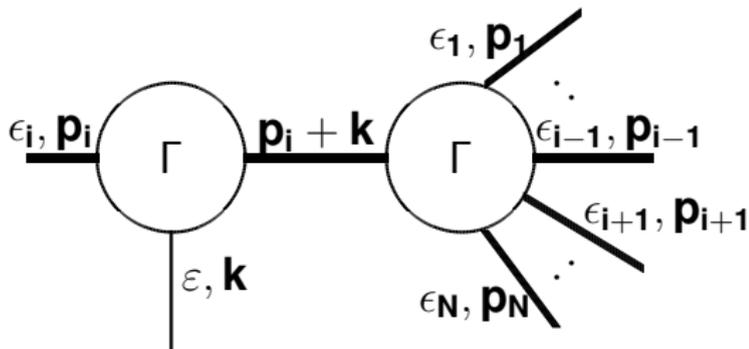
We divide the Feynman diagrams into two classes
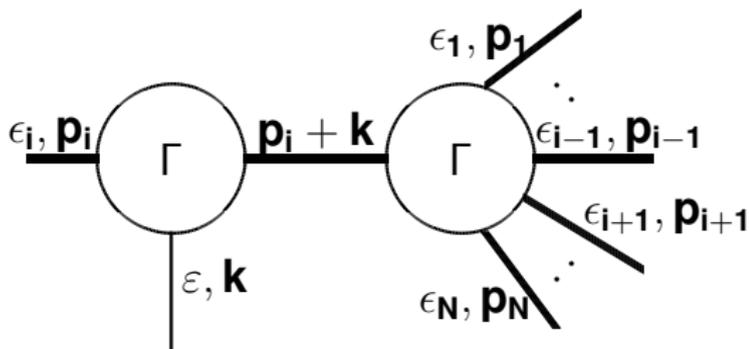
Γ: Full amputated Green's function

Internal lines: Full renormalized propagators

$\varepsilon, \mathbf{k}$: **polarization, momentum of soft graviton**

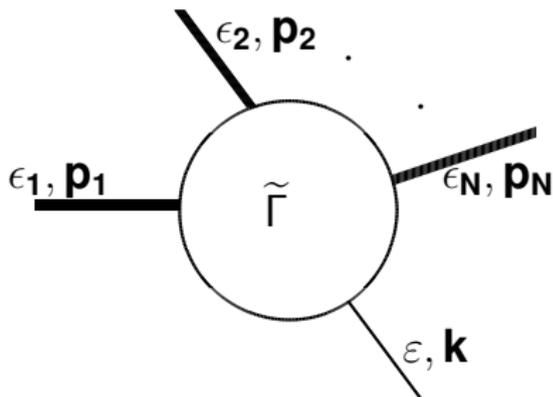$\epsilon_i, \mathbf{p}_i$: **polarization, momentum of finite energy external particles.**

The internal line carrying momentum $p_i + k$ has denominator factor

$$\{(p_i + k)^2 + M^2\}^{-1} = (2p_i \cdot k)^{-1} \quad \text{if } M = M_i$$

using $p_i^2 + M_i^2 = 0$, $k^2 = 0$.

$\Rightarrow$ this starts contributing at the leading order.

Second class of diagrams

$\widetilde{\Gamma}$: **Amputated amplitudes in which the external soft graviton <u>does not</u> get attached to an external line**

– has no pole as k → 0

⇒ the contribution from this diagram begins at the subleading order.

Strategy for computation

1. Consider the gauge invariant one particle irreducible (1PI) effective action of the theory

2. Expand the action in powers of all fields, including the metric fluctuations, around the extremum of the action

3. Add manifestly Lorentz invariant gauge fixing terms.

4. This action is used to compute vertices and propagators of finite energy external states but <u>not</u> of soft external gravitons.

5. To calculate the coupling of the soft graviton $S_{\mu\nu}$ to the rest of the fields, we <u>covariantize</u> the gauge fixed action.

a. Replace the background metric $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

b. Replace all derivatives by covariant derivatives computed with the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$

This misses terms involving Riemann tensor computed from the metric $\eta_{\mu\nu} + 2\,S_{\mu\nu}$ but that contains two derivatives and hence is sub-subleading.

1. We choose

$$\mathbf{S}_{\mu\nu} = \varepsilon_{\mu\nu}\mathbf{e}^{\mathbf{ik}\cdot\mathbf{x}}, \quad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}, \quad \varepsilon^{\mu}{}_{\mu} = \mathbf{k}^{\mu}\varepsilon_{\mu\nu} = \mathbf{0}$$

All indices raised and lowered by η.

2. All fields representing finite energy external states are taken to carry tangent space Lorentz indices

– allows us to give uniform treatment to fermions and bosons.

