Scattering Amplitudes from Ambitwistor Strings

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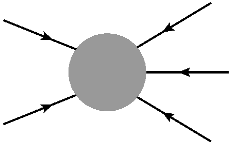
Current Themes in High Energy Physics and Cosmology,

Copenhagen, 16 August 2018

with Y. Geyer (IAS), L. Mason (Oxford), P. Tourkine (CERN):
1507.00321, 1511.06315, 1607.08887

with Y. Geyer (IAS): 1711.09923, 1805.05344

Scattering Amplitudes



Calculable with Feynman diagrams:

- good: general, clear physical picture.
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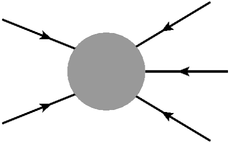

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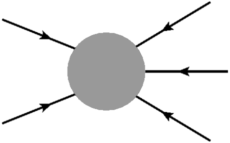

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Modern approaches:

- analyticity, kinematic variables (e.g. on-shell), symmetries.
- relations between theories (e.g. gravity vs gauge theory).
- new formulations of QFTs.

Why Strings?

QFT from low-energy string theory: alternative to Feynman expansion.

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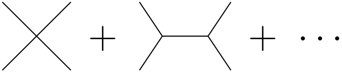
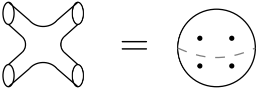


particle scattering
(many Feynman diagrams)



string scattering
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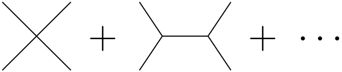

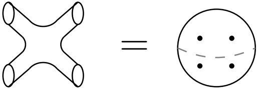

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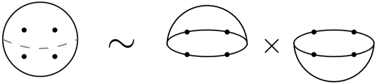
Old idea: calculate QFT amplitudes from string theory. [Green, Schwarz, Brink 82]

Insights: UV divergences, algebraic structure,

gravity (closed strings) v.s. gauge theory (open strings), . . .



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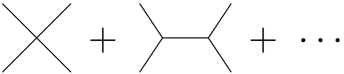

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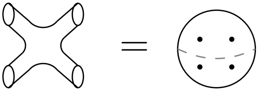

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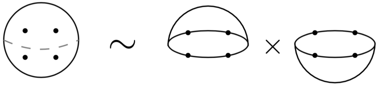
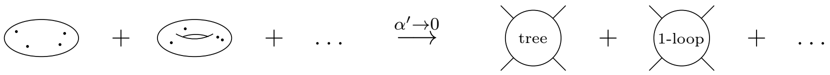


Hard: higher loop corrections (simpler at low energy), dropping susy.

World-sheet Models of (Massless) QFTs

String theory: field theory is $\alpha' \to 0$, massive modes decouple, $m_n^2 = c_n/\alpha'$.



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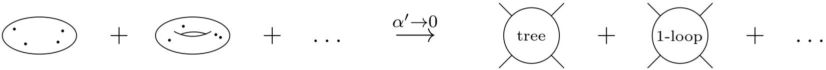
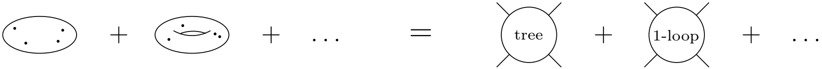


Target space is spacetime.

Is there truncated version just for QFT?

Ambitwistor strings: no α', only massless states.



Target space is space of (complex) null geodesics = *ambitwistor space*.

Formulas for amplitudes based on *scattering equations*.

Worldsheet Models $\longleftrightarrow$ Scattering Equations

Old story

Twistor string theory [Witten 03] $\longrightarrow$ RSV formula [Roiban, Spradlin, Volovich 04]

$D = 4$. SYM, SUGRA [Hodges, Cachazo, Geyer, Skinner, Mason 12].

Only tree level (unwanted states). [Berkovitz, Witten 04]

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New story

Ambitwistor string theory [Mason, Skinner 13] $\longleftarrow$ CHY formulas [Cachazo, He, Yuan 13-14]

Any D. Many theories of massless particles.

Loop-level progress! Unwanted states absent or projected out.

