Current Themes in the Analytic Conformal Bootstrap

Luis Fernando Alday

University of Oxford

Current Themes in High Energy Physics and Cosmology

What will this talk be about?

Conformal field theories in $D > 2$

- Very relevant for Physics.
- Interesting interplay with Mathematics.
- Ubiquitous in dualities in string and gauge theory.

Unfortunately, studying CFT in $D > 2$ is not so easy...

- Symmetries are less powerful than in $D = 2$...
- In general they do not have a Lagrangian description...
- In a Lagrangian theory we can use Feynman diagrams:

$$A(g) = A^{(0)} + gA^{(1)} + \cdots$$

- But generic CFTs don't have a small coupling constant!

In spite of all this, progress can be made!

Conformal bootstrap

- Conformal bootstrap: resort to consistency conditions!
  - Conformal symmetry
  - Properties of the OPE
  - Unitarity
  - Crossing symmetry
- Successfully applied to 2d CFT in the eighties! [Ferrara, Gatto, Grillo; Belavin, Polyakov, Zamolodchikov]
- 25 years later it was finally implemented in $D > 2$! [Rattazzi, Rychkov, Tonni, Vichi '08] The original approach was numeric.

Today: Analytic results for CFTs in the spirit of the bootstrap!
[Work with Bissi, Lukowski, Zhiboedov, Henriksson and van Loon]

Analytic conformal bootstrap

Which kind of analytic results will you get today?

Analytic bootstrap

- Results for operators with spin in a generic CFT!

$$\mathcal{O} \sim \varphi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi$$

- Study their scaling dimension Δ for large values of the spin ℓ:

$$\Delta(\ell) = \ell + 2\Delta_\varphi + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots$$

- We will obtain analytic results to all orders in $1/\ell$ resorting only to consistency conditions.
- Even valid for finite values of the spin!

CFT - Symmetries

Conformal algebra:

- Scale transformations $\rightarrow$ dilatation D
- Poincare Algebra: P_μ and $M_{\mu\nu}$
- Special conformal transformations: K_μ

Specific CFTs may have extra symmetries but we will keep the discussion very general.

CFT - Ingredients

Main ingredient:

- Conformal Primary local operators: $\mathcal{O}_{\Delta,\ell}(x)$, $[K_\mu, \mathcal{O}(0)] = 0$

  Dimension (Δ) — Lorentz spin (ℓ)

In addition we have descendants $P_{\mu_k}...P_{\mu_1}\mathcal{O}_{\Delta,\ell} = \partial_{\mu_k}...\partial_{\mu_1}\mathcal{O}_{\Delta,\ell}$.

Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k\in\text{prim.}} C_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\left(\mathcal{O}_k(0)\underbrace{+x^\mu\partial_\mu\mathcal{O}_k(0)+\cdots}_{\text{all fixed}}\right)$$

- CFT data: The set Δ_i and C_{ijk} characterizes the CFT.

CFT - Basics

Main observable:

Correlation functions of primary operators

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

$$\langle \mathcal{O}_i(1)\mathcal{O}_j(2)\mathcal{O}_k(3)\rangle = \frac{C_{ijk}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

Four-point function of identical operators:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}} x_{34}^{2\Delta_{\mathcal{O}}}}$$

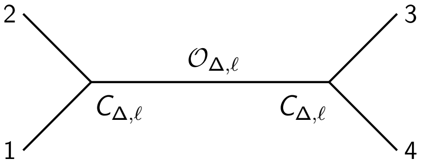
where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

Four-point function - properties

Conformal partial wave decomposition

- OPE: $\mathcal{O} \times \mathcal{O} = \sum_i \mathcal{O}_i + \textit{descendants}$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Delta,\ell}$$



$$\mathcal{G}(u,v) = 1 + \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)$$

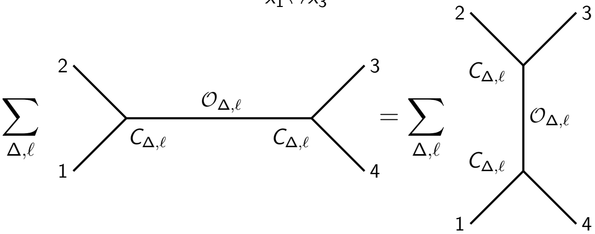
Identity operator Conformal primaries Conformal blocks

- Conformal blocks: For a given primary, take into account the contribution from all its descendants. Fully fixed function!