3. To first order in $\mathbf{S}_{\mu\nu}$, we take the vielbeins to be

$$\mathbf{e}_{\mu}{}^{\mathbf{a}} = \delta_{\mu}{}^{\mathbf{a}} + \mathbf{S}_{\mu}{}^{\mathbf{a}}, \quad \mathbf{E}_{\mathbf{a}}{}^{\mu} = \delta_{\mathbf{a}}{}^{\mu} - \mathbf{S}_{\mathbf{a}}{}^{\mu}, \quad \mathbf{g}^{\mu\nu} = \eta^{\mu\nu} - 2\,\mathbf{S}^{\mu\nu}$$

Covariantization: Acting on a field ϕ_α:

$$\partial_{a_1} \cdots \partial_{a_n} \Rightarrow E_{a_1}{}^{\mu_1} \cdots E_{a_n}{}^{\mu_n} D_{\mu_1} \cdots D_{\mu_n}, \quad E_a{}^\mu \equiv \left(\delta_a{}^\mu - S_a{}^\mu\right)$$

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha + \frac{1}{2} \omega_\mu^{ab} (J_{ab})_\alpha{}^\gamma \phi_\gamma$$

$$D_{(\mu} D_{\nu)} \phi_\alpha = \partial_\mu \partial_\nu \phi_\alpha + \cdots + \frac{1}{2} \partial_{(\mu} \omega_{\nu)}^{ab} (J_{ab})_\alpha{}^\gamma \phi_\gamma + \left\{ \begin{array}{c} \rho \\ \mu \ \nu \end{array} \right\} \partial_\rho \phi_\alpha$$

etc.

$$\omega_\mu^{ab} = \partial^b S_\mu{}^a - \partial^a S_\mu{}^b, \quad S_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik\cdot x}$$

$$\left\{ \begin{array}{c} \rho \\ \mu \ \nu \end{array} \right\} = \frac{1}{2} \left[\partial_\mu S_\nu{}^\rho + \partial_\nu S_\mu{}^\rho - \partial^\rho S_{\mu\nu} \right]$$
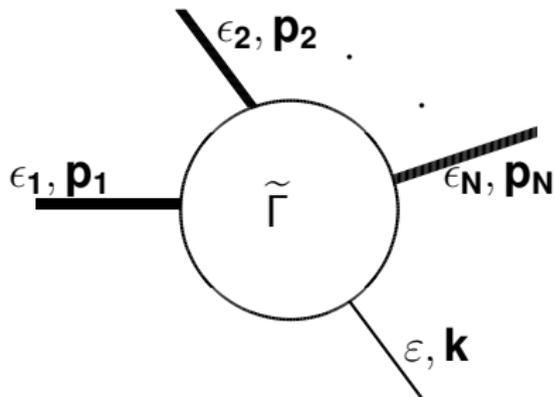
Consider a Lorentz invariant functional

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})$$

$$\delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N})\, \mathbf{F}^{\alpha_1 \cdots \alpha_N}(\mathbf{p_1}, \ldots \mathbf{p_N})$$
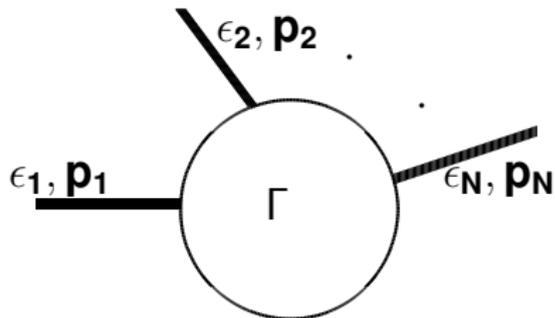
Covariantization produces an additional term

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})\, \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N} + \mathbf{k})$$

$$\sum_{\mathbf{i=1}}^{\mathbf{N}} \left[-\delta_{\beta_i}^{\ \alpha_i} \varepsilon_\mu^{\ \nu} \mathbf{p_{i\nu}} \frac{\partial}{\partial \mathbf{p_{i\mu}}} + \frac{\mathbf{1}}{\mathbf{2}} (\mathbf{k^b} \varepsilon_\mu^{\ a} - \mathbf{k^a} \varepsilon_\mu^{\ b})(\mathbf{J_{ab}})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p_{i\mu}}} \right.$$

$$\left. -\frac{\mathbf{1}}{\mathbf{2}} \delta_{\beta_i}^{\ \alpha_i} \left\{ \mathbf{k_\mu} \varepsilon_\nu^{\ \rho} + \mathbf{k_\nu} \varepsilon_\mu^{\ \rho} - \mathbf{k^\rho} \varepsilon_{\mu\nu} \right\} \mathbf{p_{i\rho}}\, \frac{\partial^2}{\partial \mathbf{p_{i\mu}} \partial \mathbf{p_{i\nu}}} \right]$$

$$\mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N}) + \mathcal{O}(\mathbf{k^\mu k^\nu}).$$

