Classical double copy for strings and other extended objects

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Based on
1611.03493, 1711.09493 (w/ Ridgway)
1705.09263 (w/ Prabhu, Thompson)
1712.09250 (w/ Li, Prabhu)
w/ Li , to appear

Motivation: (1) "Gravity as the square of gauge theory"

Does the color kinematics "duality" (Bern, Carrasco, Johanson 2008) relate classical solutions in YM and GR? This was first raised by:

Monteiro, O'Connell, White (2014)

Luna, Monteiro, O'Connell, White (2015)

Luna, Monteiro, Nicholson, O'Connell, White (2016)

who pointed out connections between classical YM, GR solutions:

$$A_\mu \mapsto g_{\mu\nu} = \eta_{\mu\nu} + A_\mu A_\nu \qquad (\eta^{\mu\nu} A_\mu A_\nu = 0)$$

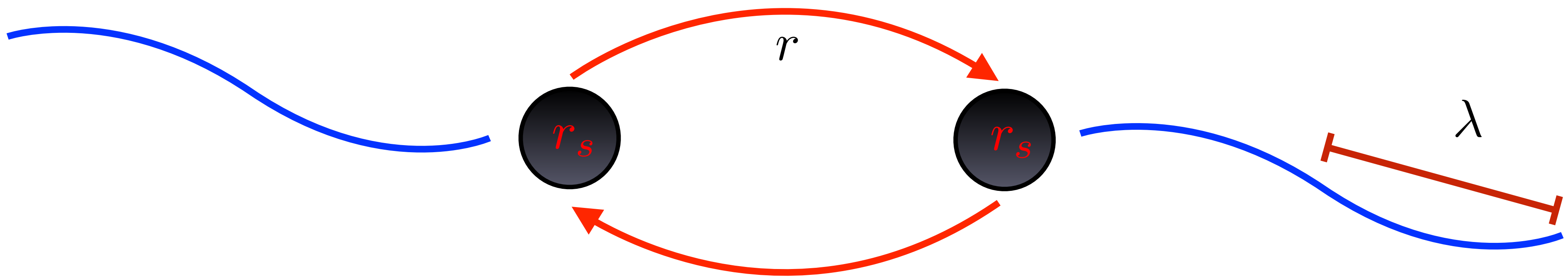
(eg: coulomb $\mapsto$ Schwarzschild, Kerr, plane waves,...)

"Double copy"

But does this Kerr-Schild double copy also appear in less symmetric configurations, such as radiation from black holes in bound orbits?

Motivation: (II) Gravitational wave sources

Gravitational dynamics of radiating classical BH (or NS) binary systems in the non-relativistic limit is experimentally relevant (LIGO/VIRGO, LISA,…)



Experiments will be sensitive to at least v^6 corrections beyond Newtonian gravity (Thorne et al 1994). Numerical GR results also motivate computing higher order corrections.

For this system, the radiation field measured by observers at infinity

$$\boxed{\lim_{|\vec{x}|\to\infty} h_{ij}^{TT}(t,\vec{x})}$$

encodes all the relevant physical info about this system (masses,spins, multipole moments,QNM frequencies,...). In perturbation theory, it is most conveniently computed by recasting Einstein's equations in the form (eg Weinberg 1972)

$$h_{\mu\nu}(x) = \frac{1}{2m_{Pl}} \int_k \frac{e^{-ik\cdot x}}{k^2} \left[\tilde{T}^{\mu\nu}(k) - \frac{1}{2}\eta_{\mu\nu}\tilde{T}^{\sigma}{}_{\sigma}\right]$$

(deDonder gauge)

$$\tilde{T}^{\mu\nu} = T^{\mu\nu}_{pp} + T^{\mu\nu}_{g} =$$ EM pseudotensor $$\partial_\mu \tilde{T}^{\mu\nu} = 0$$

$$\sim h\partial^2 h + h^2\partial^2 h + \cdots$$

$$S_{pp} = -m\int d\tau + c\int d\tau R^2_{\mu\nu\alpha\beta} + \cdots$$

$$c_{NS} \sim mR^4 \qquad c_{BH,d=4} = 0$$

(Damour et al; Poisson et al;Kol+Smolkin 2010)

The radiation field at infinity has a simple relation to the pseudo-tensor evaluated on-shell

$$\boxed{h_\pm(t,\vec{n}) = \frac{4G_N}{r}\int \frac{d\omega}{2\pi} e^{-i\omega t} \epsilon^{*ij}_\pm(k)\tilde{T}_{ij}(k)}$$

(Weinberg 1972)

w/

$$k^\mu = \omega(1, \vec{n} = \frac{\vec{x}}{r})$$

$$k^2 = 0$$

In practice computing higher order terms in perturbation theory $(v \ll 1)$ is difficult for two reasons:

Many terms in the expansion of $\tilde{T}^{\mu\nu}(x)$ at high orders in $h_{\mu\nu}$

Many physically relevant scales

Gravitational radius: $r_g = 2G_N M$

Physical radius: $r_s (= r_g \text{ for BH})$

Orbital scale: r

Radiation wavelength λ

$$r_g \sim r_s \gg r \gg \lambda$$

all correlated to the perturbative expansion parameter

$$r \sim r_g / v^2 \qquad \lambda \sim r/v \sim r_g / v^3$$

Some of these challenges can be ameliorated by employing tools borrowed from the analysis of $Q\bar{Q}$ bound states in QCD (WG+Rothstein 2005):

Many terms in the expansion of $\tilde{T}^{\mu\nu}(x)$ at high orders in $h_{\mu\nu}$

→ Organize the expansion in terms of Feynman diagrams

Many physically relevant scales

→ Treat each scale separately, by constructing a tower of gravity Effective Field Theories

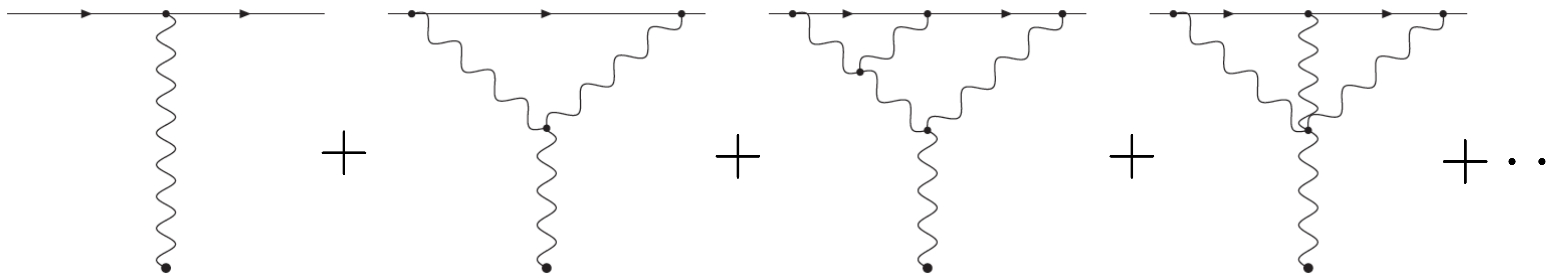
$$h_{\mu\nu} = h_{\mu\nu}^{potential} + h_{\mu\nu}^{rad}$$

The focus of this talk is the Feynman diagram expansion.

The types of Feynman diagrams that are relevant are of the same type as in Duff's (1973) perturbative construction of the Schwarzschild solution:

(NOT A PROPAGATOR!)

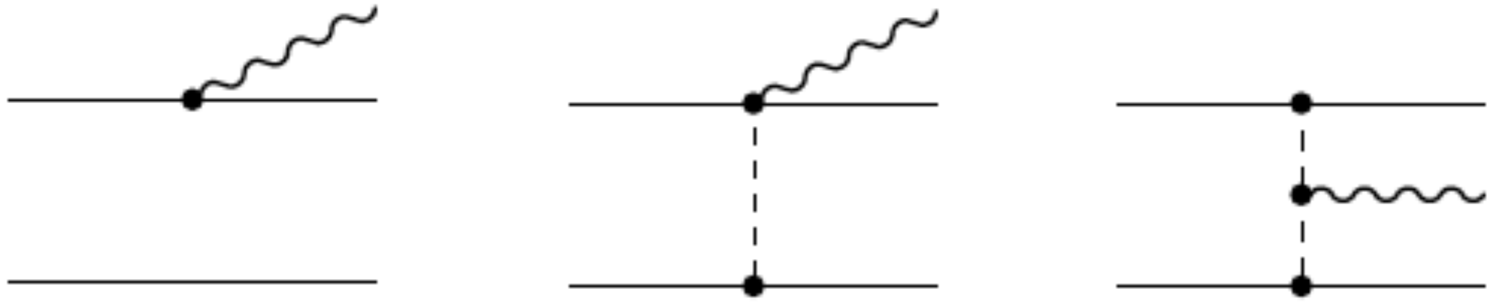
$h_{\mu\nu} =$



$$
\begin{aligned}
g_{00} &= 1 - \frac{2G_N m}{r} + 2\left(\frac{G_N m}{r}\right)^2 + 2\left(\frac{G_N m}{r}\right)^3 + \cdots \\
g_{ij} &= -\delta_{ij}\left[1 + \frac{2G_N m}{r} + 5\left(\frac{G_N m}{r}\right)^2 - \frac{2}{3}\left(\frac{G_N m}{r}\right)^3 + \cdots\right] \\
&\quad + \frac{x_i x_j}{r^2}\left[7\left(\frac{G_N m}{r}\right)^2 - \frac{4}{3}\left(\frac{G_N m}{r}\right)^3 + \cdots\right]
\end{aligned}
$$

Same type of diagrams in the two-body sector:

(WG+Ross,2010)

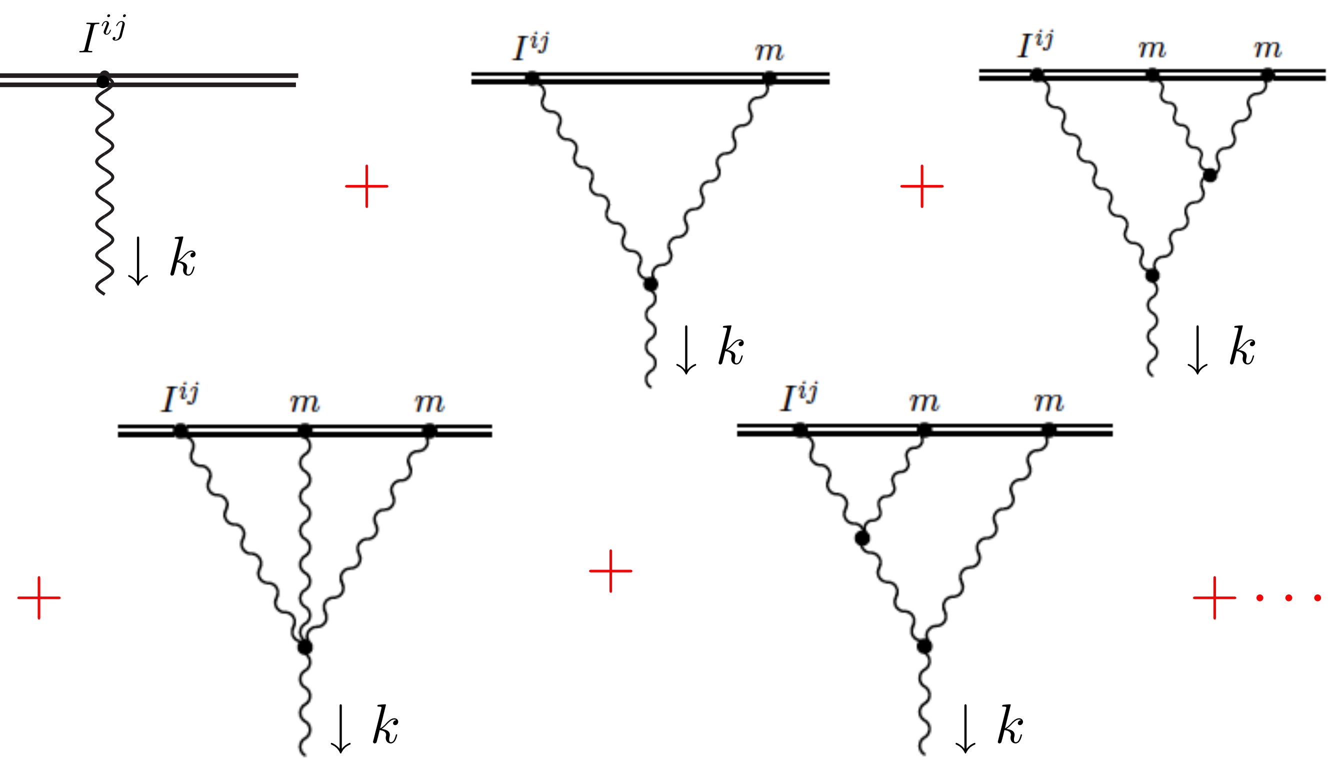


Quadrupole moment

$$I_{ij} = \sum_a m_a \mathbf{x}_a^i \mathbf{x}_a^j \left[1 + \frac{3}{2}\mathbf{v}_a^2 - \sum_b \frac{G_N m_b}{|\mathbf{x}_a - \mathbf{x}_b|}\right] + \frac{11}{42}\sum_a m_a \frac{d^2}{dt^2}(\mathbf{x_a}^2 \mathbf{x}_a^i \mathbf{x}_a^j)$$
$$-\frac{4}{3}\sum_a m_a \frac{d}{dt}(\mathbf{x_a}\cdot\mathbf{v}_a \mathbf{x}_a^i \mathbf{x}_a^j) - \text{traces} + \mathcal{O}(v^4)$$

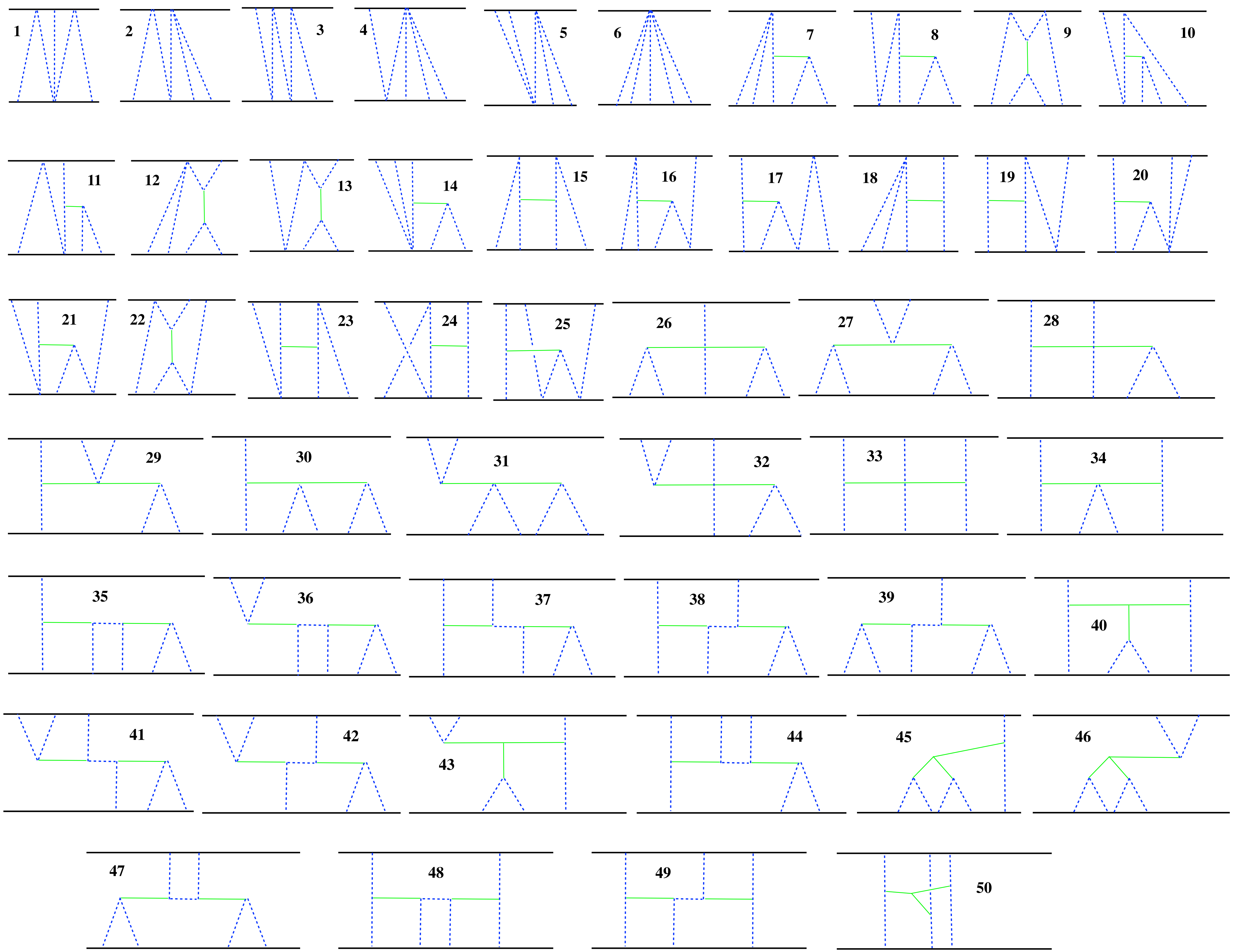
Quadrupole Radiation:

$$h_{ij}^{TT} =$$



$$\frac{\dot{E}_{3PN,LL}}{\dot{E}_{LO}} = -\frac{1712}{105}v^6 \ln v$$

State of the art is 4PN order: see Foffa et al, 1612.00482



Q: Is there a more efficient way, based on modern amplitude methods, eg color kinematics duality (BCJ) to organize the perturbative expansion?