Conformal bootstrap

Crossing symmetry

$$v^{\Delta_{\mathcal{O}}}\mathcal{G}(u,v) \underbrace{=}_{x_1 \leftrightarrow x_3} u^{\Delta_{\mathcal{O}}}\mathcal{G}(v,u)$$

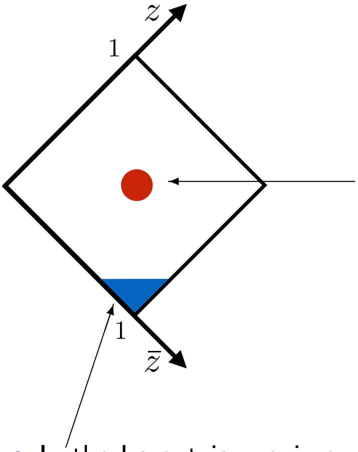


A remarkable...but hard equation!

$$\underbrace{v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)\right)}_{\text{Easy to expand around } u=0,\, v=1} = \underbrace{u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)\right)}_{\text{Easy to expand around } u=1,\, v=0}$$

Numerical vs Analytic bootstrap

Study this equation in different regions, $u = z\bar{z}, v = (1-z)(1-\bar{z})$



- In the Euclidean regime $\bar{z} = z^*$.
- We can study crossing around $u = v = \frac{1}{4}$
- Starting point of the numerical bootstrap.

- In the Lorentzian regime $z, \bar{z}$ are independent real variables and we can consider $u, v \to 0$.
- Starting point of the analytic (light-cone) bootstrap!

Analytic bootstrap

- Why is this a good idea?

$$v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)\right) = u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(v,u)\right)$$

Direct channel $\Leftrightarrow$ Crossed channel

- Very complicated interplay between l.h.s. and r.h.s. ... but:

Operators with large spin

Operators with large spin $\Leftrightarrow$ Identity operator

Conformal blocks - technicalities

- Small u limit:

$$G_{\Delta,\ell}(u,v) \sim u^{\tau/2} f^{coll}_{\tau,\ell}(v), \quad \tau = \Delta - \ell$$

We will introduce the notation

$$G_{\Delta,\ell}(u,v) \equiv u^{\tau/2} f_{\tau,\ell}(u,v)$$

- Small v limit:

$$f_{\tau,\ell}(u,v) \sim \log v$$

Necessity of a large spin sector

- Consider the $v \ll 1$ limit of the crossing equation: $C^2_{\Delta,\ell} \to a_{\tau,\ell}$

$$v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v)\right) = u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\tau,\ell} a_{\tau,\ell} v^{\tau/2} f_{\tau,\ell}(v,u)\right)$$

$$\Downarrow$$

$$1+\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \underbrace{\text{subleading terms}}_{\text{rest of operators sorted by twist}}$$

- The r.h.s. is power-law divergent as $v \to 0$.
- Each term on the l.h.s. diverges as $f_{\tau,\ell}(u,v) \sim \log v$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_{\mathcal{O}}$ (actually $\tau_n = 2\Delta_{\mathcal{O}} + 2n$)

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Example: Generalised free fields

- Simplest solution: Generalised free fields

$$\mathcal{G}^{(0)}(u,v) = 1 + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}}$$

- Intermediate ops: Double twist operators: $\mathcal{O}\Box^n\partial_{\mu_1}\cdots\partial_{\mu_\ell}\mathcal{O}$

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n \\ a_{n,\ell} &= a^{(0)}_{n,\ell} \end{aligned}$$

- Their OPE coefficients are such that the divergence of a single conformal block ($\sim \log v$) , as $v \to 0$, is enhanced!

$$1 + \sum_{\tau,\ell} a^{(0)}_{\tau_n,\ell} u^{\tau_n/2} f_{\tau_n,\ell}(u,v) = 1 + \underbrace{\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}} + u^{\Delta_{\mathcal{O}}}$$

But this divergence is quite universal!