Now consider

1. Take the amplitude without soft graviton.

2. Covariantize it to order k^μ

Denote the amplitude without the soft graviton by

$$\epsilon_{1,\alpha_1}(\mathbf{p_1}) \ldots \epsilon_{N,\alpha_N}(\mathbf{p_N})\, \Gamma^{\alpha_1 \ldots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N})$$
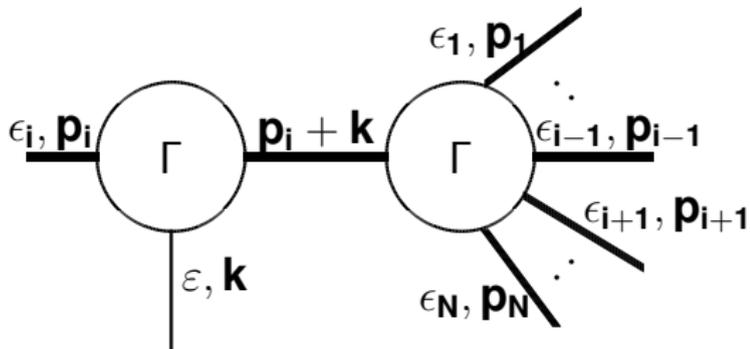
α_i**: tangent space tensor / spinor indices labelling all the fields of the theory**

$\Gamma^{\alpha_1 \cdots \alpha_N}$ **includes the** $\delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N})$ **factor.**

Then the result for $\widetilde{\Gamma}$ **is**

$$\sum_{i=1}^{N} \left[-\delta_{\beta_i}^{\alpha_i} \varepsilon_{\mu}{}^{\nu} \mathbf{p}_{i\nu} \frac{\partial}{\partial \mathbf{p}_{i\mu}} + \mathbf{k}^b \varepsilon_{\mu}{}^a (\mathbf{J_{ab}})_{\beta_i}^{\alpha_i} \frac{\partial}{\partial \mathbf{p}_{i\mu}} \right.$$

$$\left. -\frac{1}{2} \delta_{\beta_i}^{\alpha_i} \left\{ \mathbf{k}_{\mu} \varepsilon_{\nu}{}^{\rho} + \mathbf{k}_{\nu} \varepsilon_{\mu}{}^{\rho} - \mathbf{k}^{\rho} \varepsilon_{\mu\nu} \right\} \mathbf{p}_{i\rho} \frac{\partial^2}{\partial \mathbf{p}_{i\mu} \partial \mathbf{p}_{i\nu}} \right]$$

$$\Gamma^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N}) + \mathcal{O}(\mathbf{k}^{\mu} \mathbf{k}^{\nu}).$$

Next consider

Need to focus on the three point coupling computed from the 1PI action.

Begin with two point function without the soft graviton and covariantize it to order $k^{\mu}k^{\nu}$.

$$S^{(2)} = \frac{1}{2} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \, \phi_\alpha(\mathbf{q_1}) \mathcal{K}^{\alpha\beta}(\mathbf{q_2}) \phi_\beta(\mathbf{q_2}) \, (2\pi)^D \delta^{(D)}(\mathbf{q_1} + \mathbf{q_2})$$

$\{\phi_\alpha\}$**: set of all the fields**

$\mathcal{K}^{\alpha\beta}(\mathbf{q})$**: Kinetic operator, chosen to satisfy**

$$\mathcal{K}^{\alpha\beta}(\mathbf{q}) = \mathcal{K}^{\beta\alpha}(-\mathbf{q})$$

Covariantization $\Rightarrow$ coupling of ϕ_α to soft graviton
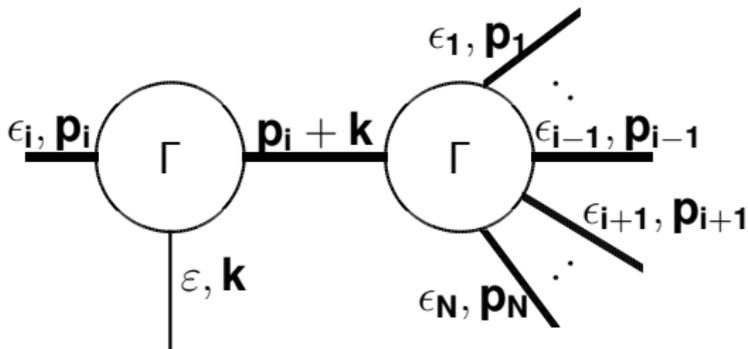
$$
\begin{aligned}
S^{(3)} \;=\; & \frac{1}{2} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \, (2\pi)^D \delta^{(D)}(\mathbf{q_1} + \mathbf{q_2} + \mathbf{k}) \\
& \times \phi_\alpha(\mathbf{q_1}) \Bigg[-\varepsilon_{\mu\nu} \mathbf{q_2}^\nu \frac{\partial}{\partial \mathbf{q_2}_\mu} \mathcal{K}^{\alpha\beta}(\mathbf{q_2}) - \mathbf{k_a}\, \varepsilon_{\mathbf{b}\mu} \frac{\partial}{\partial \mathbf{q_2}_\mu} \mathcal{K}^{\alpha\gamma}(\mathbf{q_2}) \left(\mathbf{J^{ab}} \right)_\gamma^{\ \beta} \\
& -\frac{1}{2} \delta_\beta^{\ \alpha} \left\{ \mathbf{k}_\mu \varepsilon_\nu^{\ \rho} + \mathbf{k}_\nu \varepsilon_\mu^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \, \mathbf{q_2}_\rho \, \frac{\partial^2}{\partial \mathbf{q_2}_\mu \partial \mathbf{q_2}_\nu} \Bigg] \phi_\beta(\mathbf{q_2}) + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu)
\end{aligned}
$$