(Other amplitude based approaches to perturbative binary dynamics:
Neill and Rothstein (2013)
Bjerrum-Bohr, Donoghue, Vanhove (2013)
Cachazo and Guevara (2017)
Bjerrum-Bohr, Damgaard, Festuccia, Plant, Vanhove (2018)
Cheung, Rothstein, Solon (2018))

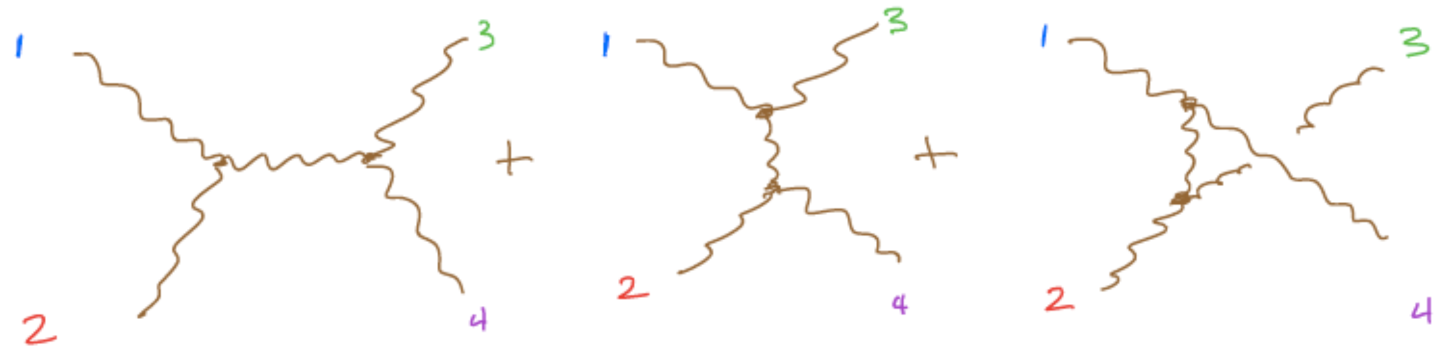
The BCJ double copy

Bern, Carrasco and Johansson (2008)

Scattering amplitudes in gauge theory and perturbative gravity obey simple "color-kinematics" relations.

E.g., $2 \to 2$ gluon scattering at tree-level. Up to overall norm, amplitude can be put into "BCJ form"

$$\mathcal{A}_4^{YM} =$$

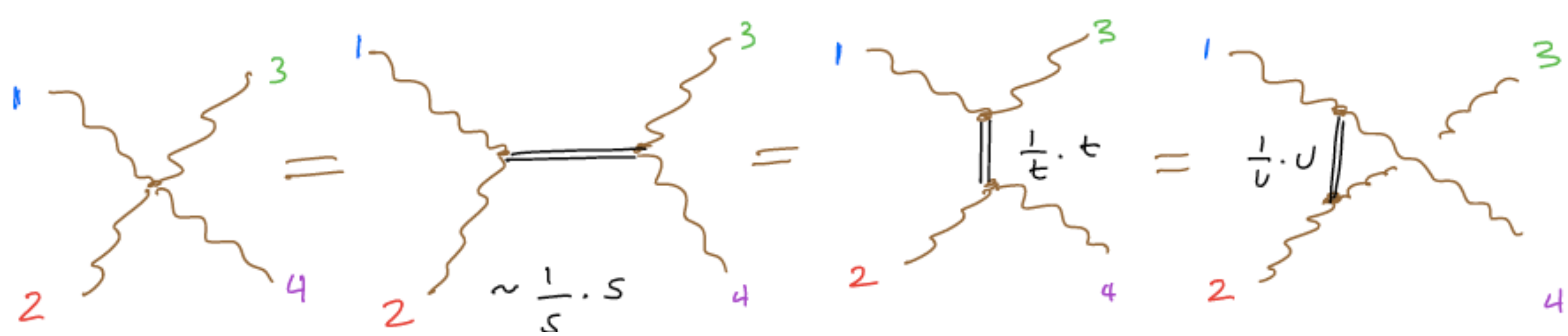


$$= \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

w/ color structures $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$ $c_t = f^{a_1 a_3 b} f^{b a_2 a_4}$ $c_u = f^{a_1 a_4 b} f^{b a_2 a_3}$

Lie alg. Jacobi identity: $c_s + c_t + c_u = 0$

(4-gluon vertex has been "blown up" and absorbed into s, t, u channels)



The "kinematic structures" $n_{s,t,u}$ are polynomials in the kinematic invariants, polarizations.
They can be chosen to satisfy a relation:

"Kinematic" Jacobi id: $$n_s + n_t + n_u = 0$$

(The decomposition is not unique due to Jacobi identities. There is a "generalized gauge transformation"

$$n_s \to n_s + s\Delta$$

$$n_t \to n_t + t\Delta \quad \Longrightarrow \quad \mathcal{A}_4^{YM} \to \mathcal{A}_4^{YM}$$

$$n_u \to n_u + u\Delta$$

that preserves the BCJ form of the amplitude.)

BCJ noticed that by applying the color-to-kinematics "duality" transformations:

$$c_s \to n_s \qquad c_t \to n_t \qquad c_u \to n_u$$

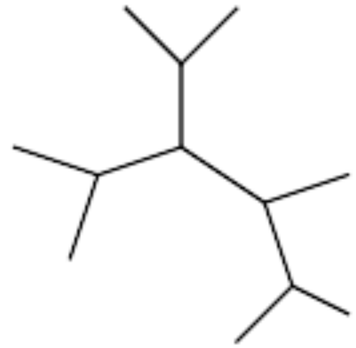
the 4-gluon amplitude maps onto

$$\mathcal{A}_4^{YM} \to \hat{\mathcal{A}}_4^{GR} = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

This precisely matches tree-level graviton scattering in Einstein-Hilbert gravity.

More generally, any tree level n-point amplitude in gauge theory can be put into the form

$$\mathcal{A}_n^{YM} = \sum_i$$



$$= \sum_i \frac{n_i c_i}{\prod \text{Propagators}}$$

(Eq. 13.2 of Elvang+Huang, w/ author's permission)

(sum is over graphs with tri-linear vertices, after blowing up 4-gluon interactions).

Then the BCJ double copy amplitude, $c_i \to \tilde{n}_i$

$$\mathcal{A}_n^{YM} \to \tilde{\mathcal{A}}_n^{grav} = \sum_i \frac{n_i \tilde{n}_i}{\prod \text{Propagators}}$$

is an n-point graviton scattering amplitude.

These BCJ relations have been established for arbitrary dimension d, and gauge group G at tree level. (Bern et al 2010)

At the loop level, BCJ remains conjectural, but many non-trivial (up to ~~4~~ 5-loop) checks suggests that these relations continue to hold.

In general the double copy gravitational theory is not pure GR. Its spectrum is the square of the gauge theory one.

In d-dimensional pure Yang-Mills, the gluon maps to the gravity d.o.f's

$$\epsilon^a_\mu(k) \to \epsilon_\mu(k)\tilde{\epsilon}_\nu(k) = \; h_{\mu\nu} \text{ (graviton)} \oplus B_{\mu\nu} \text{ (2-form gauge field; d=4, axion)} \oplus \phi \text{ (dilaton)}$$

all of which propagate in loops. This is also expected from KLT relations in string theory.

Gauge supermultiplets, e.g.:

$$(\mathcal{N}=4)\ \mathrm{SYM} \to (\mathcal{N}=4) \otimes (\mathcal{N}=4) = (\mathcal{N}=8)\ \mathrm{SUGRA}$$

etc.

Classical radiation from the double copy

Does the BCJ double copy relate other observables, not just the S-matrix? Can it be used to obtain classical perturbative solutions in gravity from (computationally simpler) solutions in YM?

In the context of radiation, evidence that the answer is "YES" was provided at lowest non-trivial order in perturbation theory in (WG+A. Ridgway, 1611.03493).

Check by explicit calculation on both sides of the color-kinematics correspondence and using the results to guess the between YM and gravity solutions.

Gauge Theory Solutions

Solve the classical Yang-Mills equations coupled to classical point color charges.

$$D_\nu F_a^{\nu\mu}(x) = g J_a^\mu(x)$$

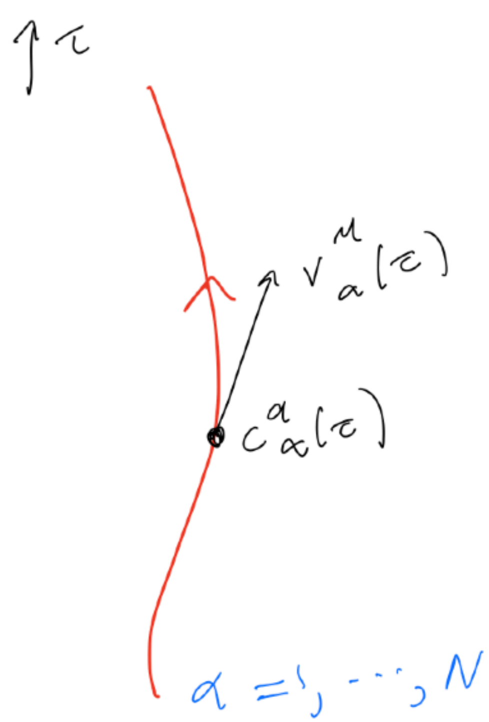
By classical color charge we mean an object whose degrees of freedom are

$x^\mu(\tau)$ =wordline coordinate

$c^a(\tau)$ =color d.o.f in adjoint at $x(\tau)$

(Sikivie and Weiss, 1978)

Current for a collection of charges



$$J_a^\mu(x) = \sum_\alpha \int d\tau c_\alpha^a(\tau) v_\alpha^\mu(\tau) \delta^d(x - x_\alpha(\tau)) + \cdots$$

(finite size terms and other color moments)

The particle equations of motion follow from conservation laws. Color conservation:

$$D_\mu J_a^\mu = 0 \Rightarrow v \cdot D c^a = 0$$

or in terms of adjoint rep. Wilson line $\quad c_\alpha^a(\tau) = W_\alpha{}^a{}_b(\tau) c_\alpha^b(-\infty)$

$$W_\alpha{}^a{}_b(\tau) = \left[P \exp\left\{ -ig \int_{-\infty}^{\tau} dx_\alpha^\mu A_\mu \cdot T_{\text{adj}} \right\} \right]^a{}_b.$$

The orbital motion is fixed by energy-momentum and gives the non-Abelian Lorentz force law:

$$\partial_\mu (T_{YM}^{\mu\nu} + T_{pp}^{\mu\nu}) = 0 \Rightarrow \frac{dp^\mu}{d\tau} = g c^a F_a^{\mu\nu} v_\nu$$

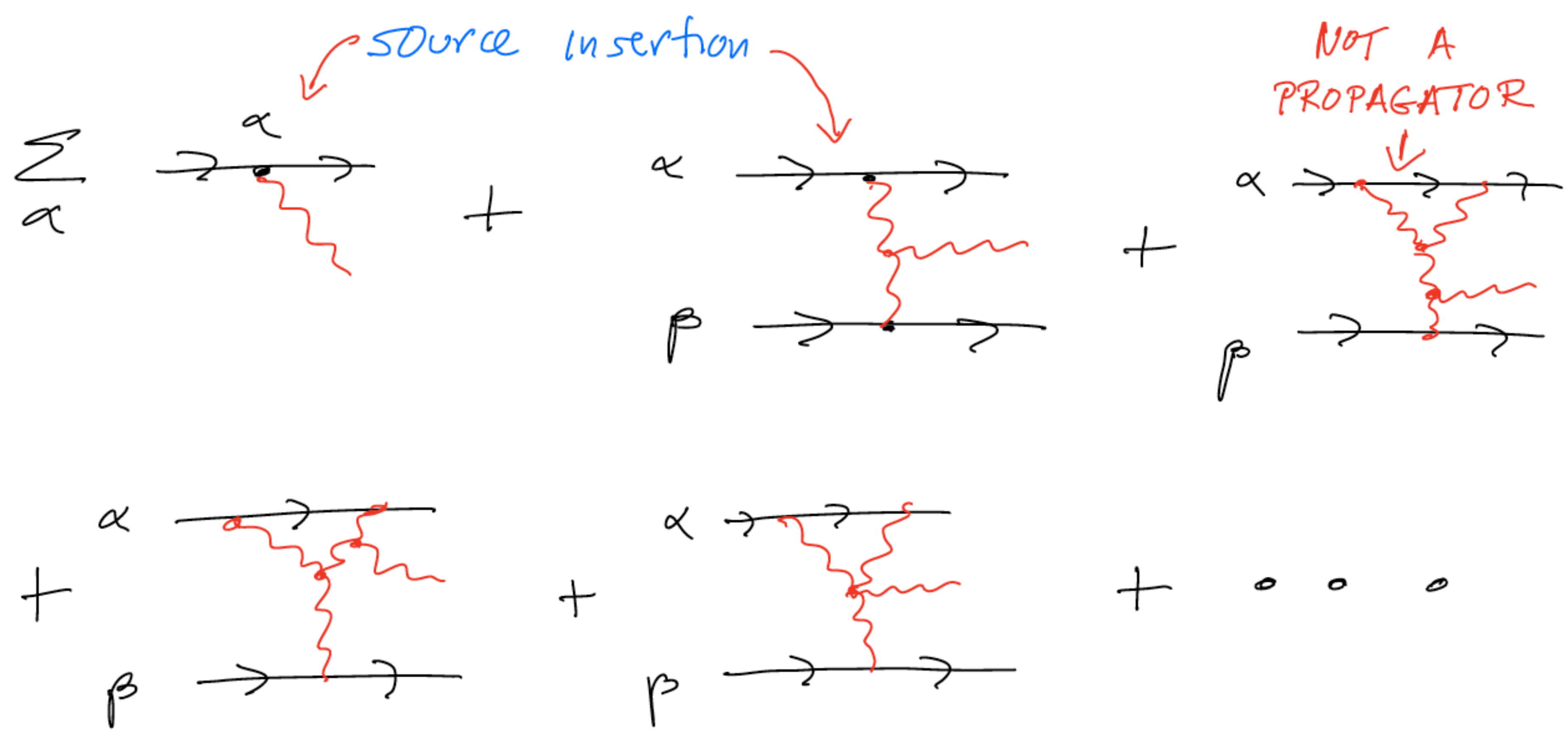
We find it convenient to solve the coupled equations in Lorentz gauge, in the form

$$\Box A_a^\mu = g\tilde{J}_a^\mu \qquad \tilde{J}_a^\mu = J_a^\mu + f^{abc} A_\nu^b (\partial^\nu A_c^\mu - F_c^{\mu\nu})$$

$$\partial_\mu \tilde{J}_a^\mu = 0$$

The classical solution is an off shell one-point function in the presence of (self-consistent) sources:

$$\langle A \rangle_J =$$



Our focus in this talk is on the radiation field measured by observers at $r \to \infty$ and fixed retarded time t (i.e. "asymptotic null future infinity"). This is related to the on-shell momentum space current

$$\tilde{J}^{\mu}_{a}(k) = \int d^d x e^{ik\cdot x} \tilde{J}^{\mu}_{a}(x)$$

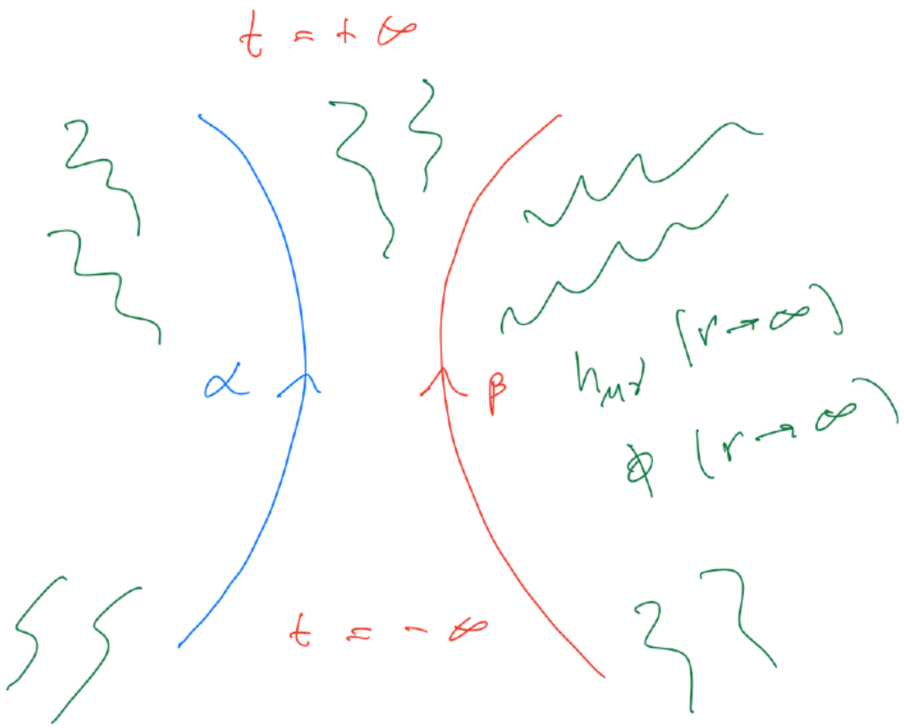
w/ $k^2 \to 0$. E.g., in four dimensions, the asymptotic gauge field is