Large spin sector

Additivity property [Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov; F.A. ,Maldacena]

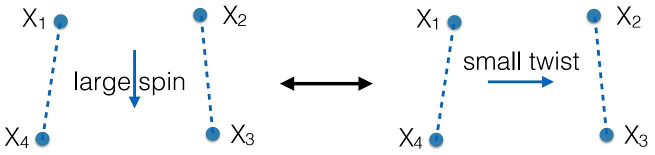
In any CFT with $\mathcal{O}$ in the spectrum, crossing symmetry implies the existence of double twist operators with arbitrarily large spin and

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n + \mathcal{O}\left(\frac{1}{\ell}\right) \\ a_{n,\ell} &= a_{n,\ell}^{(0)}\left(1 + \mathcal{O}\left(\frac{1}{\ell}\right)\right) \end{aligned}$$

- All CFTs have a large spin sector, for which the operators become "free"!
- Can we do perturbations around large spin? YES!

What's going on?

- In Minkowski space we can have $x_{23}^2 \to 0, x_{23} \neq 0$.
- When some operators become null-separated the correlator develops singularities.



Dominated by high spin operators $\Leftrightarrow$ Dominated by low twist operators

$$(\mathcal{O}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\mathcal{O}) \qquad (1, T_{\mu,\nu},\cdots)$$

Large spin perturbation theory

- We exploited the following idea

Identity in the crossed channel

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = \overset{\downarrow}{\frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}}} + \cdots$$

Behaviour at large spin $\Leftrightarrow$ Enhanced divergences as $v \to 0$

Let's take this to the next level...

Large spin perturbation theory

Given $Sing(u, v)$, find $a_{\tau,\ell}$ such that

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u, v) = Sing(u, v)$$

$$\uparrow$$

$$v^{-\Delta}, v^n \log^2 v, v^{1/2}, \cdots$$

Idea

1. Construct a complete basis of functions with specific CFT-data and prescribed singularities.
2. Express $Sing(u, v)$ in that basis.

Large spin perturbation theory

$$Sing(u, v) \to a_{\tau,\ell} = a^{(0)}_{\tau,\ell}\left(1 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots\right)$$

- LSPT fixes the CFT data to all orders in $1/\ell$, just from $Sing(u, v)$!

Inversion formula

- Is the CFT-data analytic in the spin? does LSPT give the full result?
- The answer appeared to be affirmative in all examples!
- Reformulation in terms of an inversion formula [Caron-Huot]

$$a_{\tau,\ell} \sim \int dudv\, K(u,v,\tau,\ell)\, Sing(u,v)$$

- Equivalent to LSPT but explicitly analytic in the spin!
- It can be extrapolated at least down to $spin = 2$ [but often down to spin zero!] as a consequence of the Regge behaviour.

Wider perspective on CFT

- Large spin perturbation theory allows to reconstruct the CFT-data from the enhanced singularities, but... the structure of singularities can be extremely complicated!

If two operators $\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}$ of twists τ_1 and τ_2 are part of the spectrum then there is a tower of operators $[\mathcal{O}_{\tau_1}, \mathcal{O}_{\tau_2}]_{n,\ell}$ of twist

$$\tau_{[\mathcal{O}_{\tau_1},\mathcal{O}_{\tau_2}]_{n,\ell}} = \tau_1 + \tau_2 + 2n + \mathcal{O}\left(\frac{1}{\ell}\right)$$

- This should make you happy and sad at the same time!

The spectrum of generic CFTs is hard!