– determines the coupling of the soft graviton to the finite energy particles

We also need the propagator of the particle carrying momentum $p_i + k$

– given by $\quad i\,\mathcal{K}^{-1}_{\alpha\beta}(\mathbf{p_i} + \mathbf{k}) \equiv (2\mathbf{p_i} \cdot \mathbf{k})^{-1}\mathcal{N}^i_{\alpha\beta}(\mathbf{p_i} + \mathbf{k})$

We now substitute and compute by Taylor series expanding in k

Algorithm for manipulation

1. Use $\mathcal{K}(\mathbf{p})\mathcal{N}^{\mathbf{i}}(\mathbf{p}) = \mathbf{i}\,(\mathbf{p}^2 + \mathbf{M}^2)$ to get

$$\frac{\partial \mathcal{K}(\mathbf{p})}{\partial \mathbf{p}_\mu}\mathcal{N}^{\mathbf{i}}(\mathbf{p}) = -\mathcal{K}(\mathbf{p})\frac{\partial \mathcal{N}^{\mathbf{i}}(\mathbf{p})}{\partial \mathbf{p}_\mu} + \mathbf{2}\,\mathbf{i}\,\mathbf{p}^\mu \,,$$

$$\frac{\partial^2 \mathcal{K}(\mathbf{p})}{\partial \mathbf{p}_\mu \partial \mathbf{p}_\nu}\mathcal{N}^{\mathbf{i}}(\mathbf{p}) = -\frac{\partial \mathcal{K}(\mathbf{p})}{\partial \mathbf{p}_\mu}\frac{\partial \mathcal{N}^{\mathbf{i}}(\mathbf{p})}{\partial \mathbf{p}_\nu} - \frac{\partial \mathcal{K}(\mathbf{p})}{\partial \mathbf{p}_\nu}\frac{\partial \mathcal{N}^{\mathbf{i}}(\mathbf{p})}{\partial \mathbf{p}_\mu}$$

$$-\mathcal{K}(\mathbf{p})\frac{\partial^2 \mathcal{N}^{\mathbf{i}}(\mathbf{p})}{\partial \mathbf{p}_\mu \partial \mathbf{p}_\nu} + \mathbf{2}\,\mathbf{i}\,\eta^{\mu\nu} \,,$$

2. Use rotational invariance to get

$$(\mathbf{J}^{\mathbf{ab}})^{\mathbf{T}}\mathcal{K}(\mathbf{p}) = -\mathcal{K}(\mathbf{p})\mathbf{J}^{\mathbf{ab}} + \mathbf{p}^{\mathbf{a}}\frac{\partial \mathcal{K}(\mathbf{p})}{\partial \mathbf{p}_{\mathbf{b}}} - \mathbf{p}^{\mathbf{b}}\frac{\partial \mathcal{K}(\mathbf{p})}{\partial \mathbf{p}_{\mathbf{a}}} \,,$$

$$\mathbf{J}^{\mathbf{ab}}\,\mathcal{N}^{\mathbf{i}}(\mathbf{p}) = -\mathcal{N}^{\mathbf{i}}(\mathbf{p})(\mathbf{J}^{\mathbf{ab}})^{\mathbf{T}} - \mathbf{p}^{\mathbf{a}}\frac{\partial \mathcal{N}^{\mathbf{i}}(\mathbf{p})}{\partial \mathbf{p}_{\mathbf{b}}} + \mathbf{p}^{\mathbf{b}}\frac{\partial \mathcal{N}^{\mathbf{i}}(\mathbf{p})}{\partial \mathbf{p}_{\mathbf{a}}} \,.$$

3. On-shell condition: $\epsilon_{\mathbf{i},\alpha}\,\mathcal{K}^{\alpha\beta}(\mathbf{p_i}) = \mathbf{0}$

Final result for the soft graviton amplitude to subleading order:

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j}(\mathbf{p_j}) \left[\mathbf{S^{(0)}} \Gamma^{\alpha_1 \ldots \alpha_N} + \left\{ \mathbf{S^{(1)}} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} \right]$$

$$\mathbf{S^{(0)}} \equiv \sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \, \varepsilon_{\mu\nu} \, \mathbf{p_i}^{\mu} \, \mathbf{p_i}^{\nu}$$

$$\left\{ \mathbf{S^{(1)}} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} = \sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \, \varepsilon_{\mu\nu} \, \mathbf{p_i}^{\mu} \, \mathbf{k}_{\rho} \left(\mathbf{p_i}^{\nu} \frac{\partial}{\partial \mathbf{p_{i\rho}}} - \mathbf{p_i}^{\rho} \frac{\partial}{\partial \mathbf{p_{i\nu}}} \right) \Gamma^{\alpha_1 \ldots \alpha_N}$$

$$+ \sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \, \varepsilon_{\mu \mathbf{b}} \, \mathbf{p_i}^{\mu} \, \mathbf{k_a} \left(\mathbf{J^{ab}} \right)_{\gamma}^{\alpha_i} \Gamma^{\alpha_1 \ldots \alpha_{i-1} \gamma \alpha_{i+1} \ldots \alpha_N}$$

This is the subleading soft graviton theorem

– agrees with all known results in field theory / string theory

The sub-subleading amplitude has a universal term and a non-universal term.

Universal term:

$$\frac{1}{2} \prod_{j=1}^{N} \epsilon_{j,\alpha_j}(\mathbf{p_j}) \sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \epsilon_{i,\alpha} \varepsilon_{\mathbf{ac}} \mathbf{k_b} \mathbf{k_d} \left[\left\{ \mathbf{p_i^b} \frac{\partial}{\partial \mathbf{p_{ia}}} - \mathbf{p_i^a} \frac{\partial}{\partial \mathbf{p_{ib}}} \right\} \delta_{\beta}^{\;\alpha_i} + (\mathbf{J^{ab}})_{\beta}^{\;\alpha_i} \right]$$

$$\left[\left\{ \mathbf{p_i^d} \frac{\partial}{\partial \mathbf{p_{ic}}} - \mathbf{p_i^c} \frac{\partial}{\partial \mathbf{p_{id}}} \right\} \delta_{\gamma}^{\;\beta} + (\mathbf{J^{cd}})_{\gamma}^{\;\beta} \right] \Gamma^{\alpha_1 \cdots \alpha_{i-1} \gamma \alpha_{i+1} \cdots \alpha_N}$$

Non-universal term

$$\frac{1}{2} \prod_{j=1}^{N} \epsilon_{j,\alpha_j}(\mathbf{p_j}) \left(\varepsilon^{\mu\nu} \mathbf{k}^\rho \mathbf{k}^\sigma \;-\; \varepsilon^{\mu\sigma} \mathbf{k}^\nu \mathbf{k}^\rho \;-\; \varepsilon^{\nu\rho} \mathbf{k}^\sigma \mathbf{k}^\mu \;+\; \varepsilon^{\rho\sigma} \mathbf{k}^\mu \mathbf{k}^\nu \right)$$

$$\sum_{i=1}^{N} (\mathbf{p_i} \cdot \mathbf{k})^{-1} \mathbf{M}^{\alpha_i}_{\gamma;\mu\rho\nu\sigma}(-\mathbf{p_i}) \, \Gamma^{\alpha_1 \cdots \alpha_{i-1} \gamma \alpha_{i+1} \cdots \alpha_N}$$

$$M^{\alpha}_{\delta;\mu\nu\sigma}(-\mathbf{p_i})$$

$$= \left\{ \frac{1}{3} \, i \, \mathbf{p}^{\nu} \, \frac{\partial \mathcal{K}^{\alpha\beta}(-\mathbf{p_i})}{\partial \mathbf{p_{i\mu}}} \, \frac{\partial^2 \mathcal{N}^{i}_{\beta\delta}(-\mathbf{p_i})}{\partial \mathbf{p_{i\rho}} \partial \mathbf{p_{i\sigma}}} - \frac{1}{6} \, i \, \frac{\partial^2 \mathcal{K}^{\alpha\beta}(-\mathbf{p_i})}{\partial \mathbf{p_{i\mu}} \partial \mathbf{p_{i\nu}}} \, \mathbf{p}^{\rho}_{i} \, \frac{\partial \mathcal{N}^{i}_{\beta\delta}(-\mathbf{p_i})}{\partial \mathbf{p_{i\sigma}}} \right.$$

$$+ \frac{i}{4} \, \frac{\partial \mathcal{K}^{\alpha\gamma}(-\mathbf{p_i})}{\partial \mathbf{p_{i\mu}}} \, \frac{\partial \mathcal{N}^{i}_{\gamma\beta}(-\mathbf{p_i})}{\partial \mathbf{p_{i\rho}}} (\mathbf{J}^{\nu\sigma})^{\beta}_{\delta} - \frac{1}{4} (\mathbf{J}^{\mu\rho})^{\alpha}_{\beta} (\mathbf{J}^{\nu\sigma})^{\beta}_{\delta}$$

$$\left. + i \, \mathcal{B}^{\alpha\beta;\mu\nu\sigma}(-\mathbf{p_i}) \, \mathcal{N}^{i}_{\beta\delta}(-\mathbf{p_i}) \right\}$$

$$\mathcal{N}^{i}(\mathbf{p}) = i \, (\mathbf{p}^2 + \mathbf{M}^2_i) \, \mathcal{K}^{-1}(\mathbf{p})$$