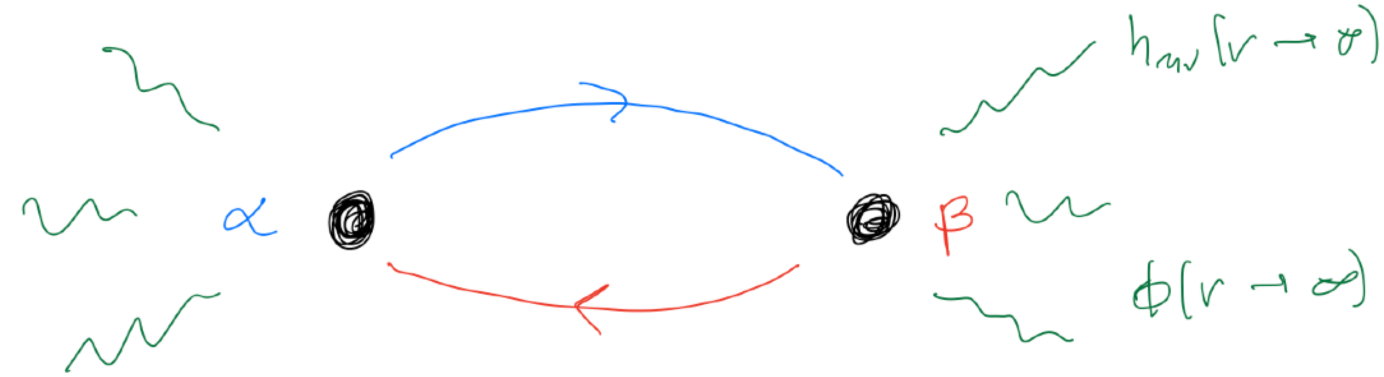
$$\lim_{r\to\infty} \langle A^{a}_{\mu} \rangle(x) = \frac{g}{4\pi r} \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{J}^{\mu}_{a}(k)$$

$$k^{\mu} = (\omega, \vec{k}) = \omega(1, \vec{x}/r)$$

This determines all observables measured by detectors at $r \to \infty$

Consider a collection of particles in generic time-dependent orbits:





$$\omega b \sim \mathcal{O}(1) \qquad\qquad \omega b \sim v \ll 1$$

or

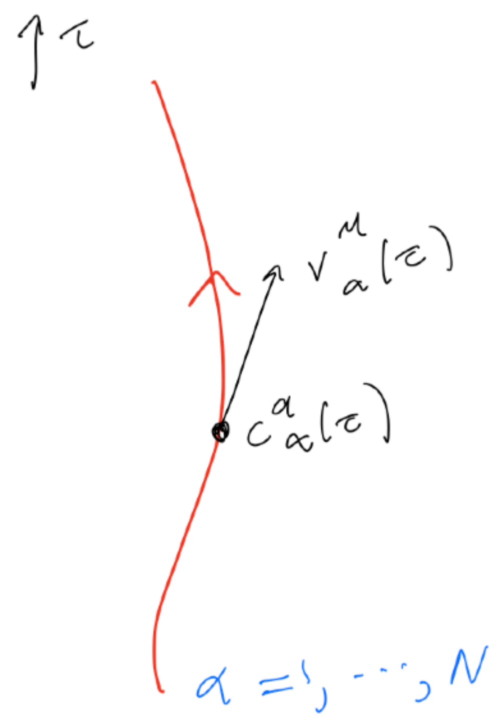
$$E/m \sim \mathcal{O}(1) \qquad\qquad E/m \sim v^2 \ll 1$$

Formally, if the particles remain well separated, the perturbative solution for the radiation field can be constructed as an expansion in powers of the gauge coupling g . (Less formally, we have a double expansion:

$$g^2 c b^{4-d} \ll 1 \qquad g^2 c^2/Eb^{d-3} \ll 1 \qquad c^a \sim Eb \gg 1 \;)$$

Perturbative solution: Leading order equations of motion:

A fixed configuration of worldlines $x^\mu(\tau), c^a(\tau)$ sources a gauge field



$$A^{a\mu}(x) = g \int \frac{d^4\ell}{(2\pi)^4} \frac{e^{-i\ell\cdot x}}{\ell^2} J^{a\mu}(\ell) = g \sum_\alpha \int d\tau \frac{d^4\ell}{(2\pi)^4} \frac{e^{-i\ell\cdot(x-x_\alpha)}}{\ell^2} c^a_\alpha(\tau) v^\mu_\alpha(\tau).$$

Feed back to get self-consistent eqns. of motion:

$$m_\alpha \dot{v}_\alpha = \quad i g^2 \sum_\beta \int d\tau_\beta \frac{d^4\ell}{(2\pi)^4} \frac{e^{-i\ell\cdot x_{\alpha\beta}}}{\ell^2} (c_\alpha \cdot c_\beta) \left[(v_\alpha \cdot v_\beta)\ell^\mu - (\ell \cdot v_\alpha) v^\mu_\beta \right],$$

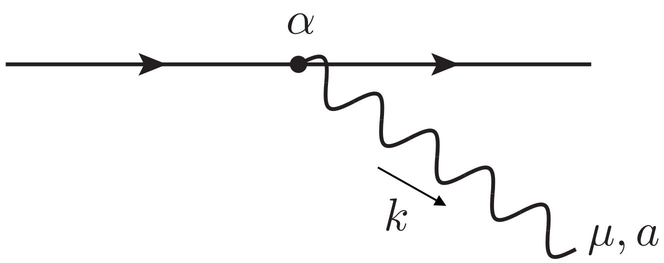
$$\dot{c}_\alpha = g^2 \sum_\beta \int d\tau_\beta \frac{d^4\ell}{(2\pi)^4} \frac{e^{-i\ell\cdot x_{\alpha\beta}}}{\ell^2} f^{abc} c^b_\alpha c^c_\beta (v_\alpha \cdot v_\beta)$$

w/ $\quad x^\mu_\alpha = x^\mu_\alpha(\tau_\alpha) \qquad v^\mu_\alpha = v^\mu_\alpha(\tau_\alpha) \qquad c^a_\alpha = c^a_\alpha(\tau_\alpha)$

Radiation:

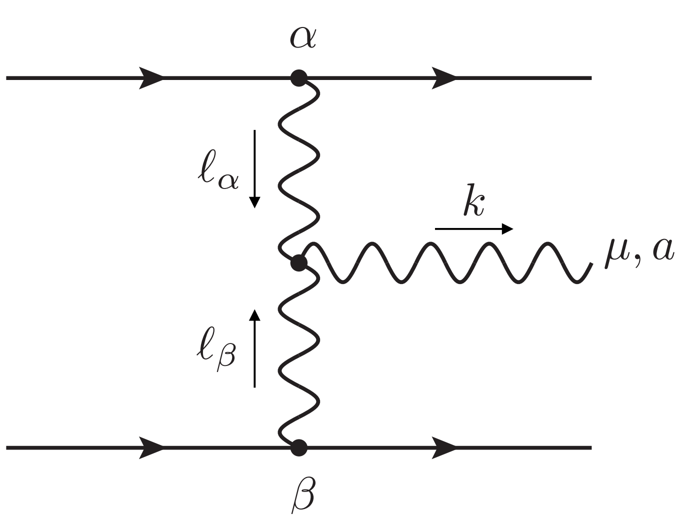
Starts at $\mathcal{O}(g^2)$. Two types of effects:

1. Direct emission by time-dependent particle current:



$$\tilde{j}^{a\mu}(k)\Big|_{\text{worldline}} = \sum_{\alpha} \int dx^{\mu}(\tau)_{\alpha} c_{\alpha}^{a}(\tau) e^{ik\cdot x_{\alpha}(\tau)}$$

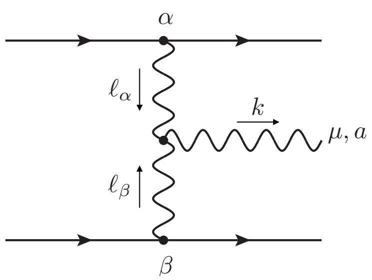
2. Gluon emission sourced by self-interaction



$$\propto f^{abc}$$

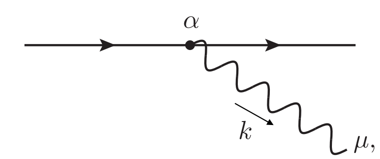
The total current has two types of color structures:

$$\tilde{J}^{a\mu}(k) = g^2 \sum_{\alpha,\beta} \int_{\alpha,\beta} d\mu_{\alpha\beta}(k) \left[i f^{abc} c_\alpha^b c_\beta^c \mathcal{A}^\mu_{adj} + (c_\alpha \cdot c_\beta) \frac{c_\alpha^a}{m_\alpha} \mathcal{A}^\mu_s \right],$$



color
eoms

orbital eom's



Integration measure:

$$d\mu_{\alpha\beta}(k) = d\tau_\alpha d\tau_\beta \left[\frac{d^4\ell_\alpha}{(2\pi)^4} \frac{e^{i\ell_\alpha \cdot x_\alpha}}{\ell_\alpha^2} \right] \left[\frac{d^4\ell_\beta}{(2\pi)^4} \frac{e^{i\ell_\beta \cdot x_\beta}}{\ell_\beta^2} \right]$$
$$\times (2\pi)^4 \delta(\ell_\alpha + \ell_\beta - k)$$

Partial amplitudes:

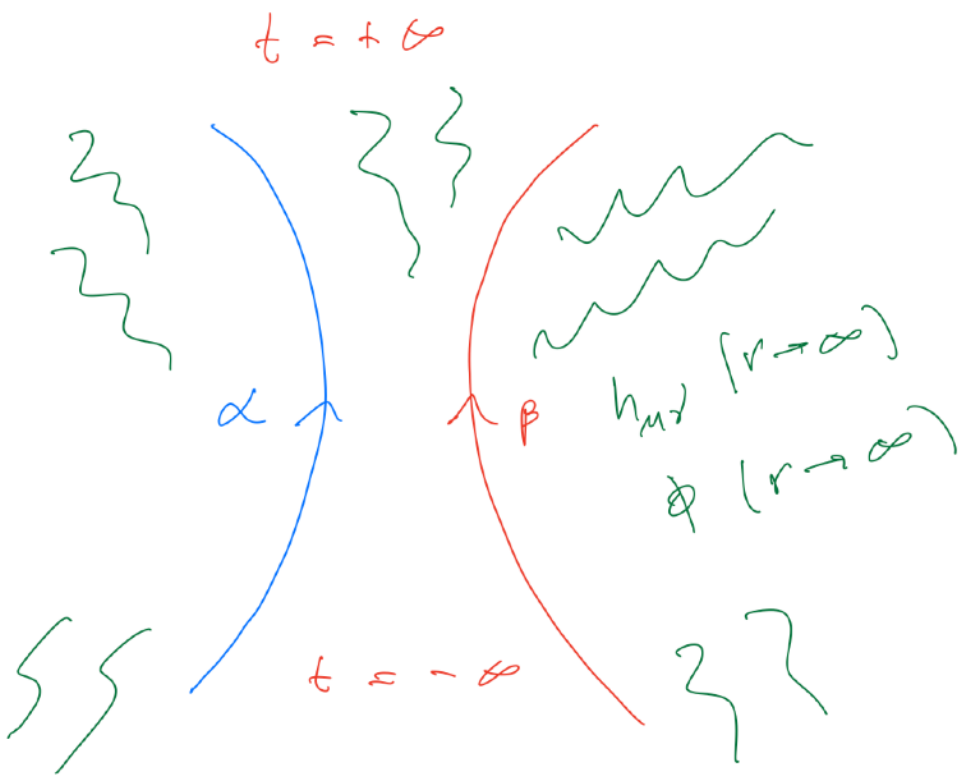
$$\boxed{\begin{aligned} \mathcal{A}^\mu_{adj} &= (v_\alpha \cdot v_\beta) \left[\frac{1}{2}(\ell_\beta - \ell_\alpha)^\mu + \frac{\ell_\alpha^2}{k \cdot v_\alpha} v_\alpha^\mu \right] + (k \cdot v_\beta) v_\alpha^\mu - (k \cdot v_\alpha) v_\beta^\mu, \\ \mathcal{A}^\mu_s &= -\frac{\ell_\alpha^2}{k \cdot v_\alpha} \left[(v_\alpha \cdot v_\beta) \left(\ell_\beta^\mu - \frac{k \cdot \ell_\beta}{k \cdot v_\alpha} v_\alpha^\mu \right) - (k \cdot v_\alpha) v_\beta^\mu + (k \cdot v_\beta) v_\alpha^\mu \right]. \end{aligned}}$$

As a check, the current obeys the Ward identity $k_\mu \tilde{J}^\mu{}_a(k) = 0$, but only after self-consistent particle eqns. of motion are plugged in to the diagrams.