Outline

- Tree level
- One loop
- Two loops

Tree Level

Scattering Equations

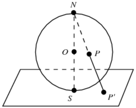
[Cachazo, He, Yuan '13]

[Previous history: Fairlie, Roberts '72, Gross, Mende '88, Witten '04]

Consider n massless particles, $k_i^2 = 0, \; i = 1, \ldots, n, \qquad \sum_{i=1}^{n} k_i = 0\,.$

$$E_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0, \qquad \forall i$$

- kinematic invariants $s_{ij} = (k_i + k_j)^2 = 2\, k_i \cdot k_j \;\longrightarrow\;$ points $\sigma_i \in \mathbb{CP}^1$



$SL(2, \mathbb{C})$ invariance, $\sigma \to \dfrac{A\,\sigma + B}{C\,\sigma + D}$

- $(n-3)!$ solutions $\sigma_i^{(A)}$.

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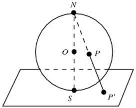
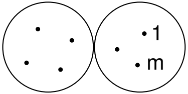


$SL(2, \mathbb{C})$ invariance, $\quad \sigma \to \dfrac{A\,\sigma + B}{C\,\sigma + D}$

- $(n-3)!$ solutions $\sigma_i^{(A)}$.
- factorisation: $\quad (k_1 + \ldots + k_m)^2 \to 0 \quad \longleftrightarrow \quad \sigma_1, \ldots, \sigma_m \to \sigma_\star$



[Dolan, Goddard 13]

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Direct evaluation:

$$\mathcal{A} = \sum_{A=1}^{(n-3)!} \frac{\mathcal{I}}{J} \Big|_{\sigma_i = \sigma_i^{(A)}}$$

Scattering equations hard to solve, but no need for that!

[Dolan, Goddard; Cachazo, Gomez; Baadsgaard at al; Huang et al; Sogaard, Zhang; Cardona, Kalousios; Fu et al; . . .]

Examples: Yang-Mills theory and Gravity

[Cachazo, He, Yuan '13]

Ingredients in Yang-Mills (gluon) $e^{ik\cdot x}\epsilon_\mu T^a$, Gravity (graviton, ...) $e^{ik\cdot x}\varepsilon_{\mu\nu}$:

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$\Rightarrow$ **Gravity $\sim$ YM^2**

cf. Kawai-Lewellen-Tye relations '86

Bern-Carrasco-Johansson relations '08

Other Examples

Many more theories
of massless particles!

e.g. Gravity-Yang-Mills,
Born-Infeld, NLSM,. . .

[Cachazo, He, Yuan 14]

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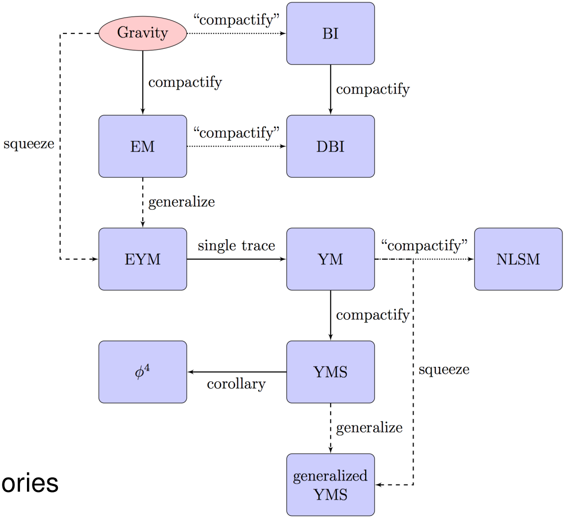

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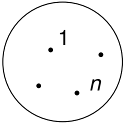
Lessons:

- relations between theories
- messy Feynman rules $\neq$ messy amplitudes

Geometry of Scattering Equations

$SL(2,\mathbb{C})$ invariant differential on $\mathbb{CP}^1$: $$P_\mu(\sigma) = d\sigma \sum_{i=1}^{n} \frac{k_{i\,\mu}}{\sigma - \sigma_i}$$



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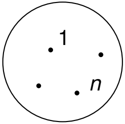

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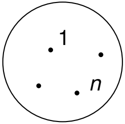

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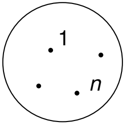

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Any D? Twistor $\rightsquigarrow$ Ambitwistor space [Mason, Skinner 13]

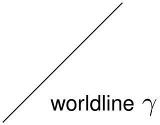
$=$ space of null geodesics of (complexified) spacetime.