- ▷ $\mathcal{O}$ is part of the spectrum.
- ▷ $[\mathcal{O}, \mathcal{O}]_{n,\ell}$ is also part of the spectrum.
- ▷ And $[[\mathcal{O}, \mathcal{O}]_{n_1,\ell_1}, [\mathcal{O}, \mathcal{O}]_{n_2,\ell_2}]_{n_3,\ell_3}$ too, and so on!

Large spin expansions for non-perturbative CFTs

In non-perturbative CFTs the spectrum is very rich. Hard (but not impossible!) to apply our method.

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \quad = \quad \text{Rich spectrum in the crossed channel}$$

$$\text{Behaviour at large spin} \quad \Leftrightarrow \quad \text{complicated divergences as } v \to 0$$

- If the CFT has a small parameter we are better of, as this parameter further organises the problem.

LSPT: Strategy for CFTs with small paramaters

Strategy

1. Use crossing symmetry to determine the enhanced singularities

$$\mathcal{G}(u,v) \leftarrow \mathcal{G}(u,v)|_{en.sing.} = \left(\frac{u}{v}\right)^{\Delta_0} \mathcal{G}(v,u)\Big|_{en.sing.}$$

   In theories with small parameters the latter follows from CFT-data at lower orders! (maybe including other correlators)

2. Use LSPT to reconstruct the CFT-data from the enhanced singularities.

3. Go to next order and repeat.

This can be turned into an efficient machinery!

- Theories with weakly broken higher spin symmetry (today)
- Weakly coupled gauge theories.
- $1/N$ corrections in holographic CFTs (IGST)

CFT with weakly broken HS symmetry

CFT with HS symmetry

- The following is part of the spectrum
  - Fundamental scalar field φ

$$\partial_\mu \partial^\mu \varphi = 0 \to \Delta_\varphi = \frac{d-2}{2}$$

  - Tower of HS conserved currents $J^{(\ell)} = \varphi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi$,

$$\text{conservation} \to \Delta_\ell = d - 2 + \ell$$

Weakly broken HS symmetry

- The fundamental field φ and HS currents $J^{(\ell)}$ acquire an anomalous dimensions:

$$\begin{aligned} \Delta_\varphi &= \frac{d-2}{2} + g\gamma_\varphi + \cdots \\ \Delta_\ell &= d - 2 + \ell + g\gamma_\ell + \cdots \end{aligned}$$

Which anomalous dimensions are consistent with crossing symmetry?

$\epsilon-$expansion from LSPT

- We will study the WF-model in $d = 4 - \epsilon$ dimensions, where $g \sim \epsilon$.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{g}{4!}\varphi^4$$

- We will make no reference to the Lagrangian.
- Instead, carry our program for the correlator $\langle\varphi\varphi\varphi\varphi\rangle$

$$\mathcal{G}(u,v) = \underbrace{1 + \left(\frac{u}{v}\right)^{\Delta_\varphi} + u^{\Delta_\varphi}}_{\text{Identity } + \ J^{(\ell)}} + g\,\mathcal{G}^{(1)}(u,v) + \cdots$$

$\epsilon-$expansion from LSPT

- Study enhanced singularities in a perturbative expansion:

$$\mathcal{G}(u,v)|_{enh.sing.} = \left(\frac{u}{v}\right)^{\Delta_\varphi} \left(1 + S_{\varphi^2}(v,u) + \underbrace{S_2(v,u)}_{\varphi\partial^\ell\varphi} + \underbrace{S_4(v,u)}_{\varphi^2\partial^\ell\varphi^2,\cdots} + \cdots\right)$$

- Order by order in g a controlled number of operators contribute.
- We only use crossing symmetry, plus conservation of the stress tensor $\rightarrow \gamma_2 = 0$.

Leading order

$$\gamma_{\varphi^2} = g, \quad \gamma_\varphi = \frac{1}{12}g^2 + \cdots, \quad \gamma_\ell = 2\gamma_\varphi\left(1 - \frac{6}{\ell(\ell+1)}\right) + \cdots$$

Known from a few decades...(but not from symmetries!)