Consistency checks I: Classical scattering and bremsstrahlung

$d=4$: Kovchegov+Rischke, 1997
Gyulassy+McLerran, 1997

1. Classical scattering and gluon bremstrhalung: Initially well separated particles with constant $b^\mu_\alpha, v^\mu_\alpha, c^a_\alpha$ at early times



$$x^\mu_\alpha(\tau\to-\infty) = b^\mu_\alpha + v^\mu_\alpha \tau$$

$$c^a_\alpha(\tau\to-\infty) = c^a_\alpha$$

$$\tilde{J}^\mu_a(k)\Big|_{\mathcal{O}(g^2)} = g^2 \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \int_{\ell_\alpha,\ell_\beta} \mu_{\alpha,\beta}(k)\left[\frac{c_\alpha\cdot c_\beta}{m_\alpha}\frac{\ell_\alpha^2}{k\cdot v_\alpha}c^a_\alpha\left\{-v_\alpha\cdot v_\beta\left(\ell^\mu_\beta - \frac{k\cdot\ell_\beta}{k\cdot v_\alpha}v^\mu_\alpha\right) + k\cdot v_\alpha v^\mu_\beta - k\cdot v_\beta v^\mu_\alpha\right\}\right.$$
$$\left. + i f^{abc} c^b_\alpha c^c_\beta \left\{2(k\cdot v_\beta)v^\mu_\alpha - (v_\alpha\cdot v_\beta)\ell^\mu_\alpha + (v_\alpha\cdot v_\beta)\frac{\ell^2_\alpha}{k\cdot v_\alpha}v^\mu_\alpha\right\}\right]$$

consistent with our earlier results in general d (WG+Ridgway, 2016)

Consistency checks II: Non-relativistic limit

$$v^\mu_\alpha(\tau) \approx (1, \vec{v}_\alpha = d\vec{x}_\alpha/dt) \qquad |\vec{v}_\alpha| \ll 1$$

Eqns of motion reduce to (in $d=4$)

$$\dot{c}_\alpha(t) = -\alpha_s \sum_\beta \frac{f^{abc} c^b_\alpha c^c_\beta}{|\vec{x}_{\alpha\beta}|}, \qquad m_\alpha \dot{\vec{v}}_\alpha(t) = \alpha_s \sum_\beta \frac{(c_\alpha \cdot c_\beta)\vec{x}_{\alpha\beta}}{|\vec{x}_{\alpha\beta}|^3},$$

Using the e.o.m's, the current simplifies to:

$$J^0_a(x) = \delta^3(\vec{x}) Q^a - \vec{p}_a(t) \cdot \nabla \delta^3(\vec{x}) + \mathcal{O}(v^2)$$
$$\vec{J}_a(x) = \delta^3(\vec{x}) \dot{\vec{p}}_a(t) + \mathcal{O}(v^2)$$

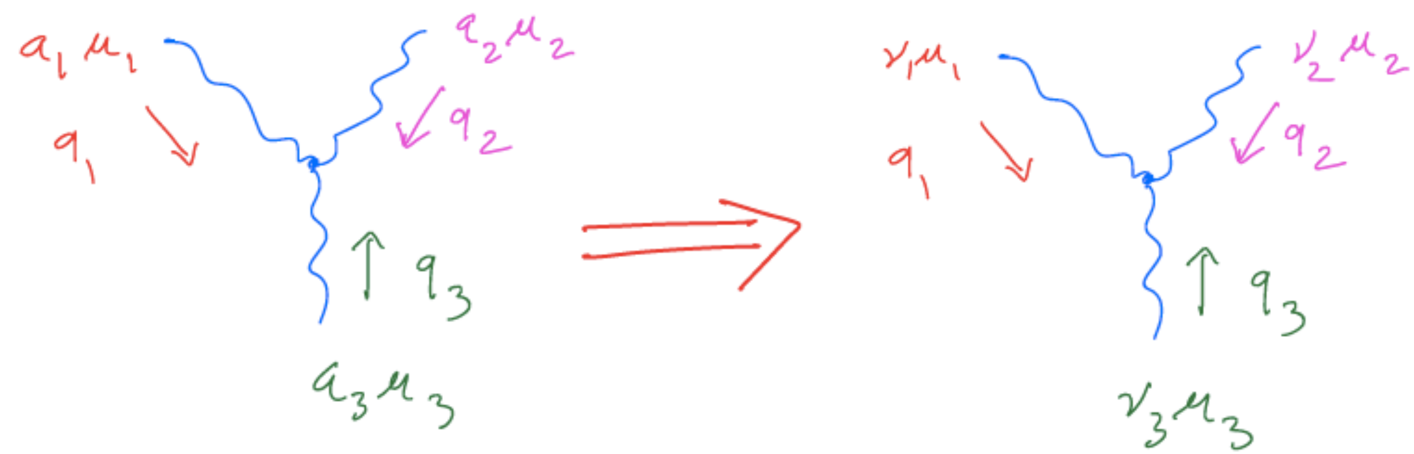
w/ color monopole moment $Q^a = \sum_\alpha c^a_\alpha(t) \quad (\dot{Q}^a = 0)$ and color electric dipole

$$\vec{p}^a(t) = \sum_\alpha c^a_\alpha(t) \vec{x}^a_\alpha(t)$$

Double copy of gauge theory?

In, 1611.03493 1711.09493, we (A. Ridgway+WG) proposed making the following formal replacements to the gauge theory result

$$\boxed{c_\alpha^a(\tau) \rightarrow ip_\alpha^\mu(\tau)}$$



$$if^{a_1a_2a_3} \rightarrow \Gamma^{\nu_1\nu_2\nu_3}(q_1,q_2,q_3) = -\frac{1}{2}\left[\eta^{\nu_1\nu_3}(q_1-q_3)^{\nu_2} + \eta^{\nu_1\nu_2}(q_2-q_1)^{\nu_3} + \eta^{\nu_2\nu_3}(q_3-q_2)^{\nu_1}\right],$$

$$\boxed{p_\alpha^\mu(\tau) \rightarrow p_\alpha^\mu(\tau)}$$

$$\boxed{g \rightarrow \frac{1}{2m_{Pl}^{d/2-1}}}$$

Under this map:

$$\tilde{J}^\mu_a(k) \to i\tilde{T}^{\mu\nu}(k)$$

w/

$$\boxed{\tilde{T}^{\mu\nu}(k) = \frac{1}{4m_{Pl}^{d-2}} \sum_{\alpha,\beta} m_\alpha m_\beta \int d\mu_{\alpha\beta}(k) \left[\left(\frac{1}{2}(v_\alpha \cdot v_\beta)(\ell_\beta - \ell_\alpha)^\nu + (v_\beta \cdot k) v_\alpha^\nu - (v_\alpha \cdot k) v_\beta^\nu \right) \mathcal{A}^\mu_{adj} - (v_\alpha \cdot v_\beta) v_\alpha^\nu \hat{\mathcal{A}}^\mu_s \right]}$$

For $k^2 = 0$ this object satisfies

$$\tilde{T}^{\mu\nu}(k) = \tilde{T}^{\nu\mu}(k) \qquad\qquad k_\mu \tilde{T}^{\mu\nu}(k) = 0$$

so it defines a consistent pseudo-tensor in some theory of gravity, with (in $d = 4$) radiation fields

$$h_\pm(t, \vec{n}) = \frac{4G_N}{r} \int \frac{d\omega}{2\pi} e^{-i\omega t} \epsilon_\pm^{*\mu\nu}(k) \tilde{T}_{\mu\nu}(k),$$

$$\phi(t, \vec{n}) = \frac{G_N}{r} \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{T}^\mu{}_\mu(k)$$

at future null infinity (retarded time t and $r \to \infty$).

Dilaton gravity:

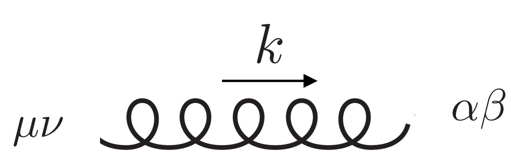
In fact, the gravity theory defined by $\tilde{T}^{\mu\nu}(k)$ is a local theory coupled to classical point sources, with Lagrangian

$$S = S_g + S_{pp}$$

$$S_g = -2m_{Pl}^{d-2} \int d^d x \sqrt{g} \left[R - (d-2) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]$$

$$S_{pp} = -m \int d\tau e^{\phi} + \cdots \,, \qquad \text{“} e^{\phi} = 1 + \phi + \frac{1}{2!}\phi^2 + \cdots \text{”}$$

(up to higher derivatives, more powers of the dilaton…). The role of the dilaton is to cancel the explicit d's from graviton exchange Bern+Grant(1999); Scherk+Schwarz (1974):



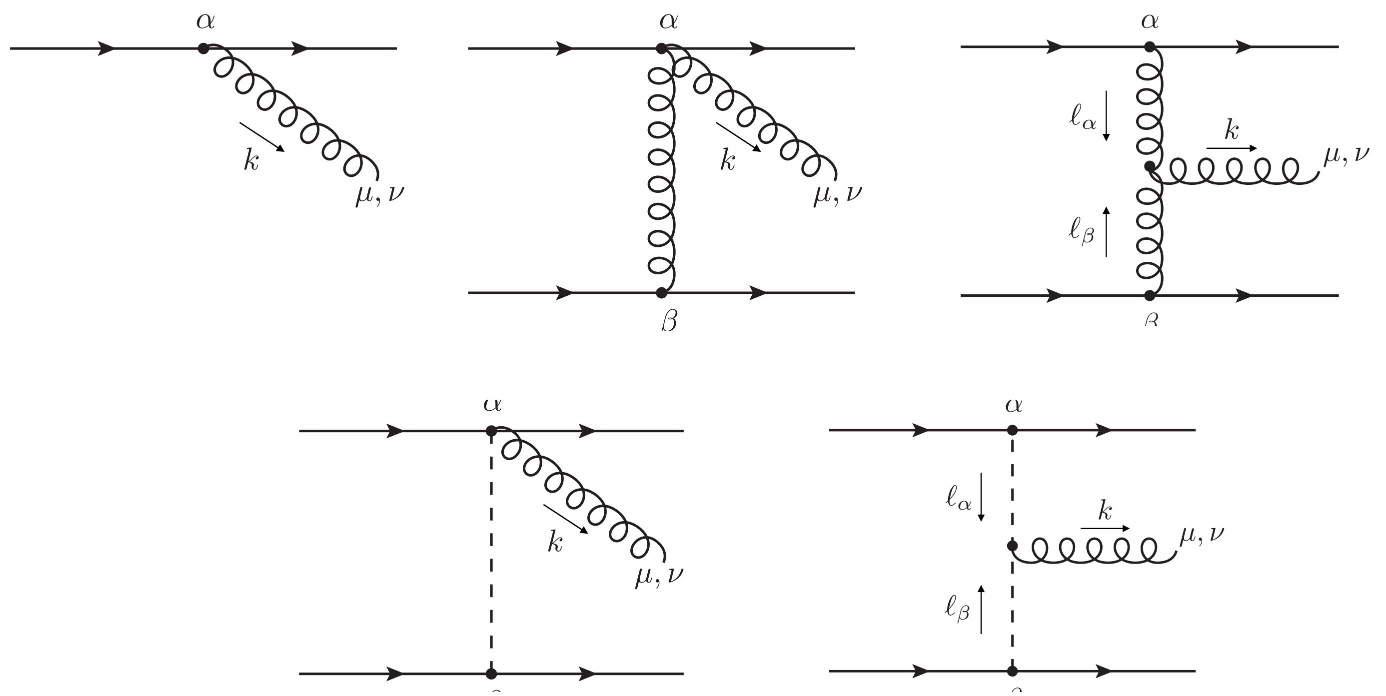
$$= \frac{i}{2m_{Pl}^{d-2}} \frac{1}{k^2} \left[\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma} \right]$$

It is also motivated by BCJ for pure gauge theory, and by results of KLT:

$$A_\mu \otimes A_\nu = \phi \oplus B_{\mu\nu} \oplus h_{\mu\nu}$$

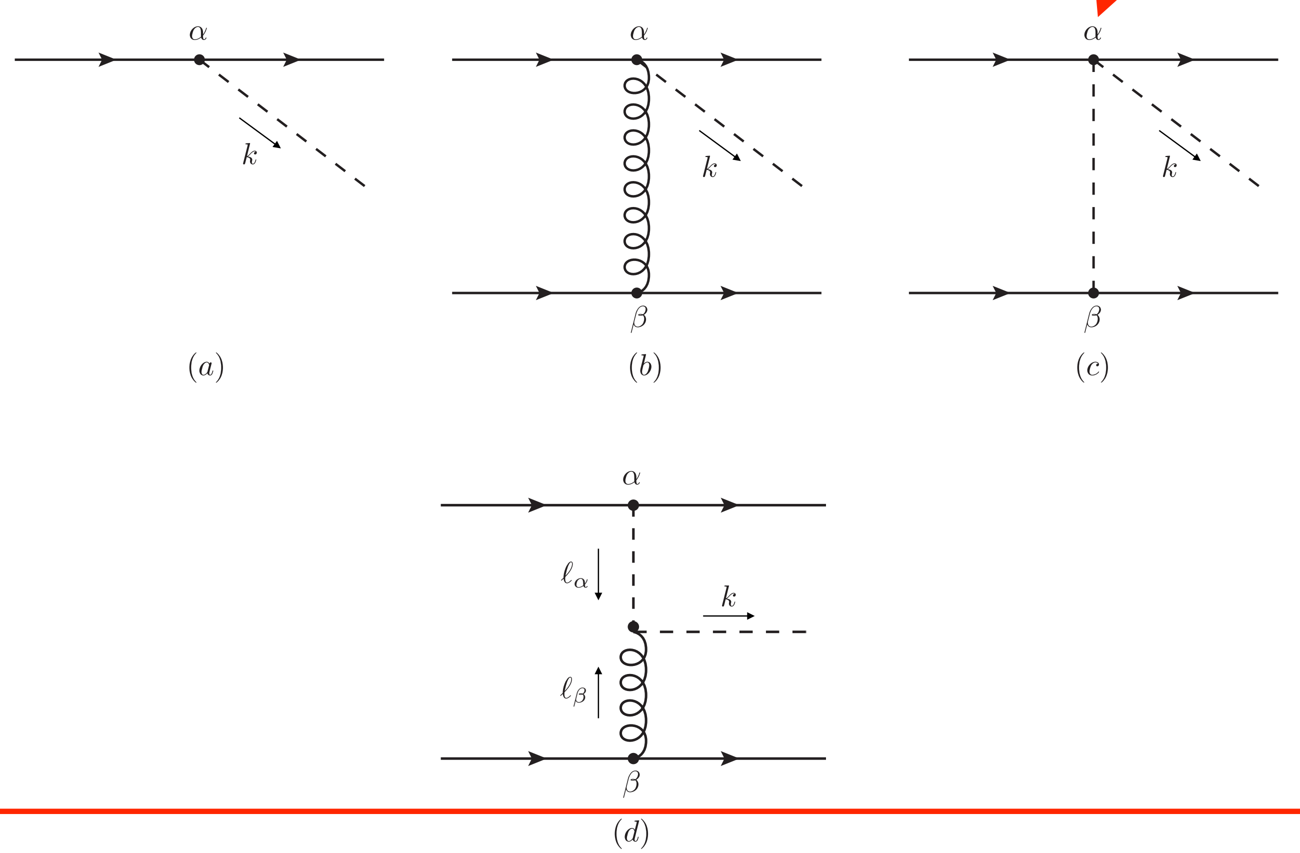
In the case of bremsstrhalung, we verified this explicitly by direct calculation (WG+Ridgway 2016) in the graviton and dilaton channels:

graviton emission channel:



$$e^{\phi} = 1 + \phi + \frac{1}{2!}\phi^2 + \cdots$$

scalar channel:



We also find that $\tilde{T}^{\mu\nu}(k)$ has the correct non-relativistic limit in $d=4$:

$$Q^{ij} = \sum_\alpha m_\alpha \left(x^i_\alpha x^j_\alpha - \frac{1}{3}\delta^{ij}\vec{x}^2_\alpha \right)$$

$$h_{ij}(t,\vec{n}) = \frac{2G_N}{r}[\ddot{Q}_{ij}(t)]^{TT}$$

$$\phi(t,\vec{n}) = \frac{G_N}{r}\sum_\alpha m_\alpha \left(\vec{v}^2_\alpha - \vec{x}_\alpha \cdot \ddot{\vec{x}}_\alpha - \frac{1}{2}\frac{d^2}{dt^2}(\vec{x}_\alpha \cdot \vec{n})^2 \right),$$

corresponding to the radiation fields sourced by a collection of interacting particles:

$$m_\alpha \dot{\vec{v}}_\alpha = -2G_N \sum_\beta \frac{m_\alpha m_\beta \vec{x}_{\alpha\beta}}{|\vec{x}_{\alpha\beta}|^3},$$

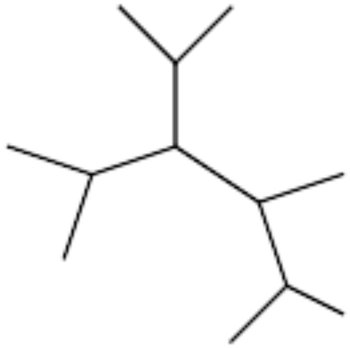
This is consistent with the post-Newtonian limit of more general scalar-tensor theories in 4D.
(Will+Zaglauer 1989; Damour+Esposito-Farrese, 1992)

Bi-adjoint double copy and classical solutions

(WG, Prabhu, Thompson arXiv:1705.09263)