Strings in Ambitwistor Space

[Mason, Skinner 13]

Worldline action for massless particle:

$$S_{\text{p}} = \int_{\gamma} p_{\mu}\, dx^{\mu} - \frac{1}{2}\, e\, p^2 \qquad d = d\lambda\, \partial_{\lambda}$$



- e enforces $p^2 = 0 \quad (m^2 = 0)$.
- gauge freedom: $\delta x^{\mu} = \alpha p^{\mu},\ \delta p_{\nu} = 0,\ \delta e = d\alpha$.

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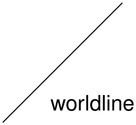

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Chiral complexification: “string”

$$S = \frac{1}{2\pi} \int_\Sigma P_\mu \, \bar{\partial} X^\mu - \frac{1}{2} e \, P^2 \qquad \bar{\partial} = d\bar{\sigma} \, \partial_{\bar{\sigma}}$$

- $P_\mu = d\sigma \, p_\mu(\sigma)$.
- e enforces $P^2 = 0$, same gauge freedom $(d\alpha \rightsquigarrow \bar{\partial}\alpha)$.

Ambitwistor space: (X^μ, P_ν) with $P^2 = 0, \; (X^\mu, P_\nu) \sim (X^\mu + \alpha \, P^\mu, P_\nu)$.

Quantisation of Ambitwistor String

[Mason, Skinner 13]

Action: $S = \frac{1}{2\pi}\int_\Sigma P_\mu\,\bar\partial\,X^\mu - \frac{1}{2}\,e\,P^2$ Amplitude: $\mathcal{A} = \left\langle \prod_{i=1}^{n} V_i \right\rangle$

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Integrate $X^\mu \Rightarrow \bar{\partial} P_\mu = 2\pi i \sum_i k_{i\,\mu} \delta^2(\sigma - \sigma_i)$

$\Rightarrow P_\mu$ is meromorphic with simple poles at σ_i with residues $k_{i\,\mu}$.

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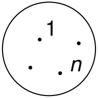
$\Rightarrow$ P_μ is meromorphic with simple poles at σ_i with residues $k_{i\,\mu}$.

Riemann sphere $\mathbb{CP}^1$ $\Rightarrow$ $P_\mu = d\sigma \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i}$



$\Rightarrow$ $\mathrm{Res}_{\sigma_i} P^2 = 2E_i = 0$ are scattering equations $\Rightarrow$ $\mathcal{A} = \int_{\mathfrak{M}_{0,n}} d\mu\; \mathcal{I}$

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[Mason, Skinner 13]

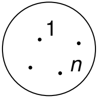
Action: $S = \frac{1}{2\pi}\int_\Sigma P_\mu\,\bar\partial\,X^\mu - \frac{1}{2}\,e\,P^2$ Amplitude: $\mathcal{A} = \left\langle \prod_{i=1}^{n} V_i \right\rangle$

Gauge fix $e = 0$: $V_i = \int d\sigma_i\,\delta(\mathrm{Res}_{\sigma_i} P^2)\; e^{ik_i\cdot X}\;\ldots$ $k_i^2 = 0$ massless

Integrate $X^\mu \;\Rightarrow\; \bar\partial P_\mu = 2\pi i \sum_i k_{i\,\mu}\,\delta^2(\sigma - \sigma_i)$

$\Rightarrow P_\mu$ is meromorphic with simple poles at σ_i with residues $k_{i\,\mu}$.

Riemann sphere $\mathbb{CP}^1 \;\Rightarrow\; P_\mu = d\sigma \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i}$



$\Rightarrow\; \mathrm{Res}_{\sigma_i} P^2 = 2E_i = 0$ are scattering equations $\Rightarrow\; \mathcal{A} = \int_{\mathfrak{M}_{0,n}} d\mu\;\mathcal{I}$

Combine with world-sheet matter to reproduce various CHY formulas.

[also Ohmori 15; Casali, Geyer, Mason, RM, Roehrig 15]

E.g., add system of fermions $\rightsquigarrow$ Pfaffian.