$\mathbf{M_i}$: mass of the i-th external particle

$\mathcal{B}$: non-minimal coupling of ϕ_{α} to the soft graviton

$$\frac{1}{2} \int \frac{\mathbf{d^D q_1}}{(2\pi)^{\mathbf{D}}} \frac{\mathbf{d^D q_2}}{(2\pi)^{\mathbf{D}}} (2\pi)^{\mathbf{D}} \delta^{(\mathbf{D})}(\mathbf{q_1} + \mathbf{q_2} + \mathbf{k}) \, \mathbf{R}_{\mu\rho\nu\sigma} \, \phi_{\alpha}(\mathbf{q_1}) \, \mathcal{B}^{\alpha\beta;\mu\nu\sigma}(\mathbf{q_2}) \, \phi_{\beta}(\mathbf{q_2})$$

$$\mathbf{R}_{\mu\rho\nu\sigma} = (\varepsilon_{\mu\nu}\mathbf{k}_{\rho}\mathbf{k}_{\sigma} \, - \, \varepsilon_{\mu\sigma}\mathbf{k}_{\nu}\mathbf{k}_{\rho} \, - \, \varepsilon_{\nu\rho}\mathbf{k}_{\sigma}\mathbf{k}_{\mu} \, + \, \varepsilon_{\rho\sigma}\mathbf{k}_{\mu}\mathbf{k}_{\nu})$$

– Riemann tensor of the soft graviton

Consistency checks:

1. Known results in string theory and field theory agree with this general formula

2. The final result depends only on the on-shell data of $\Gamma^{\alpha_1 \cdots \alpha_N}$ even though the individual terms depend on off-shell data.
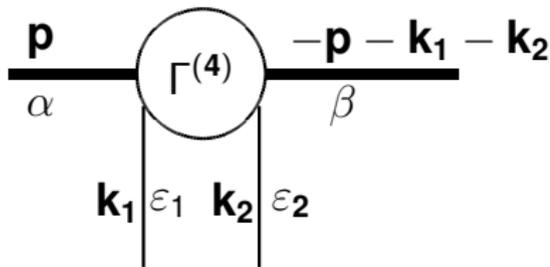
Example: Suppose we change

$$\Gamma^{\alpha_1 \cdots \alpha_N} \Rightarrow \Gamma^{\alpha_1 \cdots \alpha_N} + (p_i^2 + M_i^2) A$$

Change in $\partial \Gamma^{\alpha_1 \cdots \alpha_N} / \partial p_{i\mu}$ does not vanish on-shell.

However when we add all the terms in the soft graviton theorem, the result is unchanged.

Multiple soft gravitons

Power counting shows that to subleading order we need two additional type of vertices