Starting from the tree level gauge theory amplitude

$$\mathcal{A}_n^{YM} = \sum_i$$



$$= \sum_i \frac{n_i c_i}{\prod \text{Propagators}}$$

we can apply BCJ backwards $n_i \rightarrow \tilde{c}_i$ to obtain a ``zeroth copy'' scattering amplitude

$$\mathcal{A}_n^{YM} \rightarrow \sum_i \frac{c_i \tilde{c}_i}{\prod \text{Propagators}} \equiv \mathcal{A}_n^{\phi}$$

The corresponding amplitudes are those of a scalar field theory with $G \times \tilde{G}$ global symmetry

$$\phi^{a\tilde{a}} = \quad \text{bi-adjoint scalar}$$

and cubic interaction

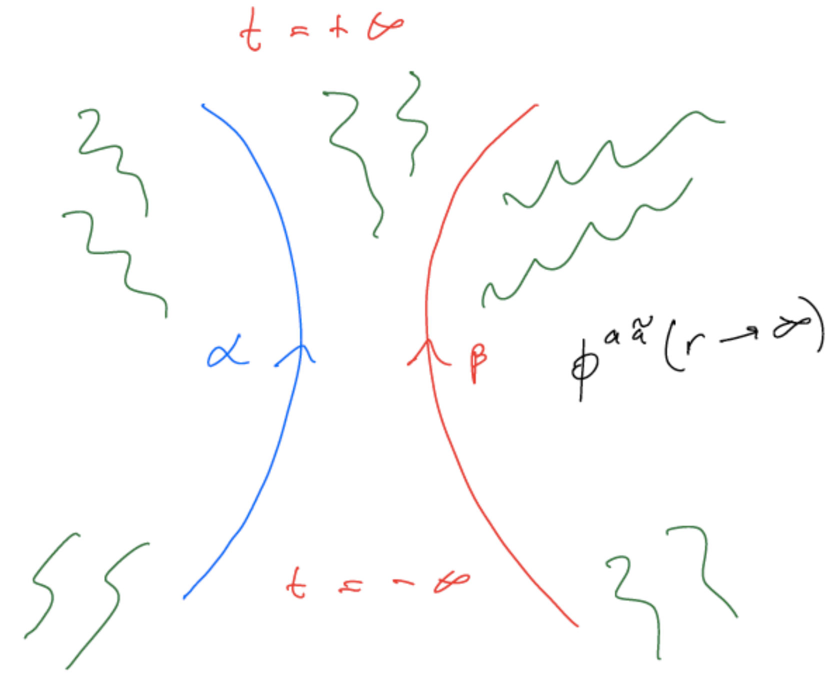
$$\mathcal{L}_{int} = -y f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \phi^{a\tilde{a}} \phi^{b\tilde{b}} \phi^{c\tilde{c}}$$

(NOT technically natural as an EFT)

Color-kinematics relates all tree-level amplitudes in this theory to gluon scattering in pure Yang-Mills (Cachazo, He, Yuan (2013))

We find that the color-kinematics also relates classical solutions of this theory coupled to bi-color charges

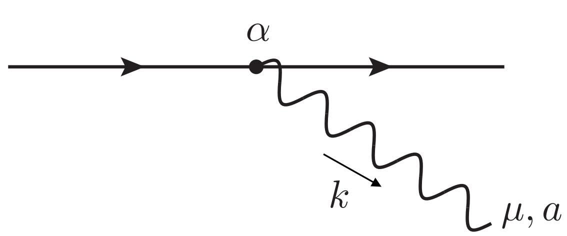
$$S_{pp} = \int d\tau c_a \tilde{c}_{\tilde{a}} \phi^{a\tilde{a}} + \cdots$$



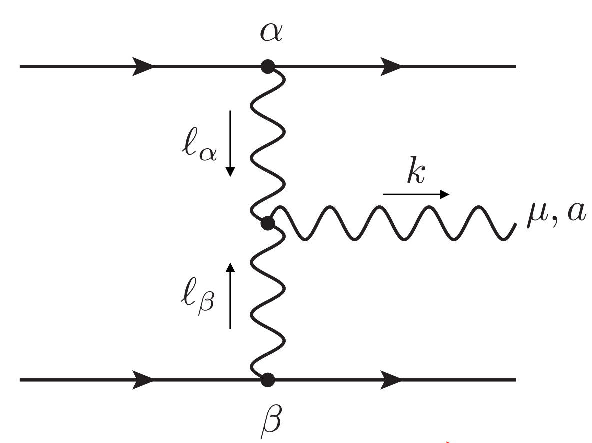
to the corresponding classical gauge theory solutions discussed earlier

$$\phi^{a\tilde{a}}(x) = -y \int_k \frac{e^{-ik\cdot x}}{k^2} \mathcal{J}^{a\tilde{a}}(k)$$

$$\mathcal{J}^{a\tilde{a}}(k) =$$



$+$



$+ \cdots$

$$-y^2 \sum_{\alpha\beta} \int_{\ell_\alpha,\ell_\beta} \mu_{\alpha\beta}(k) \frac{\ell_\alpha^2}{k\cdot p_\alpha} \left[\frac{k\cdot \ell_\beta}{k\cdot p_\alpha} ((c_\alpha \cdot c_\beta) c_\alpha)^a ((\tilde{c}_\alpha \cdot \tilde{c}_\beta)\tilde{c}_\alpha)^{\tilde{a}} \right.$$
$$\left. + [c_\alpha, c_\beta]^a ((\tilde{c}_\alpha \cdot \tilde{c}_\beta)\tilde{c}_\alpha)^{\tilde{a}} + ((c_\alpha \cdot c_\beta)c_\alpha)^a [\tilde{c}_\alpha, \tilde{c}_\beta]^{\tilde{a}} \right].$$

$$y^2 \sum_{\alpha,\beta} \int_{\ell_\alpha,\ell_\beta} \mu_{\alpha\beta}(k) [c_\alpha, c_\beta]^a [\tilde{c}_\alpha, \tilde{c}_\beta]^{\tilde{a}}$$

Applying color-kinematics to $\tilde{G}$:

$$\boxed{\tilde{c}^a_\alpha \to p^\mu_\alpha}$$

$$\boxed{[\tilde{c}_\alpha, \tilde{c}_\beta]^a \mapsto \Gamma^{\mu\nu\rho}(-k, \ell_\alpha, \ell_\beta) p_{\nu\alpha} p_{\rho\beta}}$$

and $y \to g$, we reproduce the classical radiation gauge field

$$\boxed{\mathcal{J}^{a\tilde{a}}(k) \to \epsilon^a_\mu(k) \tilde{J}^\mu_a(k)}$$

A further color-kinematics transformation then yields the gravity solutions

$$\boxed{\mathcal{J}^{a\tilde{a}}(k) \to \epsilon^a_\mu(k) \tilde{J}^\mu_a(k) \to \epsilon_\mu(k) \tilde{\epsilon}_\nu(k) \tilde{T}^{\mu\nu}(k)}$$

directly from the much simpler Feynman rules in a scalar field theory.

Classical double copy and NLO Radiation

The rules proposed in 1611.03493 1711.09493 are puzzling from the viewpoint of BCJ duality. Eg, where do the (Lie, kinematic) Jacobi identities make an appearance?

The situation was very recently clarified by C-H Shen (1806.07388) who worked out NLO radiation in bi-adj., YM, dilaton gravity.

Start with bi-adjoint scalar and YM radiation fields, written in the BCJ-like form at integrand level:

$$\mathcal{A}^{a\tilde{a}}(k) = \sum_{i,j} \int d\mu(k) C_i^a P_{ij}(k) \tilde{C}_j^{\tilde{a}},$$

$(P = P^T)$ "propagator matrix"

$$\mathcal{A}^{a,\mu}(k) = \sum_{i,j} \int d\mu(k) C_i^a P_{ij}(k) N_j^{\mu},$$

$(k_\mu (P \cdot N)^\mu = 0)$

"Ward id."

Color-kinematics: $C_i^a \mapsto N_i^\nu$

$$\mathcal{A}^{a,\mu}(k) \mapsto \mathcal{A}^{\mu\nu}(k) = \sum_{i,j} \int d\mu(k) ds N_i^\nu P_{ij}(k) N_j^\mu,$$

$(k_\mu \mathcal{A}^{\mu\nu} = k_\nu \mathcal{A}^{\mu\nu} = 0)$

LO radiation:

Read off color structures and propagators from bi-adjoint scalar radiation:

$$C^a = \begin{pmatrix} (c_\alpha \cdot c_\beta) c_\alpha^a \\ [c_\alpha, c_\beta]^a \end{pmatrix} \qquad P = \begin{pmatrix} \frac{\ell_\alpha^2 (k\cdot\ell_\beta)}{(k\cdot p_\alpha)^2} & \frac{\ell_\alpha^2}{k\cdot p_\alpha} \\ \frac{\ell_\alpha^2}{k\cdot p_\alpha} & -1 \end{pmatrix} \qquad \tilde{C}^{\tilde{a}} = \begin{pmatrix} (\tilde{c}_\alpha \cdot \tilde{c}_\beta) \tilde{c}_\alpha^{\tilde{a}} \\ [\tilde{c}_\alpha, \tilde{c}_\beta]^{\tilde{a}} \end{pmatrix}$$

Read off kinematic structures from YM radiation:

$$C^a = \begin{pmatrix} (c_\alpha \cdot c_\beta) c_\alpha^a \\ [c_\alpha, c_\beta]^a \end{pmatrix} \qquad N^\mu = \begin{pmatrix} (p_\alpha \cdot p_\beta) p_\alpha^\mu \\ (k\cdot p_\alpha) p_\beta^\mu - (k\cdot p_\beta) p_\alpha^\mu + \frac{1}{2}(p_\alpha \cdot p_\beta)\,(\ell_\alpha - \ell_\beta)^\mu \end{pmatrix}$$

Predict dilaton gravity amplitude in factorized form

$$(N^T)^\nu P N^\mu$$

Note that at LO, the relation between color and kinematics is what one would have obtained from the naive replacement rules introduced earlier:

$$\begin{matrix} c^a \mapsto p^\mu \\ f^{abc} \mapsto \Gamma^{\mu\nu\rho} \end{matrix} \quad \Longrightarrow \quad C^a \mapsto N^\mu$$

However, at NLO the naive rules no longer work. From Shen 1806.07388

Color factors:

(full set)

$C_1 = (c_1\cdot c_2)(c_1\cdot c_3)c_1^a$	$C_7 = (c_1\cdot c_2)(c_2\cdot c_3)c_1^a$	$C_{13} = (c_1\cdot c_3)\,[c_1,c_2]^a$	$C_{19} = [c_1,\,[c_2,\,c_3]]^a$
$C_2 = (c_1\cdot c_2)(c_2\cdot c_3)c_2^a$	$C_8 = (c_1\cdot c_3)(c_2\cdot c_3)c_1^a$	$C_{14} = (c_2\cdot c_3)\,[c_1,c_2]^a$	$C_{20} = [c_2,\,[c_3,\,c_1]]^a$
$C_3 = (c_1\cdot c_3)(c_2\cdot c_3)c_3^a$	$C_9 = (c_1\cdot c_2)(c_1\cdot c_3)c_2^a$	$C_{15} = (c_1\cdot c_2)\,[c_2,c_3]^a$	$C_{21} = [c_3,\,[c_1,\,c_2]]^a$
$C_4 = (c_1\cdot [c_2,c_3])\; c_1^a$	$C_{10} = (c_1\cdot c_3)(c_2\cdot c_3)c_2^a$	$C_{16} = (c_1\cdot c_3)\,[c_2,c_3]^a$	
$C_5 = (c_1\cdot [c_2,c_3])\; c_2^a$	$C_{11} = (c_1\cdot c_2)(c_2\cdot c_3)c_3^a$	$C_{17} = (c_1\cdot c_2)\,[c_3,c_1]^a$	
$C_6 = (c_1\cdot [c_2,c_3])\; c_3^a$	$C_{12} = (c_1\cdot c_2)(c_1\cdot c_3)c_3^a$	$C_{18} = (c_2\cdot c_3)\,[c_3,c_1]^a$	

related by color Jacobi ids.

Kinematic factors:

(representative sample)

$$N_1 = (p_1\cdot p_2)(p_1\cdot p_3)(p_1\cdot e)$$

$$N_4 = \Big((p_1\cdot[p_3,p_2]\cdot l_{23}) + \tfrac{1}{2}(p_2\cdot p_3)(p_1\cdot q_{23})\Big)\,(p_1\cdot e) + \tfrac{1}{2}\,(p_1\cdot l_{23})\,(p_1\cdot[p_2,p_3]\cdot e)$$

$$N_7 = (p_1\cdot p_2)(p_2\cdot p_3)(p_1\cdot e)$$

$$N_{13} = (p_1\cdot p_3)\,((l_{123}\cdot[p_1,p_2]\cdot e) - (p_1\cdot p_2)(l_2\cdot e))$$

$$\begin{aligned} N_{19} = {} & \Big((l_1\cdot[p_2,p_3]\cdot l_{23}) - \tfrac{1}{2}(p_2\cdot p_3)(l_{123}\cdot q_{23})\Big)\,(p_1\cdot e) + (p_1\cdot l_{23})\,(l_{23}\cdot[p_2,p_3]\cdot e) \\ & + (p_1\cdot[p_2,p_3]\cdot l_{23})\,(l_{23}\cdot e) - (p_2\cdot p_3)\,(p_1\cdot[l_2,l_3]\cdot e) - \tfrac{1}{2}(l_{23}\cdot l_{123})\,(p_1\cdot[p_2,p_3]\cdot e) \end{aligned}$$

$$N_{19} + N_{20} + N_{21} = 0$$

Kinematic Jacobi id.

(Shen only considered classical solutions with scattering BC's w/ $\ell_\alpha \cdot p_\alpha = 0$. More recently Plefka et al 1807. 09859 considered the classical double copy map of particle EOMS at NLO for general orbits.

Spin, axions, and other finite size effects:

WG, Li, Prabhu, 1712.09250
Li, Prabhu, 1803.02405
WG+Li, to appear

So far, we have treated the sources on both sides of the classical double copy as point objects. Shen's proposal also sheds light on the internal structure of the radiating sources:

I. Spin effects and axion couplings.

II. Linear chromodynamic and tidal responses

Axion radiation and particle spin:

For spinless particles, there is no radiation in the anti-symmetric channel at LO or NLO.

$$a_{\mu\nu}(k)\tilde{T}^{\mu\nu}(k) = 0$$

$$a_{\mu\nu}(k) = -a_{\mu\nu}(k)$$

$$k^{\mu}a_{\mu\nu}(k) = 0$$

This is expected on the basis of symmetry: There are no linear pt. particle couplings of the $B_{\mu\nu}$ field that are consistent w/.