Ambitwistor Strings vs. Ordinary Strings

Ambitwistor strings:

- chiral, massless states
- $\mathcal{A} = \int \left(\prod_i d\sigma_i \, \delta(E_i)\right) \ \mathcal{I}_L(\sigma_i) \, \mathcal{I}_R(\sigma_i)$

Ordinary (closed) strings:

- not chiral, infinite tower of massive states
- $\mathcal{A}_{\text{st}}(\alpha') = \int \left(\prod_i d\sigma_i d\bar{\sigma}_i\right) F(|\sigma_i - \sigma_j|) \ \mathcal{I}_L(\sigma_i) \, \mathcal{I}_R(\bar{\sigma}_i)$

Formulas: field theory limit is $\mathcal{A}_{\text{st}}(\alpha') \xrightarrow{\alpha' \to 0} \mathcal{A}$. Non-trivial!

[e.g. Bjerrum-Bohr, Damgaard, Tourkine, Vanhove 14, Mizera 17]

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Theories:

Surprise understanding from $\alpha' \to \infty$ with alternative quantisation.

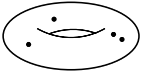
[Siegel 15, +Huang, Yuan 16, Casali, Tourkine 16, +Herfray 17, Azevedo, Jusinskas 17, . . .] [Gross, Mende 88]

Open problem type II superstring $\xrightarrow{\alpha' \to 0}$ type II ambitwistor string.

Higher Genus

Geometry at Higher Genus

Good theory. Loop corrections? [Adamo, Casali, Skinner 13]

Example: one loop needs torus  τ $(\sigma \sim \sigma+1 \sim \sigma+\tau)$

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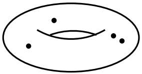
Example: one loop needs torus  τ $(\sigma \sim \sigma+1 \sim \sigma+\tau)$

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- torus: $P_\mu = \ell_\mu \, d\sigma + \sum_i k_{i\,\mu} \, \varpi_{i,*}$

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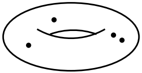

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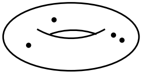
Example: one loop needs torus  τ $(\sigma \sim \sigma+1 \sim \sigma+\tau)$

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Scattering equations $P^2 = 0$ should localise both σ_i and parameters τ_{IJ}.

Higher-genus surfaces hard (Jacobi θ functions)… Practical?

One Loop

From the Torus to the Nodal Sphere

[Geyer, Mason, RM, Tourkine 15]

Torus scattering equations should localise $\{\sigma_i\,,\tau\}$, but too hard to solve.

Loop integrand should be easier, like tree level.

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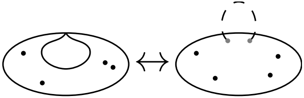

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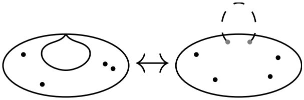
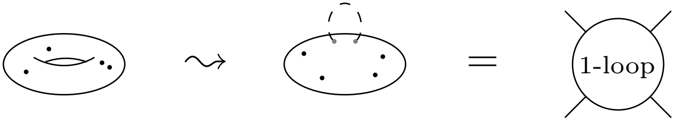


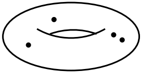
How to get there? Residue theorem on τ integration: localises on $\tau = i\infty$.



Different approach: use elliptic parametrisation. [Cardona, Gomez 16]

Residue Argument

Torus amplitude: $\mathcal{A}^{(1)} = \int d^D\ell \int_{\mathfrak{M}_{1,n}} d\mu \; \mathcal{I}^{\text{torus}}$



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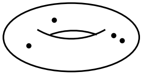

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- $E_i = 0$ (no poles at σ_i),
- $P^2|_{E_i=0} = u(\ell, \sigma_i, \tau)\, d\sigma^2 \;\; \Rightarrow \;\; u = 0$ (extra).

Measure: $d\mu \sim d\tau\, \delta(u)\, (\prod_i d\sigma_i\, \delta(E_i))$

View as residue integral, e.g., $\int d\sigma\, \delta(F(\sigma)) \, \cdots = \oint \frac{d\sigma}{2\pi i\, \underline{F(\sigma)}} \, \cdots$

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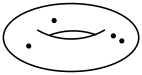

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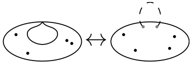
Use $q = e^{2\pi i \tau}$ and localise at $q = 0$ $(\tau = i\infty)$.