–needed to leading order in soft momenta

– can be obtained by covariantizing the kinetic term to quadratic order in the soft graviton field.
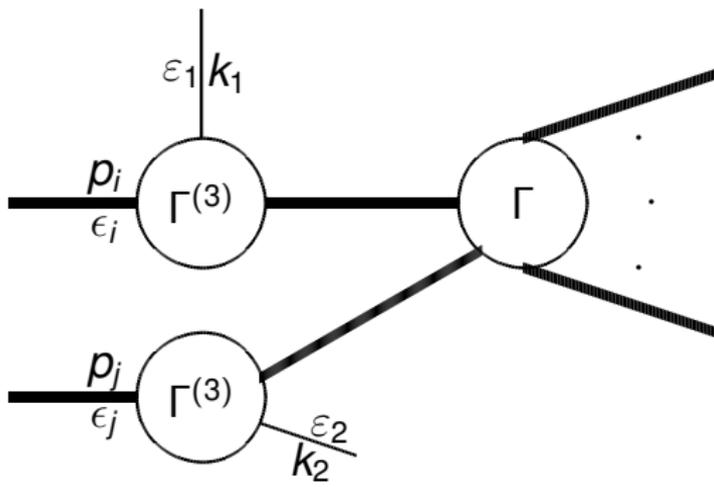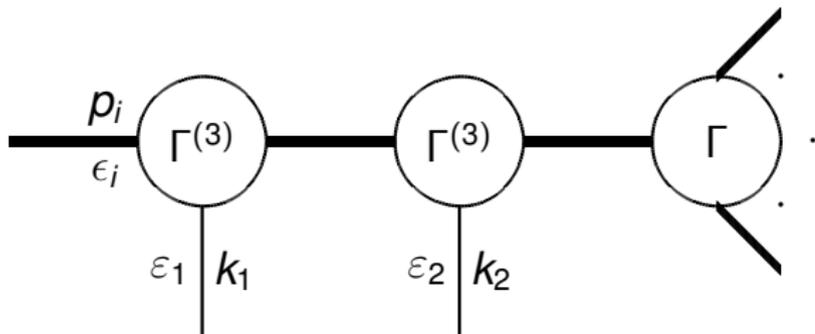
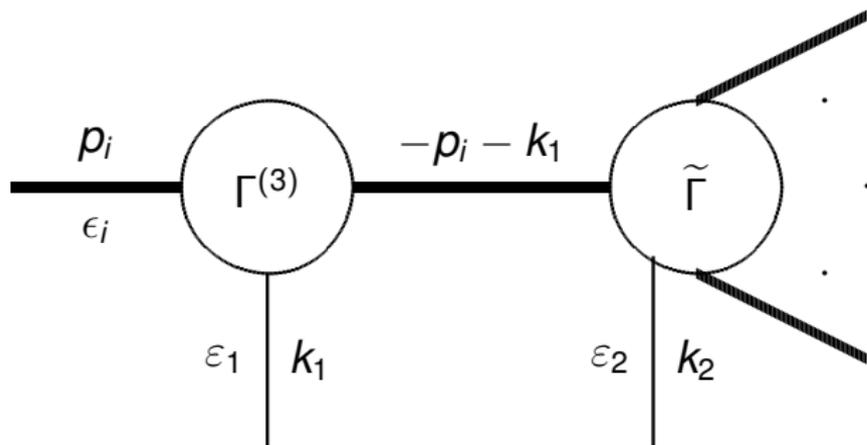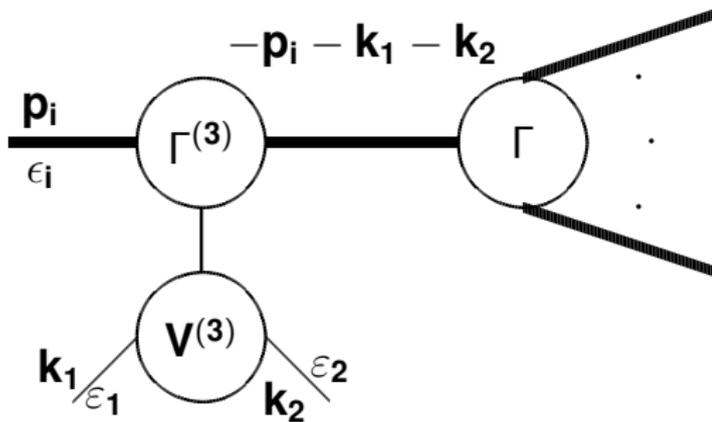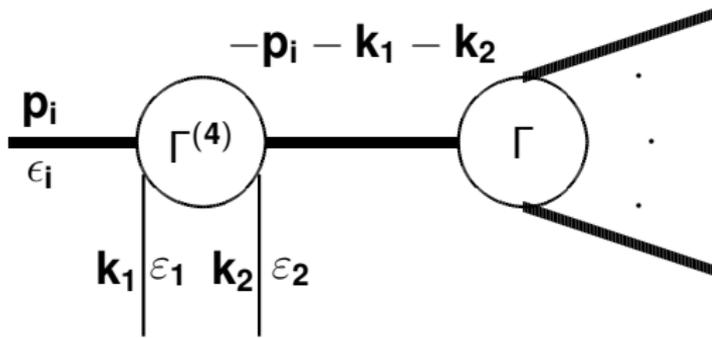**The other new vertex is three point coupling of soft gravitons.**
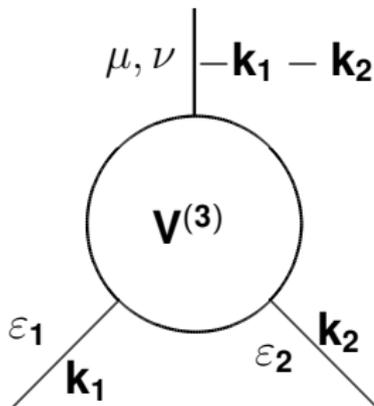
We need this to leading order in soft momenta

– can be obtained by expanding the Einstein-Hilbert action

For two soft gravitons, the following five kinds of diagrams need to be computed to subleading order

Result for M soft gravitons and N finite energy particles

Chakrabarti, Kashyap, Sahoo, A.S, Verma

$$\left\{ \prod_{i=1}^{N} \epsilon_{i,\alpha_i}(\mathbf{p}_i) \right\} \left[\left\{ \prod_{r=1}^{M} \mathbf{S}_r^{(0)} \right\} \Gamma^{\alpha_1 \cdots \alpha_N} + \sum_{s=1}^{M} \left\{ \prod_{\substack{r=1 \\ r \neq s}}^{M} \mathbf{S}_r^{(0)} \right\} \left[\mathbf{S}_s^{(1)} \Gamma \right]^{\alpha_1 \cdots \alpha_N} \right.$$

$$\left. + \sum_{\substack{r,u=1 \\ r < u}}^{M} \left\{ \prod_{\substack{s=1 \\ s \neq r,u}}^{M} \mathbf{S}_s^{(0)} \right\} \left\{ \sum_{j=1}^{N} \{ \mathbf{p}_j \cdot (\mathbf{k}_r + \mathbf{k}_u) \}^{-1} \ \mathcal{M}(\mathbf{p}_j; \varepsilon_r, \mathbf{k}_r, \varepsilon_u, \mathbf{k}_u) \right\} \Gamma^{\alpha_1 \cdots \alpha_N} \right]$$