Two-form gauge invariance: $B \to B + d\lambda$

Diff. invariance

unless the particles carry a spin degree of freedom $S^{\mu\nu}(\tau) = -S^{\nu\mu}(\tau)$:

$$S_{pp} \supset \int dx^{\mu}\tilde{\kappa}(\phi)S^{\nu\sigma}H_{\mu\nu\sigma}$$

Does the spin interaction w/ $B_{\mu\nu}$ arise as double copy of classical YM? Add a chromomagnetic interaction to the classical color charge:

$$S_{pp} = -\int dx^\mu c_a(s) A^a_\mu + \frac{g\kappa}{2}\int dse c_a(s) S^{\mu\nu} F^a_{\mu\nu} + \cdots \,,$$

Color current is modified due to spin:

$$J^\mu_a(x) = -\frac{1}{g_s}\frac{\delta}{\delta A^a_\mu(x)} S_{int} = \sum_\alpha \int dx^\mu_\alpha c^a_\alpha(s)\delta(x - x(s)) - \kappa_\alpha \int dse S^{\mu\sigma} D_\sigma \left[c^a_\alpha \delta(x - x(s))\right].$$

As is the energy-momentum tensor

$$T^{\mu\nu}_{pp}(x) = \int dx^{(\mu} p^{\nu)} \delta(x - x(s)) + \int dx^{(\mu} S^{\nu)\sigma} \partial_\sigma \delta(x - x(s) - \kappa g_s \int dse \frac{\delta(x - x(s))}{\sqrt{g}} c_a F^a{}_\sigma{}^{(\mu} S^{\nu)\sigma},$$

with spin $S^{\mu\nu}(s) = -S^{\nu\mu}(s)$ defined to obey the constraint $p_\mu S^{\mu\nu} = 0$. For $F^{\mu\nu}_a = 0$:

$$P^\mu = \int d^3\mathbf{x} T^{0\mu}(\mathbf{x}, x^0)$$

= "CM" particle momentum

$$J^{\mu\nu} = \int d^3\mathbf{x} x^{[\mu} T^{0\nu]}(\mathbf{x}, x^0) = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}$$

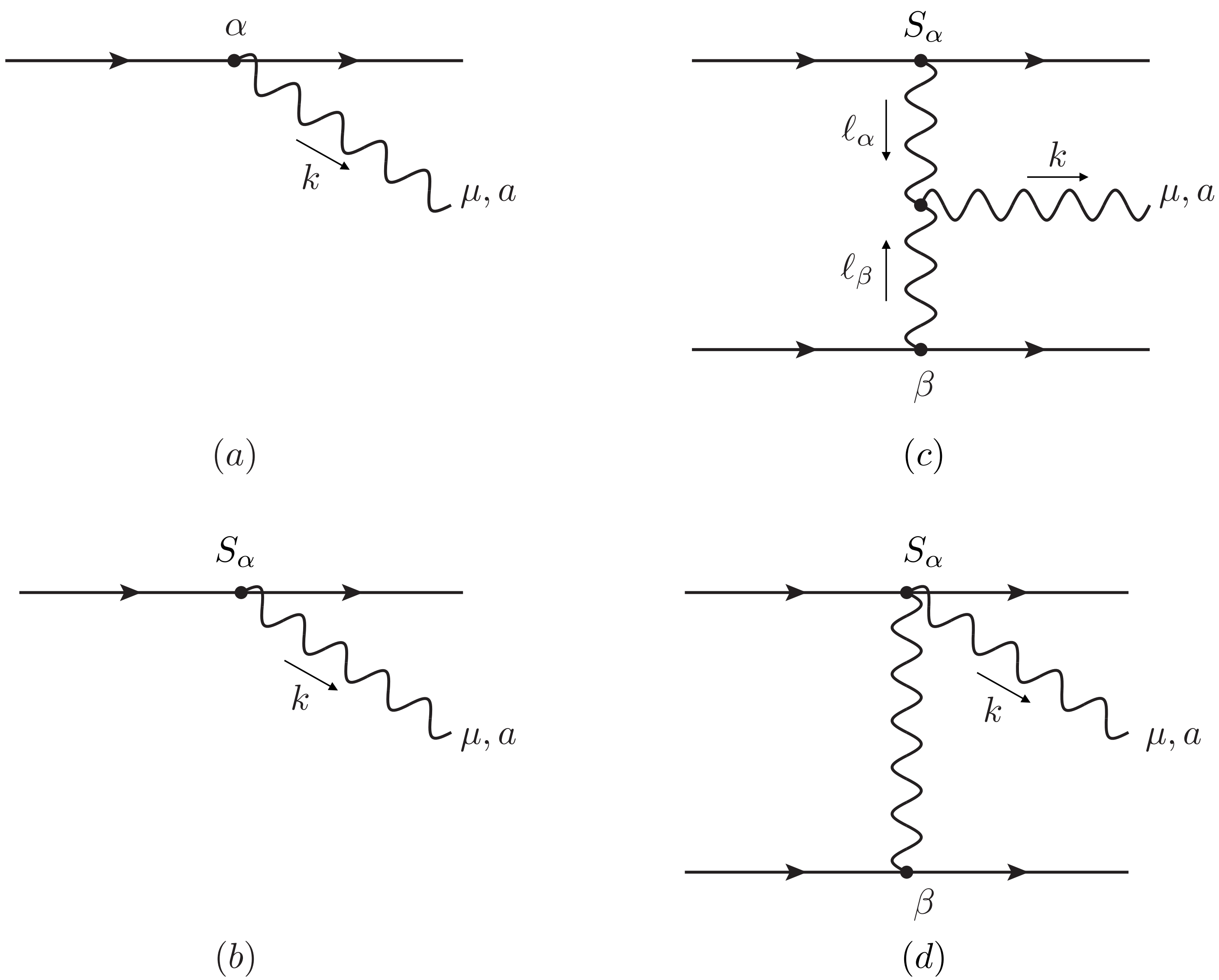
= "CM" particle angular momemtum

Eqns of motion from conservation laws:

$$D_\mu J^\mu_a = 0 \quad \Longrightarrow \quad (v \cdot D)c^a = \frac{i}{2} g_s \kappa [S^{\mu\nu} F_{\mu\nu}, c]^a$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \begin{aligned} \frac{d}{ds} p^\mu &= g_s c^a F^{\mu\nu}_a v_\nu - \frac{1}{2} \kappa g_s c_a S^{\alpha\beta} D^\mu F^a_{\alpha\beta} \\ \frac{d}{ds} S^{\mu\nu} &= p^\mu v^\nu - p^\nu v^\mu - 2\kappa g_s c^a F^{\alpha[\mu}_a S_\alpha{}^{\nu]} \end{aligned}$$

Then the total current at $\mathcal{O}(g^2)$, working to linear order in spin:



Then the total current at $\mathcal{O}(g^2)$, working to linear order in spin:

$$\tilde{J}^\mu_a(k)\Big|_{\mathcal{O}(S^1)} = ig_s^2 \sum_{\alpha,\beta} \int \mu_{\alpha,\beta}(k) \left((c_\alpha \cdot c_\beta) c^a_\alpha \mathcal{A}^\mu_s + [c_\alpha, c_\beta]^a \mathcal{A}^\mu_a \right)$$

where now the partial amplitudes are

$$\boxed{\begin{aligned} \mathcal{A}^\mu_a = \kappa_\alpha \Bigg[&(\ell_\alpha \wedge p_\beta)_\alpha (\ell_\beta - \ell_\alpha)^\mu - \frac{\ell_\alpha^2}{k \cdot p_\alpha} (\ell_\beta \wedge p_\beta)_\alpha p^\mu_\alpha - \frac{\ell_\beta^2}{k \cdot p_\beta} (\ell_\alpha \wedge p_\beta)_\alpha p^\mu_\beta + \ell_\alpha^2 (S_\alpha \wedge p_\beta)^\mu \Bigg] \\ &-2\kappa_\alpha \left[(\ell_\alpha \wedge \ell_\beta)_\alpha p^\mu_\beta + (k \cdot p_\beta)(S_\alpha \wedge \ell_\alpha)^\mu \right] - \kappa_\alpha \frac{\ell_\alpha^2}{k \cdot p_\alpha} (p_\alpha \cdot p_\beta)(S_\alpha \wedge k)^\mu . \end{aligned}}$$

$$\boxed{\begin{aligned} \mathcal{A}^\mu_s = \frac{(1+\kappa_\alpha)^2}{m_\alpha^2} \ell_\alpha^2 &\left[(k \cdot p_\alpha) \left\{ (S_\alpha \wedge p_\beta)^\mu - \frac{(k \wedge p_\beta)_\alpha}{k \cdot p_\alpha} p^\mu_\alpha \right\} + (p_\alpha \cdot p_\beta) \left\{ (S_\alpha \wedge \ell_\beta)^\mu - \frac{(\ell_\alpha \wedge \ell_\beta)_\alpha}{k \cdot p_\alpha} p^\mu_\alpha \right\} \right] \\ -\kappa_\alpha^2 \frac{\ell_\alpha^2}{k \cdot p_\alpha} &\left[\left\{ (\ell_\alpha \wedge \ell_\beta)_\alpha p^\mu_\beta - (k \cdot p_\beta)(S_\alpha \wedge \ell_\beta)^\mu \right\} + \left\{ (k \cdot \ell_\beta)(S_\alpha \wedge p_\beta)^\mu - (k \wedge p_\beta)_\alpha \ell^\mu_\beta \right\} \right] \\ +\kappa_\alpha \frac{\ell_\alpha^2}{k \cdot p_\alpha} &\left[(\ell_\beta \wedge p_\beta)_\alpha \left\{ \ell^\mu_\beta - \frac{k \cdot \ell_\beta}{k \cdot p_\alpha} p^\mu_\alpha \right\} + (k \cdot p_\beta)(S_\alpha \wedge k)^\mu \right] + \kappa_\beta \frac{\ell_\alpha^2}{k \cdot p_\alpha} (\ell_\beta \wedge p_\alpha)_\beta \left[\ell^\mu_\beta - \frac{k \cdot \ell_\beta}{k \cdot p_\alpha} p^\mu_\alpha \right] \\ &-\kappa_\alpha \frac{\ell_\alpha^2}{(k \cdot p_\alpha)^2} (p_\alpha \cdot p_\beta)(k \cdot \ell_\beta)(S_\alpha \wedge k)^\mu \end{aligned}}$$

$$(S_\alpha \wedge a)^\mu = S^{\mu\nu}_\alpha a_\nu \qquad (a \wedge b)_\alpha = a \cdot (S_\alpha \wedge b) \qquad (k_\mu \tilde{J}^\mu_a(k) = 0)$$

To linear order in spin, the amplitude can be put in Shen's factorized form

$$\mathcal{A}^{a,\mu}(k) = \sum_{i,j} \int d\mu(k) C_i^a P_{ij}(k) N_j^\mu ,$$

w/

$$C^a = \begin{pmatrix} (c_\alpha \cdot c_\beta) c_\alpha^a \\ [c_\alpha, c_\beta]^a \end{pmatrix} \qquad P = \begin{pmatrix} \frac{\ell_\alpha^2 (k\cdot \ell_\beta)}{(k\cdot p_\alpha)^2} & \frac{\ell_\alpha^2}{k\cdot p_\alpha} \\ \frac{\ell_\alpha^2}{k\cdot p_\alpha} & -1 \end{pmatrix}$$

same as in the spinless case, but now

$$N^\mu = N^\mu_{(S^0)} + N^\mu_{(S^1)} + \mathcal{O}(S^2)$$

with spin-dependent kinematic factor:

$$N^\mu_{(S^1)} = \begin{pmatrix} (\ell_\beta \wedge p_\alpha)_\beta p_\alpha^\mu - (\ell_\beta \wedge p_\beta)_\alpha p_\alpha^\mu - (p_\alpha \cdot p_\beta)(S_\alpha \wedge k)^\mu + (k \cdot p_\alpha)(S_\alpha \wedge p_\beta)^\mu \\ (k \cdot p_\beta)(S_\alpha \wedge \ell_\alpha)^\mu + (\ell_\alpha \wedge \ell_\beta)_\beta p_\alpha^\mu - \frac{1}{2}(\ell_\alpha \wedge p_\beta)_\alpha (\ell_\beta - \ell_\alpha)^\mu - (\alpha \longleftrightarrow \beta) \end{pmatrix} .$$

However, this factorization is only possible for a specific choice of chromomagnetic coupling:

$$\boxed{\kappa = -1}$$

(In the NR limit, for $d=4$ the dipole interaction is

$$S_{int} = -\frac{g_s \kappa}{m} \int dt c_a \vec{S} \cdot \vec{B}^a$$

$\kappa = -1 \longrightarrow$ g-factor $g = g_{Dirac} = 2$

So this is a classical spin $S \gg \hbar$ with the magnetic properties of a Dirac particle.)

Double copy

For the choice $\kappa = -1$, we can define a gravitational double copy via the replacement

$$C^a \mapsto N^{\mu}_{(S^0)}$$

which yields a consistent radiation field

$$\mathcal{A}^{a,\mu}(k) \mapsto \mathcal{A}^{\mu\nu}(k) = \sum_{i,j} \int d\mu(k) \left(N^T_{(S^0)}\right)^{\nu} \cdot P(k) \cdot \left(N_{(S^0)} + N_{(S^1)}\right)^{\mu}$$

with

$$k_\mu \mathcal{A}^{\mu\nu} = k_\nu \mathcal{A}^{\mu\nu} = 0$$

and

$$\mathcal{A}^{\mu\nu} - \mathcal{A}^{\nu\mu} \neq 0 \longrightarrow \text{Axion radiation}$$

For the choice of magnetic coupling where the double copy works, this $\mathcal{A}^{\mu\nu}$ describes radiation in a local theory of massless particles $\phi, B_{\mu\nu}, h_{\mu\nu}$

At the two derivative level, the form of this theory is fixed by diffeomorphism and two-form gauge invariance to be :

$$S_g = -2m_{Pl}^{d-2} \int d^d x \sqrt{g} \left[R - (d-2) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{12} f(\phi) H_{\mu\nu\sigma} H^{\mu\nu\sigma} \right]$$

$f(\phi) = 1 + c\phi + \cdots$

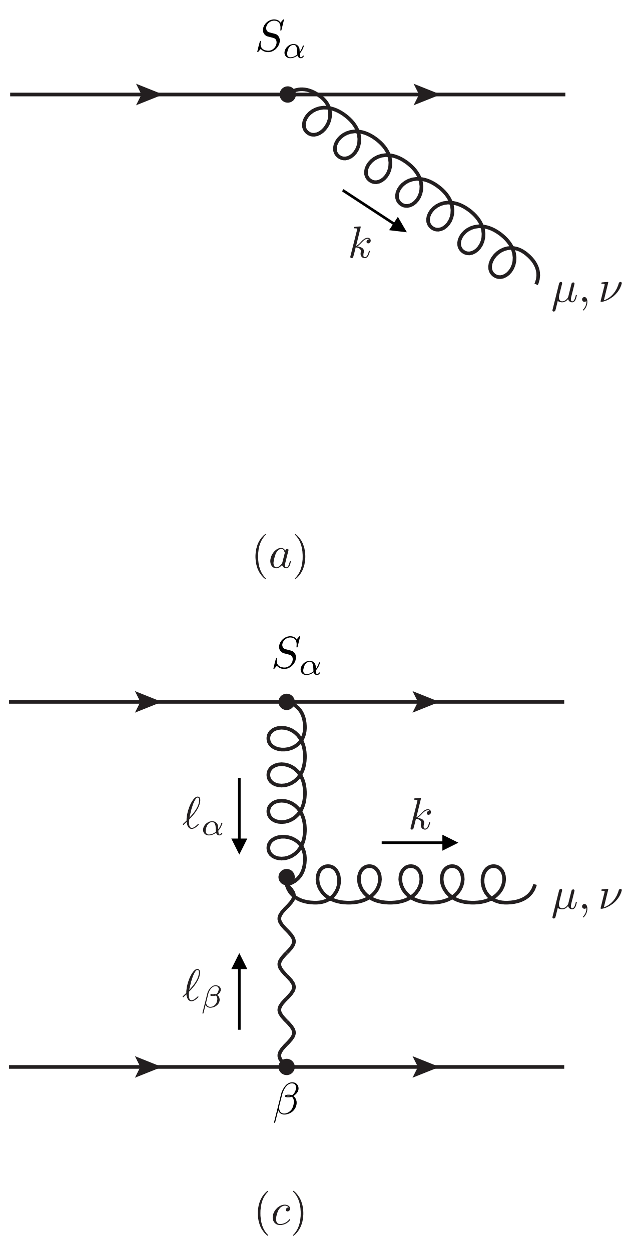
with couplings to point sources of the form

$$S_{pp} = \int dx^\mu \tilde{\kappa}(\phi) H_{\mu\nu\sigma} S^{\nu\sigma}$$

$\tilde{\kappa}(\phi) = \tilde{\kappa} + \tilde{\kappa}' \phi + \cdots$

We'll fix the unknown constants by comparing the double copy to the gravitational emission amplitude.

In addition to spin corrections to particle eqns. of motion, must calculate the diagrams



In order to cancel the explicit d dependence of the diagrams, need to fix dilaton couplings

$$\boxed{c = -4, \tilde{\kappa}' = 0}$$

to the axion and point particles respectively.