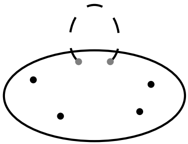


$$\int_{\mathfrak{M}_{1,n}} d\mu\ \mathcal{I}^{\text{torus}} \sim \oint \frac{dq\, \prod_i d\sigma_i}{q\ u\ \prod_i E_i}\ \mathcal{I}^{\text{torus}} = -\oint \frac{dq\, \prod_i d\sigma_i}{q\ u\ \prod_i E_i}\ \mathcal{I}^{\text{torus}} = -\left[\oint \frac{\prod_i d\sigma_i}{u\ \prod_i E_i}\ \mathcal{I}^{\text{torus}}\right]_{q=0}$$

New One-Loop Formula

[Geyer, Mason, RM, Tourkine 15]

Final result: $$\mathcal{A}^{(1)} = \int d^D\ell \, \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} d\mu^{(1)} \; \mathcal{I}^{(1)}$$



Like tree level, but now for one-loop integrand!

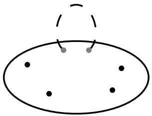
- two new "particles": loop momentum insertions $\pm\ell$
- $$P_\mu = d\sigma \left(\frac{\ell_\mu}{\sigma - \sigma_{+\ell}} + \frac{-\ell_\mu}{\sigma - \sigma_{-\ell}} + \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i} \right) = \ell_\mu \, \omega + d\sigma \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i}$$
- $$\mathcal{I}^{(1)} = \lim_{\tau \to i\infty} \mathcal{I}^{\text{torus}}$$
- one-loop scattering equations depend on ℓ_μ

$$P^2 - \ell^2 \omega^2 = 0 \quad \Leftrightarrow \quad P^2 - \ell^2 \omega^2 \text{ has no poles at } \sigma_i \,, \; \sigma_\pm$$

New Formalism

CHY-type expression for loop integrand:

$$\mathcal{A}^{(1)} = \int d^D\ell \, \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} d\mu^{(1)} \;\; \mathcal{I}^{(1)}$$



Previous derivation applied to “type II supergravity in 10D”.

More general?

- Higher genus is very restrictive (modular invariance).
- Perturbative QFT less restricted than a string theory!

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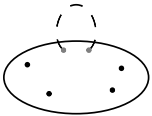

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Proposal: generic loop expansion is nodal expansion on sphere.

Look directly for $\mathcal{I}^{(1)}$ in nodal sphere, in any D.

New CHY-type Formulas

[Geyer, Mason, RM, Tourkine 15]

Yang-Mills: $\mathcal{I}^{(1)}_{\text{YM}} = \mathcal{I}^{(1)}_{\text{kin}}(\epsilon)\, \mathcal{I}^{(1)}_{\text{colour}}$

Gravity: $\mathcal{I}^{(1)}_{\text{Grav}} = \mathcal{I}^{(1)}_{\text{kin}}(\epsilon)\, \mathcal{I}^{(1)}_{\text{kin}}(\tilde{\epsilon})$

Double copy!

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Can write "vertex operator" for node. [Roehrig, Skinner 17]

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Explicit integrand $\mathfrak{I}(\ell)$? No need to solve the scattering equations.

[Baadsgaard, Bourjaily, Bjerrum-Bohr, Damgaard, Feng 15; He, Yuan 15, . . .]

New Propagator Structure

New formula: $\mathcal{A}^{(1)} = \int d^D\ell \, \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} d\mu^{(1)} \; \mathcal{I}^{(1)}$

Puzzle! Only one $\frac{1}{\ell^2}$, rest depends only on $\ell \cdot K, \; \ell \cdot \epsilon \; \ldots$

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Shifted Integrand

- use $\frac{1}{\prod_i D_i} = \sum_i \frac{1}{D_i \prod_{j\neq i}(D_j - D_i)}$, $\quad D_i = (\ell + K_i)^2$
- shift each term $\frac{1}{D_i} \to \frac{1}{\ell^2}$

Bubble:

$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\,\ell\cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\,\ell\cdot K - K^2)}$$

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Democratic placement of ℓ in “canonical” integrand.