$\mathbf{S}_r^{(0)}, \mathbf{S}_r^{(1)}$: Soft factors defined earlier for r-th soft graviton

$\mathcal{M}$: 'contact term'

$$\mathcal{M}(\mathbf{p_i}; \varepsilon_1, \mathbf{k_1}, \varepsilon_2, \mathbf{k_2})$$

$$= (\mathbf{p_i} \cdot \mathbf{k_1})^{-1}(\mathbf{p_i} \cdot \mathbf{k_2})^{-1} \Bigg\{ - \mathbf{k_1} \cdot \mathbf{k_2} \, \mathbf{p_i} \cdot \varepsilon_1 \cdot \mathbf{p_i} \, \mathbf{p_i} \cdot \varepsilon_2 \cdot \mathbf{p_i}$$

$$+ 2\, \mathbf{p_i} \cdot \mathbf{k_2} \, \mathbf{p_i} \cdot \varepsilon_1 \cdot \mathbf{p_i} \, \mathbf{p_i} \cdot \varepsilon_2 \cdot \mathbf{k_1} + 2\, \mathbf{p_i} \cdot \mathbf{k_1} \, \mathbf{p_i} \cdot \varepsilon_2 \cdot \mathbf{p_i} \, \mathbf{p_i} \cdot \varepsilon_1 \cdot \mathbf{k_2}$$

$$-2\, \mathbf{p_i} \cdot \mathbf{k_1} \, \mathbf{p_i} \cdot \mathbf{k_2} \, \mathbf{p_i} \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \mathbf{p_i} \Bigg\}$$

$$+ (\mathbf{k_1} \cdot \mathbf{k_2})^{-1} \Bigg\{ - (\mathbf{k_2} \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \mathbf{p_i})(\mathbf{k_2} \cdot \mathbf{p_i}) - (\mathbf{k_1} \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \mathbf{p_i})(\mathbf{k_1} \cdot \mathbf{p_i})$$

$$+ (\mathbf{k_2} \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \mathbf{p_i})(\mathbf{k_1} \cdot \mathbf{p_i}) + (\mathbf{k_1} \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \mathbf{p_i})(\mathbf{k_2} \cdot \mathbf{p_i})$$

$$- \varepsilon_1^{\gamma\delta} \varepsilon_{2\gamma\delta} (\mathbf{k_1} \cdot \mathbf{p_i})(\mathbf{k_2} \cdot \mathbf{p_i}) - 2(\mathbf{p_i} \cdot \varepsilon_1 \cdot \mathbf{k_2})(\mathbf{p_i} \cdot \varepsilon_2 \cdot \mathbf{k_1})$$

$$+ (\mathbf{p_i} \cdot \varepsilon_2 \cdot \mathbf{p_i})(\mathbf{k_2} \cdot \varepsilon_1 \cdot \mathbf{k_2}) + (\mathbf{p_i} \cdot \varepsilon_1 \cdot \mathbf{p_i})(\mathbf{k_1} \cdot \varepsilon_2 \cdot \mathbf{k_1}) \Bigg\},$$

– agrees with results for two soft gravitons in specific theories

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Infrared issues

Consider the case of two soft gravitons and consider the following diagram:
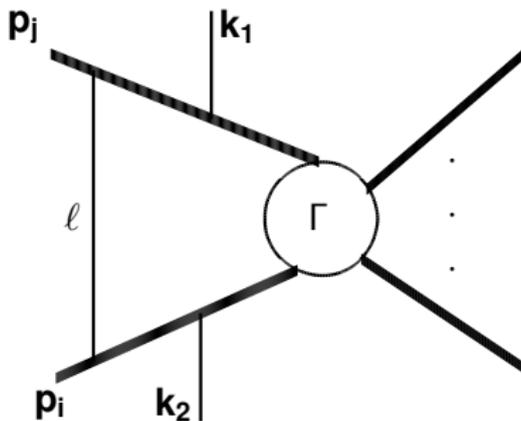
Naive power counting shows that the contribution of this diagram is subsubleading since it has no soft factor in the denominator

However if ℓ becomes soft then the loop integral can produce an inverse power of soft momentum for D=5

Standard soft factorization theorems tell us that in the sum over graphs, the region where ℓ is soft is suppressed in $D \geq 5$

Nevertheless, our analysis is based on covariantization of 1PI vertices which do receive contribution from soft loop integrals in D=5

$\Rightarrow$ **our results for multiple soft gravitons can be taken to be proved only for $D \geq 6$ for loop amplitudes**

For single soft gravitons one can show using independent argument that the results up to sub-subleading order hold for all $D \geq 5$.

Laddha, A.S.