Furthermore, we need specific choice of axion-spin coupling $\tilde{\kappa} = 1/4$

$$\boxed{S_{pp} = \frac{1}{4}\int dx^\mu e^{-2\phi} S^{\nu\sigma} H_{\mu\nu\sigma}}$$

in which case we find agreement between axion radiation in gravity and its double copy. At linear order in dilaton couplings, the gravity theory is consistent with the form

$$\boxed{S_g = -2m_{Pl}^{d-2}\int d^d x\sqrt{g}\left[R - (d-2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{12}e^{-4\phi}H_{\mu\nu\sigma}H^{\mu\nu\sigma}\right]}$$

which precisely matches the the action for the $(\phi, g_{\mu\nu}, B_{\mu\nu})$ sector of (oriented) non-critical string theory. This can also written as (Scherk, Schwarz 1974)

$$S_g = -2m_{Pl}^{d-2}\int d^d x\sqrt{g}R(g,\hat{\Gamma})$$

$\hat{\Gamma}^\mu_{\alpha\beta} =$ non-Riemannian connection w torsion

(Spin correction to dilaton +graviton channels are also consistent with the double copy for this choice of parameters (Li, Prabhu 1803.02405))

Generalization of spin double copy

WG+Li, to appear

The connection with classical strings can be made even more explicit via the following generalization of the double copy procedure. Start with the YM spinning particle

$$\mathcal{A}^{a,\mu}(k) = \sum_{i,j} \int d\mu(k) C_i^a P_{ij}(k) N_j^\mu ,$$

$$C^a = \begin{pmatrix} (c_\alpha \cdot c_\beta) c_\alpha^a \\ [c_\alpha, c_\beta]^a \end{pmatrix} \qquad P = \begin{pmatrix} \frac{\ell_\alpha^2 (k\cdot\ell_\beta)}{(k\cdot p_\alpha)^2} & \frac{\ell_\alpha^2}{k\cdot p_\alpha} \\ \frac{\ell_\alpha^2}{k\cdot p_\alpha} & -1 \end{pmatrix}$$

$$N^\mu = N^\mu_{(S^0)} + N^\mu_{(S^1)} + \mathcal{O}(S^2)$$

and formally introduce two independent copies of the spin variable

$$S_L^{\mu\nu}, S_R^{\mu\nu}$$

$$(S^{\mu\nu} = S_L^{\mu\nu} + S_R^{\mu\nu})$$

$$N_L^\mu = N^\mu_{(S^0)} + N^\mu_{(S_L)} + \mathcal{O}(S^2)$$

$$N_R^\mu = N^\mu_{(S^0)} + N^\mu_{(S_R)} + \mathcal{O}(S^2)$$

(motivated by analogy with BCJ/scattering amplitudes)

The double copy is now:

$$\mathcal{A}^{a,\mu}(k) \mapsto \mathcal{A}^{\mu\nu}(k) = \int d\mu(k)\, (N_L^\nu)^T \cdot P(k) \cdot N_R^\mu$$

$$(k_\mu \mathcal{A}^{\mu\nu} = k_\nu \mathcal{A}^{\mu\nu} = 0)$$

This generalized procedure yields the same bulk theory, but modified axion couplings to the worldline:

$$\boxed{S_{int} = \frac{1}{4}\int dx^{\lambda} e^{-2\phi} H_{\mu\nu\lambda}\left(S_R^{\mu\nu} - S_L^{\mu\nu}\right)}$$

We recover the previous results of 1712.09250 1803.02405 by taking $S_L^{\mu\nu} = 0$ or $S_R^{\mu\nu} = 0$, but there are other interesting possibilities:

(I) $S_L^{\mu\nu} = S_R^{\mu\nu} = \frac{1}{2} S^{\mu\nu}$: axion decoupling.

(II) $S_L^{\mu\nu}, S_R^{\mu\nu}$ independent: open string $\Longrightarrow$ closed string?

Support for (II) comes from taking the multipole expansion of bosonic strings coupled to background fields…

String couplings:

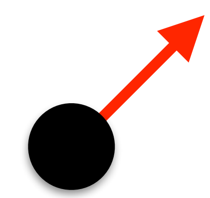
The double copy is only consistent if on the gauge theory side, the sources have chromomagnetic coupling $\kappa = -1$.

Are there fundamental classical objects with such couplings? Look at the open (bosonic) string with Chan-Paton color charges. Taking point particle limit of the string

$c_a(\ell)$

$c_a(0)$

$$\ell_s \to 0$$



$S^{\mu\nu}$

$$c_a = c_a(0) + c_a(\ell)$$

$$S_{int} = \sum_{\sigma=0,\ell} \int dX^\mu(\tau,\sigma) c^a(\sigma,\tau) A^a_\mu(X)$$

$$-g_s \int dx^\mu c_a A^a_\mu - \frac{1}{2} g_s \int d\tau c_a S^{\mu\nu} F^a_{\mu\nu}$$

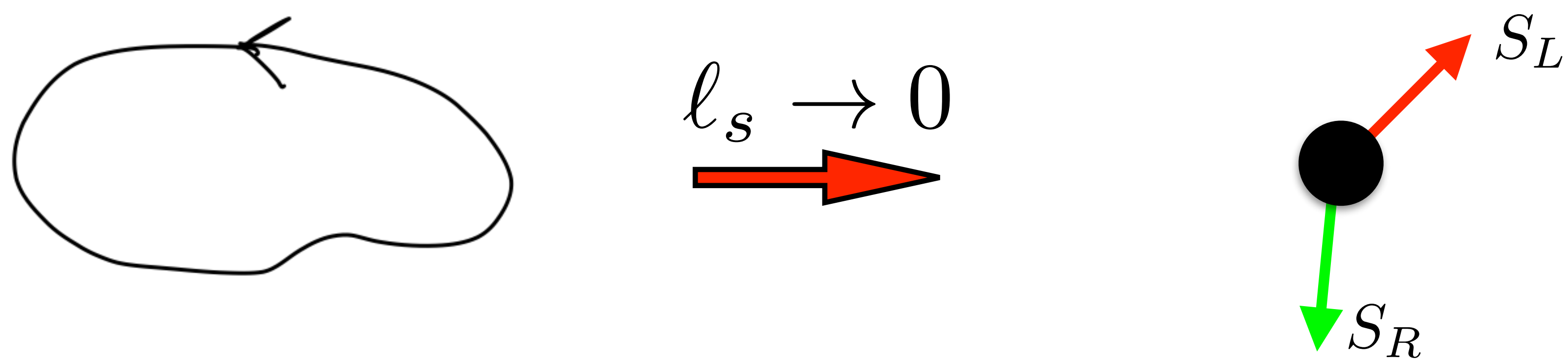
$$S^{\mu\nu} = \sum_{n>0} \frac{i}{n} \left(\alpha^\mu_n \alpha^\nu_{-n} - \alpha^\mu_{-n} \alpha^\nu_n\right)$$

So the open string is an example of a fundamental massive object with $g = g_D = 2$

Consistent with quantum analysis in Ferrara, Porrati, Telegdi 1992.

(See also d=4 Kerr-Newman BH, which has $g_D = 2$ (Horowitz, 1992))

We can also do this multipole expansion for the oriented (bosonic) closed string



$$S_{int} = \frac{1}{2\pi\alpha'} \int d^2\sigma \epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \xrightarrow{\ell_s \to 0} S_{int} = \frac{1}{4} \int dx^\lambda e^{-2\phi} H_{\mu\nu\lambda} \left(S_R^{\mu\nu} - S_L^{\mu\nu}\right) + \cdots$$

in agreement with the double copy of the Yang-Mills source.

(relative sign consistent with worldsheet parity $P : L \leftrightarrow R$ of oriented string)

In terms of the string oscillator modes:

$$S_R^{\mu\nu} = \sum_{n>0} \frac{i}{n} \left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_{-n}^\mu \alpha_n^\nu\right)$$

$$S_L^{\mu\nu} = \sum_{n>0} \frac{i}{n} \left(\tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu\right)$$

are independently conserved "zero modes" (in the absence of background fields) of the closed string.

Polarizabilities and finite size corrections

WG+Li, to appear

By including non-minimal interactions in the worldline EFT, we can systematically describe the effects of finite size while retaining a point particle description. Eg:

Electromagnetic polarizability: Most general linear response to long wavelength external EM fields

$$S_{int} = \cdots + \frac{\alpha}{2}\int d\tau F_{\mu\nu}F^{\mu\mu} + \frac{\beta}{2}\int d\tau \dot{x}^{\mu}\dot{x}^{\nu}F_{\mu\sigma}F^{\sigma}{}_{\nu} + \cdots$$

Wilson $\alpha, \beta \sim R^3$ coefficients are obtained by matching to the UV theory. (Eg. perfect spherical conductor $\beta = 6\alpha = 6\pi R^3$)

Tidal response: Most general (parity invariant) Lagrangian in 4D encoding tidal polarizability ("Love numbers")

$$S_{pp} = -m\int d\tau + c_E\int d\tau E_{\mu\nu}E^{\mu\nu} + c_B\int d\tau B_{\mu\nu}B^{\mu\nu} + \cdots$$

$E_{\mu\nu} = R_{\mu\alpha\nu\beta}v^{\alpha}v^{\beta}$

$B_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\rho\sigma\lambda}v^{\rho}v^{\alpha}R_{\nu\alpha}{}^{\sigma\lambda}$

(Flanagan+Hinderer, 2007) $c^{NS} \sim mR^4$

$c_{BH,d=4} = 0$ (Damour et al; Poisson et al;Kol+Smolkin 2010)

(NOTE: Ignoring dissipation/absorption. See WG+Rothstein (2006))

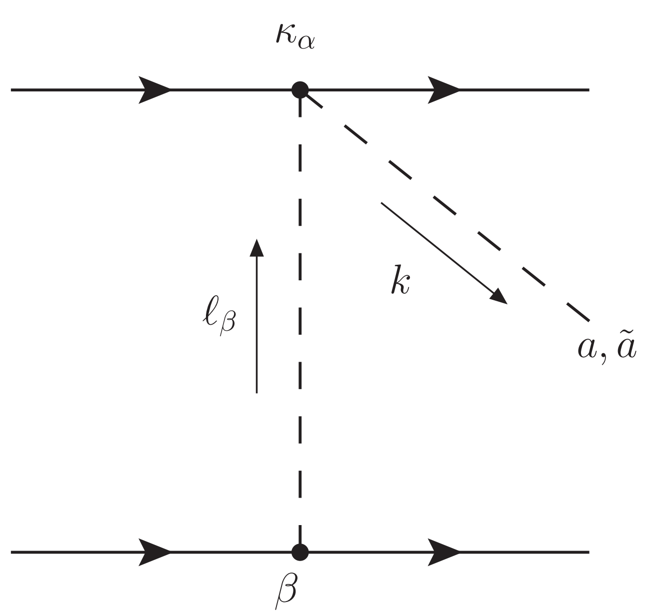
Color kinematics relates finite size sources in YM + gravity. Finite size object in scalar bi-adjoint theory:

$$S_{pp} = \sum_n \kappa_n \int d\tau \mathcal{O}_n$$

At leading order in derivatives

$$\mathcal{O}_1 = \phi^{a\tilde{a}}\phi^{a\tilde{a}} \qquad \mathcal{O}_2 = (c\cdot\phi)^{\tilde{a}}(c\cdot\phi)^{\tilde{a}} \qquad \mathcal{O}_3 = (\tilde{c}\cdot\phi)^{a}(\tilde{c}\cdot\phi)^{a} \qquad \mathcal{O}_4 = (c\cdot\phi\cdot\tilde{c})^2$$

Radiation from induced multipole moments in classical scattering process:



$$\mathcal{A}^{a\tilde{a}}(k) = \sum_{i,j} \int d\mu(k) C^a_i P_{ij}(k) \tilde{C}^{\tilde{a}}_j,$$

$$C^a = \begin{pmatrix} c^a_\beta \\ (c_\alpha\cdot c_\beta)c^a_\alpha \end{pmatrix}, \quad \tilde{C}^{\tilde{a}} = \begin{pmatrix} \tilde{c}^{\tilde{a}}_\beta \\ (\tilde{c}_\alpha\cdot\tilde{c}_\beta)\tilde{c}^{\tilde{a}}_\alpha \end{pmatrix}$$

$$P_{ij} = \ell^2_\alpha \delta_{ij}$$

This is in direct correspondence with finite size terms in gauge theory:

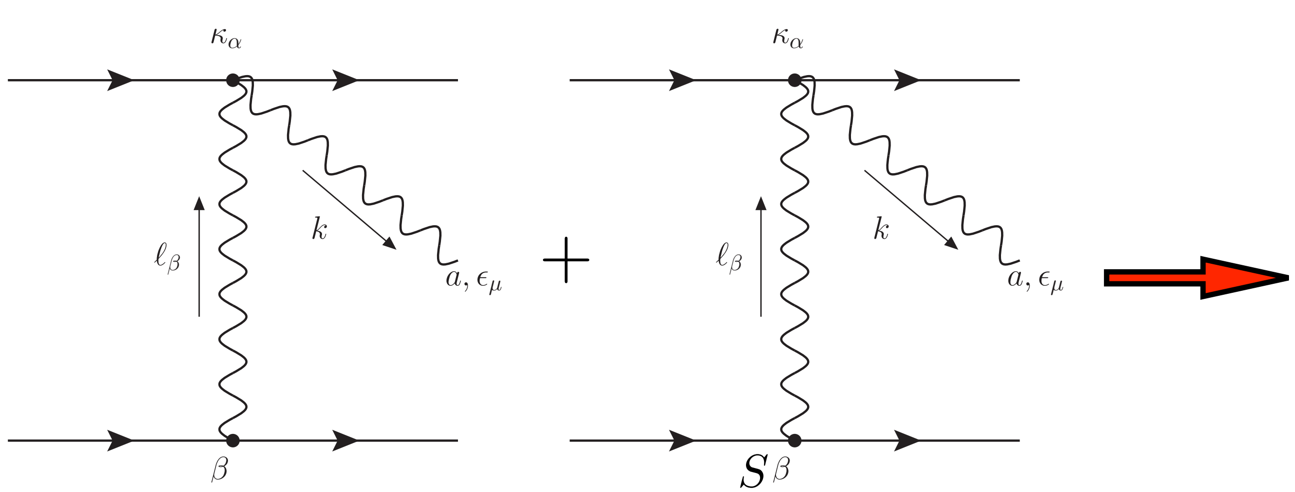
$$\mathcal{O}_n \mapsto \tilde{\mathcal{O}}_n \qquad S^{YM}_{pp} = \sum_n \kappa_n \int d\tau \tilde{\mathcal{O}}_n$$

Leading finite size terms for each source:

$$\tilde{\mathcal{O}}_1 = F^a_{\mu\nu} F^{\mu\nu}_a \qquad \tilde{\mathcal{O}}_2 = (c\cdot F)_{\mu\nu}(c\cdot F)^{\mu\nu}$$

$$\tilde{\mathcal{O}}_3 = (\dot{x}^\rho F^a_{\rho\mu})(\dot{x}^\sigma F_{a\sigma}{}^{\mu}) \qquad \tilde{\mathcal{O}}_4 = (c\cdot F)_{\mu\rho}\dot{x}^\rho (c\cdot F)^{\mu}{}_{\sigma}\dot{x}^\sigma$$

Radiation to linear order in spin:



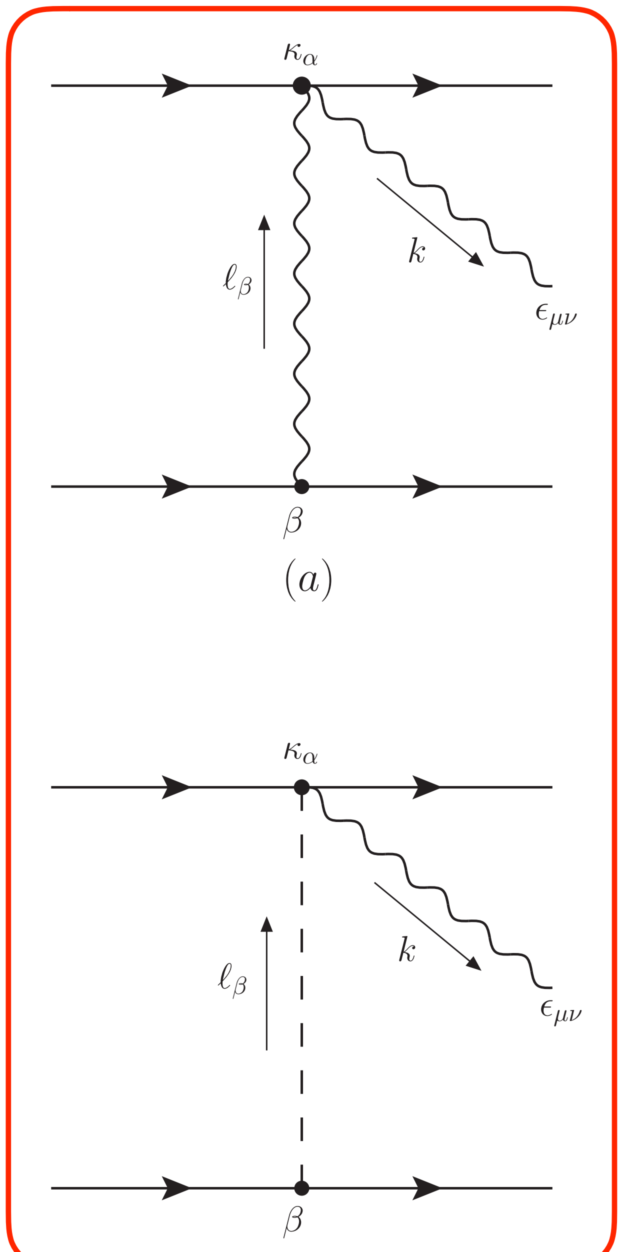
$$\mathcal{A}^{a,\mu}(k) = \sum_{i,j} \int d\mu(k) C^a_i P_{ij}(k) N^\mu_j,$$

$$C^a = \begin{pmatrix} c^a_\beta \\ (c_\alpha \cdot c_\beta) c^a_\alpha \end{pmatrix} \qquad P_{ij} = \ell^2_\alpha \delta_{ij}$$