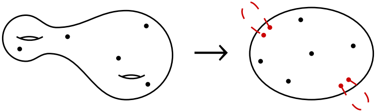
Good for obtaining loop integrands from trees! “Q-cuts” [Baadsgaard et al 15, CHY 15]

Consequences for colour-kinematics duality. [He, Schlotterer 16-17, Geyer, RM 17]

Two Loops

Status at Two Loops

Idea:



Done

- heuristic: 4-pts in SUGRA, SYM [Geyer, Mason, RM, Tourkine 16] [genus 2: Adamo, Casali 15]
- detailed: *n*-pts in SUGRA, SYM [Geyer, RM 18]

  genus 2 follows closely superstring [D'Hoker, Phong 01, 05]

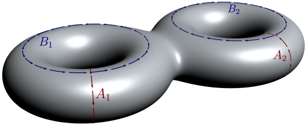
In progress

- *n*-pts in non-SUSY gravity, YM

Genus 2

There are 2 holomorphic differentials ω_I , $I = 1, 2$.

Pick homology basis:

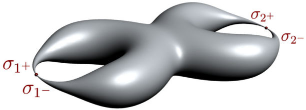


Normalise ω_I with A-cycles: $\oint_{A_I} \omega_J = \delta_{IJ}$

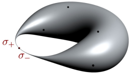
Period matrix: $\tau_{IJ} = \oint_{B_I} \omega_J = \tau_{JI}$ $\longrightarrow$ 3 modular parameters

$$q_{11} = e^{i\pi\tau_{11}} \ , \quad q_{22} = e^{i\pi\tau_{22}} \ , \quad q_{12} = e^{2i\pi\tau_{12}}$$

Bi-nodal sphere: $q_{11} = q_{22} = 0$



Recall genus 1: $q = 0$



From Genus 2 to the Bi-Nodal Sphere

Solution for P_μ : $$P_\mu = \ell_{1\,\mu}\,\omega_1 + \ell_{2\,\mu}\,\omega_2 + \sum_i k_{i\,\mu}\,\varpi_{i,*}$$

$P^2(\sigma) = 0$ at genus 2:

- $E_i = 0$ (no poles at σ_i),
- $P^2|_{E_i=0} = u_{IJ}\,\omega_I\,\omega_J \quad \Rightarrow \quad u_{IJ} = 0$ (3 extra).

Genus-2 scattering equations would fix $\{\sigma_i, q_{IJ}\}$ but too hard to solve.

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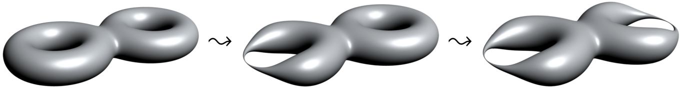
Genus 1: from $u = 0$ into $q = 0$.

Genus 2: from $u_{11} = u_{22} = 0$ into $q_{11} = q_{22} = 0$. **(more subtle)**



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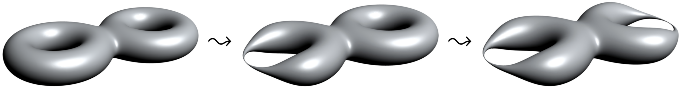
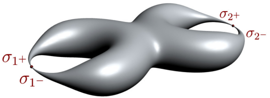


Amplitude: $$\mathcal{A}^{(2)} = \int d^D\ell_1 d^D\ell_2 \int_{\mathfrak{M}_{2,n}} d\mu \;\; \mathcal{I}^{\text{genus-2}} = \int \frac{d^D\ell_1 d^D\ell_2}{(\ell_1)^2(\ell_2)^2} \int_{\mathfrak{M}_{0,n+4}} d\mu^{(2)} \;\; \mathcal{I}^{(2)}$$

From Genus 2 to the Bi-Nodal Sphere

New formula: $\mathcal{A}^{(2)} = \int \frac{d^D\ell_1 \, d^D\ell_2}{(\ell_1)^2(\ell_2)^2} \int_{\mathfrak{M}_{0,n+4}} d\mu^{(2)} \; \mathcal{I}^{(2)}$



Subtle points of residue argument:

- leftover equation of type $u_{IJ} = 0$: $\quad u_{11} + u_{22} + u_{12} = 0$
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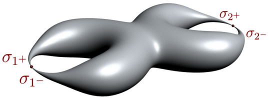

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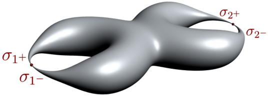

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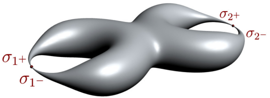