Results

- Analyticity down to spin zero fixes the relation between g and ϵ:

$$g(3g-\epsilon)=0$$

$g=0$	$g=\epsilon/3$
Usual free theory!	Interacting WF model!

- Without any reference to a Lagrangian or equations of motion we have rediscovered the two expected CFTs!

Results

- We can go to much higher order and compute the CFT-data to order ϵ^4!

New: central charge to order ϵ^4!

$$\frac{C_T}{C_{free}} = 1 - \frac{5}{324}\epsilon^2 - \frac{233}{8748}\epsilon^3 - \left(\frac{100651}{3779136} - \frac{55}{2916}\zeta_3\right)\epsilon^4 + \cdots$$

- We have catch up with decades of diagrammatic computations!

Large spin expansions for non-perturbative CFTs

Q: What about a truly non-perturbative CFT?

A: Hard...but still possible to make progress!

- The most singular part of $Sing(u, v)$ arises from low twist operators in the dual channel.

$$Sing(u,v) \to a_{\tau,\ell} = a^{(0)}_{\tau,\ell}\Big(\underbrace{1}_{\text{from the identity}} + \underbrace{\frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots}_{\text{from low twist operators}} \Big)$$

- Can we extrapolate the results to small spin!?

3d Ising Model

- No Lagrangian description.
- No large N.
- No integrability.

But: Conformal symmetry!

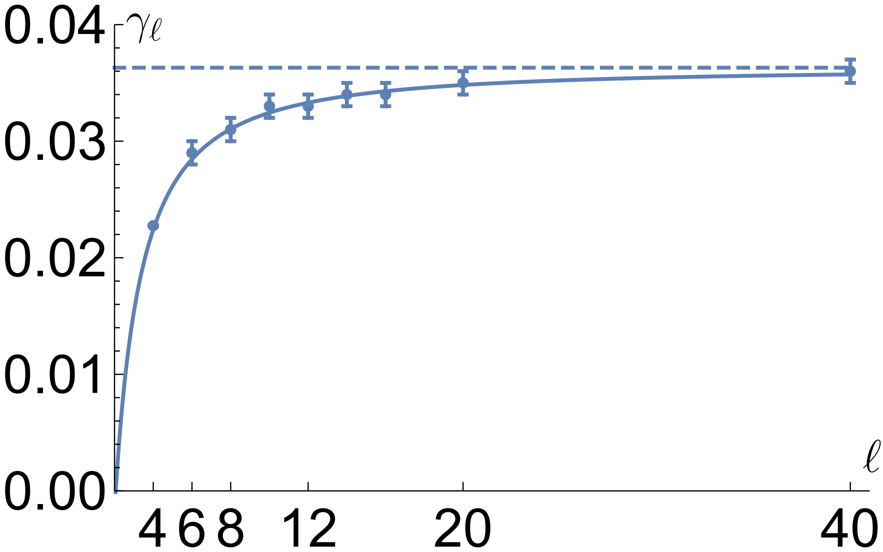
- Spin operator $\Delta_\sigma = 0.518151 = 1/2 + \gamma_\sigma$.
- Large spin-operators $\sigma \partial_{\mu_1} \cdots \partial_{\mu_s} \sigma$.

We predict

$$\gamma_\ell \sim 2\gamma_\sigma - \frac{0.0027}{\ell} - \frac{0.0926}{\ell^{1.4126}}$$

- Next corrections are computable and can be seen to be small, even for $\ell \sim 1$.

3d Ising - Comparison with numerical results



Conclusions

- Generic CFTs have a large spin sector, which becomes essentially free. We have shown how to perform a perturbation around that sector.
- This applies to vast families of CFTs and provides an alternative to diagrammatic computations.
- Evidence for a transcendentality principle for WF.
- There are many other analytic approaches to CFT, in the spirit of the conformal bootstrap, and the connection to these, in particular numerical bootstrap, is a fascinating question.