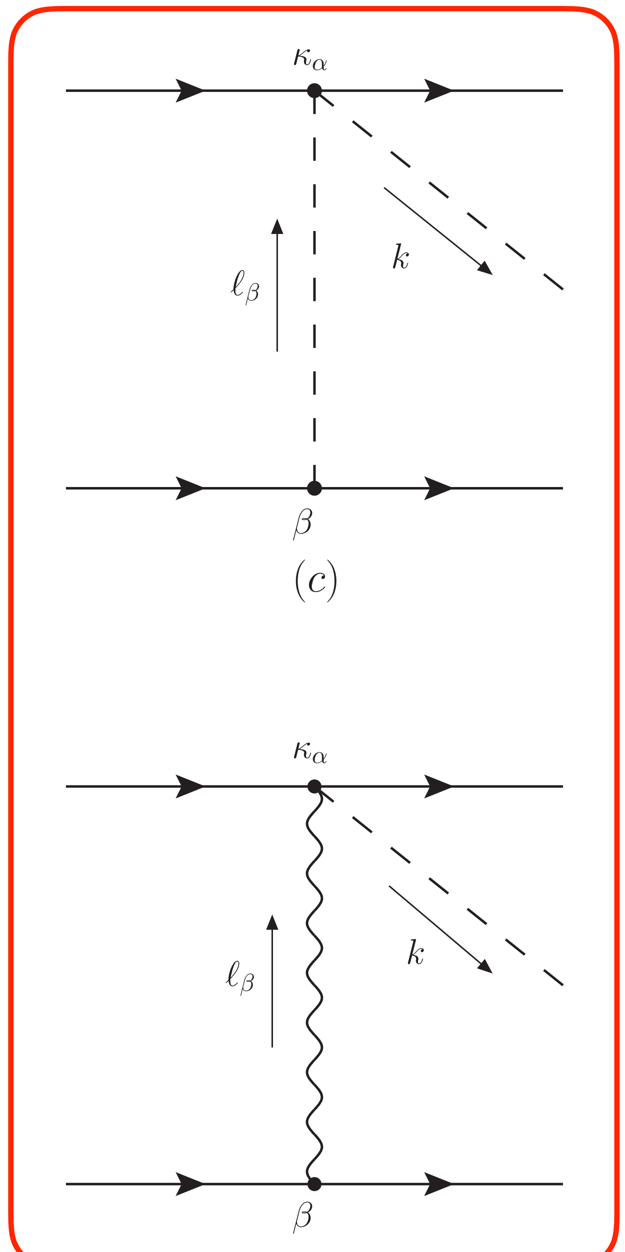
The structure of the gauge theory result predicts the form of the double copy:

$$\mathcal{A}^{a,\mu}(k) \mapsto \mathcal{A}^{\mu\nu}(k) = \sum_{i,j} \int d\mu(k) ds N_i^{\nu} P_{ij}(k) N_j^{\mu},$$

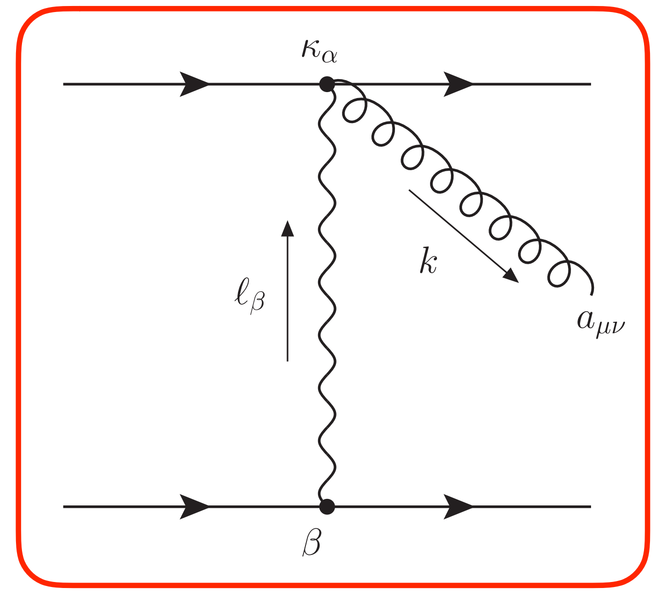
Compare to radiation axion-dilaton gravity. Spin-independent terms:



graviton

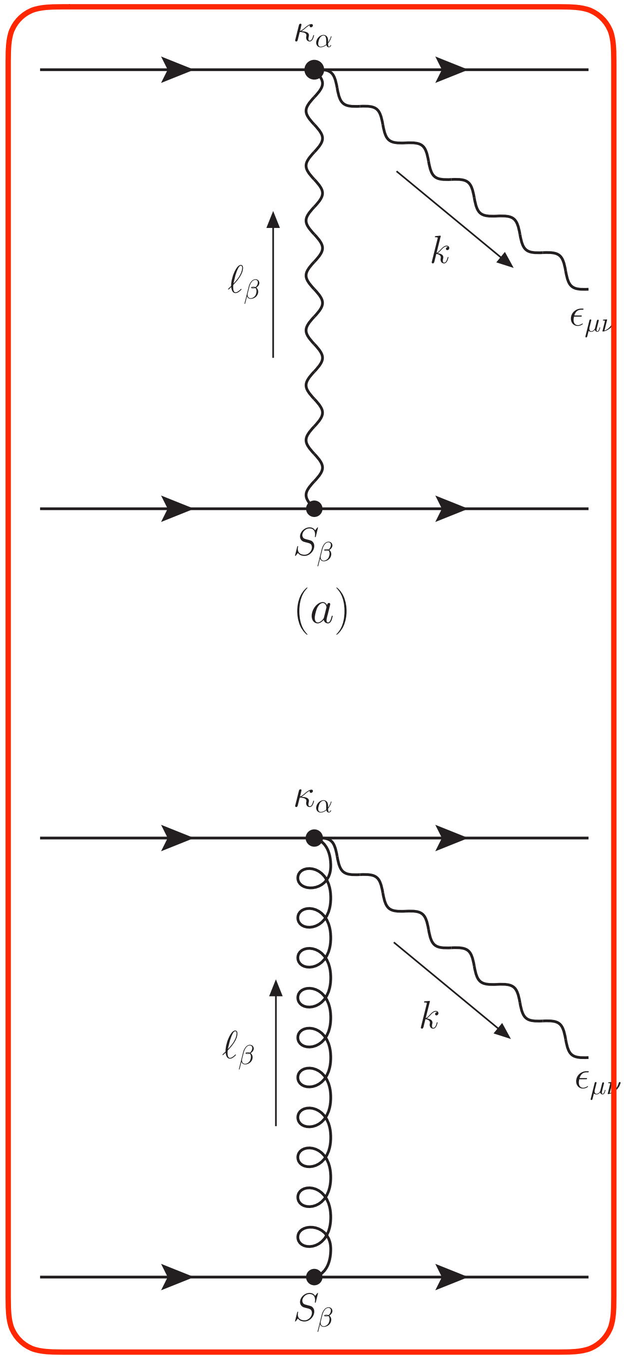


dilaton

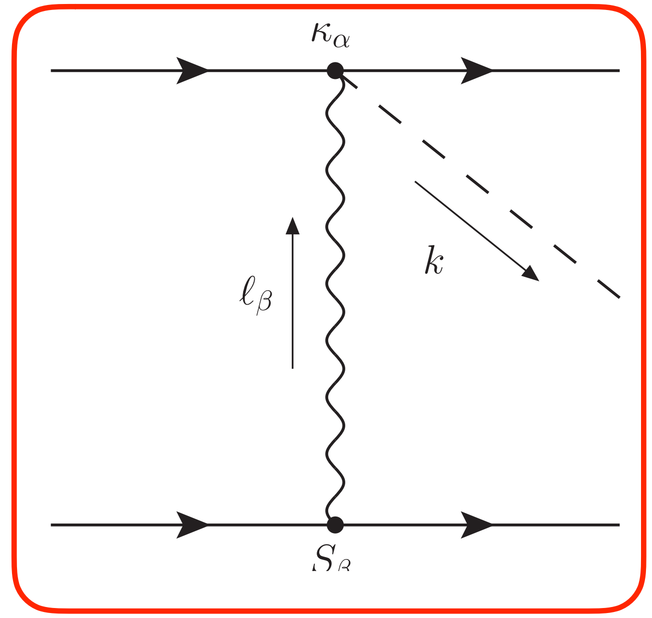


axion

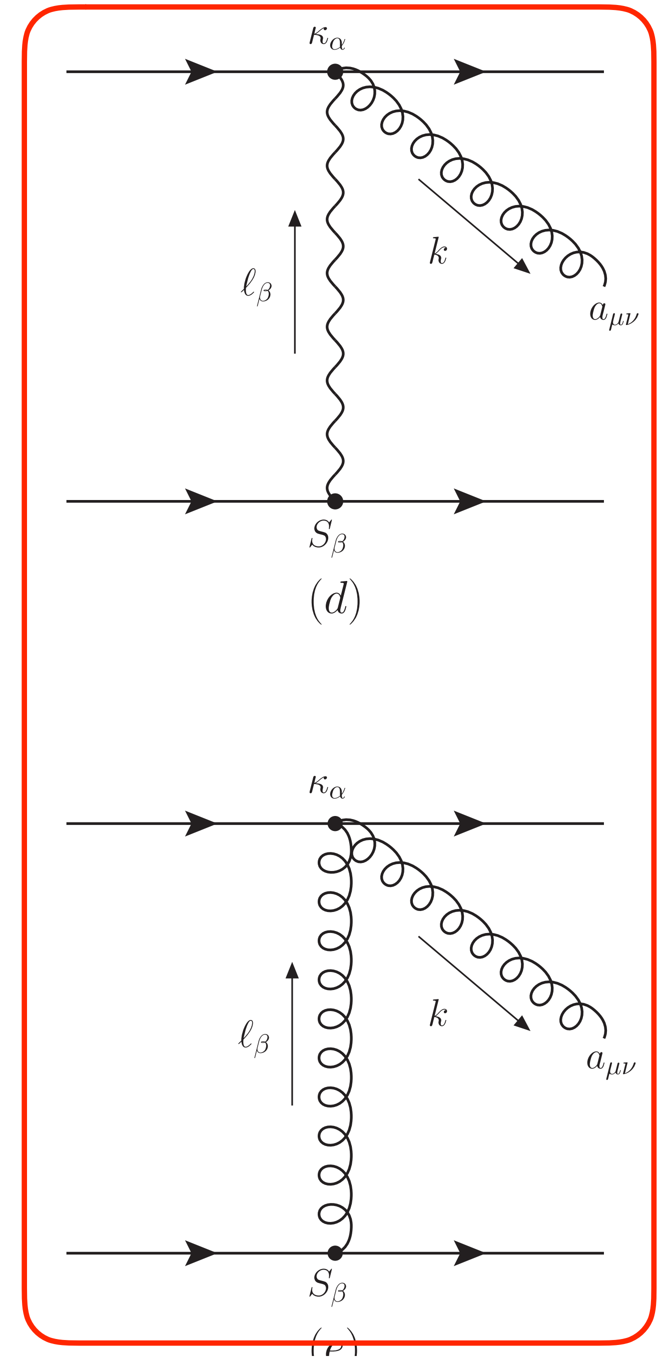
Also need to go to linear order in spin to probe axion couplings



graviton



dilaton



axion

Find that the gravitational amplitude is consistent with finite size effects in axion-dilaton gravity

$$\tilde{\mathcal{O}}_n \mapsto \mathcal{O}^g_n \qquad\qquad S^{grav}_{pp} = \sum_n \kappa_n \int d\tau \mathcal{O}^g_n$$

But the gravitating sources are not generic. Four-parameter space of finite size interactions:

$$\mathcal{O}^g_1 = \frac{1}{4}\left[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 4(d-2)\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi + \frac{1}{6}\nabla_\sigma H_{\mu\nu\rho}\nabla^\sigma H^{\mu\nu\rho}\right]$$

$$\mathcal{O}^g_\pm = \tfrac{1}{4}\dot{x}^\sigma\dot{x}^\lambda\left[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\lambda} - 2R_{\lambda\mu\sigma\nu}\nabla^\mu\nabla^\nu\phi - 2g_{\sigma\lambda}\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi + 2(d-4)\nabla_\mu\nabla_\lambda\phi\nabla^\mu\nabla_\sigma\phi\right.$$
$$\left.\pm 2R_{\mu\nu\rho\sigma}\nabla^\mu H^{\nu\rho\lambda} + \tfrac{1}{4}\nabla_\rho H_{\mu\nu\lambda}\nabla^\rho H^{\mu\nu}{}_\sigma + \tfrac{1}{12}\nabla_\sigma H_{\mu\nu\rho}\nabla_\lambda H^{\mu\nu\rho}\right]$$

$$\mathcal{O}^g_4 = \tfrac{1}{4}\dot{x}^\mu\dot{x}^\nu\dot{x}^\sigma\dot{x}^\lambda\left[R_{\tau\mu\rho\sigma}R^{\tau\nu\rho\lambda} - 2g_{\sigma\lambda}R_{\mu\rho\nu\tau}\nabla^\rho\nabla^\tau\phi + \nabla_\mu\nabla_\nu\phi\nabla_\lambda\nabla_\sigma\phi + \tfrac{1}{4}\nabla_\sigma H_{\tau\rho\lambda}\nabla_\mu H^{\tau\rho}{}_\nu\right]$$

Geometrical significance of relative coefficients?

$$\mathcal{O}^g_1 = \frac{1}{4}\tilde{R}^2_{\mu\nu\rho\sigma} \qquad\qquad \mathcal{O}^g_4 = \frac{1}{4}(\tilde{R}_{\mu\nu\rho\sigma}\dot{x}^\nu\dot{x}^\sigma)^2$$

$\tilde{R}^\mu{}_{\nu\rho\sigma}$ = curvature for connection of $\tilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu}$ and torsion $T^\lambda{}_{\mu\nu} = \frac{1}{2}e^{-2\phi}H^\lambda{}_{\mu\nu}$

This is the same geometry implied by mass and spin couplings
(NOT equivalent to Sherk-Schwarz connection)

Conclusions:

There is evidence (now at NLO) of color-kinematics relations between bi-adjoint scalar, YM and gravity radiation solutions:

$$\mathcal{A}^{a\tilde{a}} \to \mathcal{A}^{a\mu} \to \mathcal{A}^{\mu\nu}$$

Under color-kinematics:

Bi-adjoint charge $\Rightarrow$ YM color charge $\Rightarrow$ $S_{pp} = -m \int d\tau e^{\phi} + \cdots ,$

(at least to quadratic order in the scalar)

$$\int d\tau c_a S^{\mu\nu} F^a_{\mu\nu} \Rightarrow \int dx^\mu S^{\nu\sigma} H_{\mu\nu\sigma}$$

Also consistent for finite size objects (strings?) at the linear response level. Couples naturally to "string frame" connection.

Open questions:

All orders proof of Shen's proposal? Relation to (loop level) BCJ or strings (KLT)?

Assuming it persists at higher orders in PT, is the classical double copy useful for gravity wave calculations:

NLO and beyond for bound orbits? See: Plefka et al, 1807.09859

Axion can be projected out by setting $S_L^{\mu\nu} = S_R^{\mu\nu}$

Systematic procedure for removing the dilaton?
See: Luna, Nicholson, O'Connell, White, 1711.03901

Classical double copy as formulated here only works for specific (spinning+finite size) objects in QCD and gravity. These are effective point particle limits of more fundamental classical objects. What are they?