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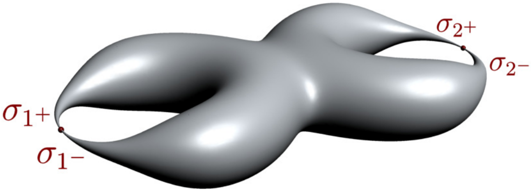
$$\mathcal{I}^{(2)}_{\text{SUGRA}} = \frac{1}{1-q_{12}}\,\mathcal{I}^{(2)}_{\text{kin}}(\epsilon)\,\mathcal{I}^{(2)}_{\text{kin}}(\tilde{\epsilon}) \quad \rightsquigarrow \quad \mathcal{I}^{(2)}_{\text{SYM}} = \mathcal{I}^{(2)}_{\text{kin}}(\epsilon)\,\mathcal{I}'^{(2)}_{\text{colour}}$$

Get $\mathcal{I}'^{(2)}_{\text{colour}}$ from current algebra on bi-nodal sphere (cf. heterotic string).

All Genus Plausibility



genus g	$\dim_{\mathbb{C}}(\mathfrak{M}_{g,n}) = n + 3g - 3$
degenerations	$-g$
g-nodal sphere	$\dim_{\mathbb{C}}(\mathfrak{M}_{0,n+2g}) = (n + 2g) - 3$

Conclusion

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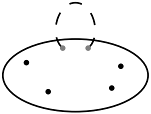
- Formulas for YM and gravity, SUSY/no-SUSY.



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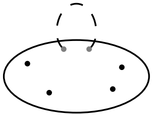

Many open questions

- All-loop story?
- Double copy from gauge theory to gravity?
- 4D formalism? Efficient manipulation, inc. loop integration?
- Beyond scattering amplitudes? Beyond perturbation theory?
- Insights into string theory?

Extra slides

KLT Relations: String Theory Origin

[Kawai, Lewellen, Tye '86]

Vertex operators: $V_{\text{closed}}(\varepsilon^{\mu\nu} = \epsilon^{\mu}\tilde{\epsilon}^{\nu}) \sim V_{\text{open}}(\epsilon^{\mu})\,\bar{V}_{\text{open}}(\tilde{\epsilon}^{\nu})$

$s_{ij} = (k_i + k_j)^2$

$$\mathcal{A}_3^{\text{grav}} = A^{\text{YM}}(123)\,\tilde{A}^{\text{YM}}(123) \qquad \mathcal{A}_4^{\text{grav}} = \frac{\sin \pi\alpha' s_{12}}{\pi\alpha'}\,A^{\text{YM}}(1234)\,\tilde{A}^{\text{YM}}(1243)$$

Field theory limit is $\alpha' \to 0$.

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In general (tree level)

[Bern, Dixon, Perelstein, Rozowsky '98]

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Useful at loop level via unitarity cuts.

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Recall YM colour decomposition: colour traces or colour factors.

$$\mathcal{A}_n^{\text{YM}} = \sum_{\text{non cyclic}} A^{\text{YM}}(1, 2, \ldots, n)\; \mathrm{tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) = \sum_{\alpha \in \text{cubic}} N_\alpha\, c_\alpha$$

with $c_\alpha = f^{abc} f^{\cdots} \cdots f^{\cdots}$, $f^{abc} = \mathrm{tr}([T^a, T^b]T^c)$,

but Jacobi identities: $c_\alpha \pm c_\beta \pm c_\gamma = 0$

Summary of Double Copy

$$\boxed{\mathcal{A}_{\text{grav}}(\varepsilon_i^{\mu\nu}) \sim (\text{prop})^{-1}\ \mathcal{A}_{\text{YM}}(\epsilon_i^{\mu}) \times \mathcal{A}_{\text{YM}}(\tilde{\epsilon}_i^{\nu})\ \Big|_{\text{colour stripped}}}$$

KLT relations

$$\mathcal{A}_{\text{grav}} = \sum_{P_n, P'_n} A_{\text{YM}}(\epsilon, P_n)\ \ S_{\text{KLT}}[P_n, P'_n]\ \ A_{\text{YM}}(\tilde{\epsilon}, P'_n)$$

BCJ double copy

$$\mathcal{A}_{\text{YM}} = \sum_{\alpha \in \text{cubic}} \frac{n_\alpha(\epsilon)\, c_\alpha}{D_\alpha} \qquad \mathcal{A}_{\text{grav}} = \sum_{\alpha \in \text{cubic}} \frac{n_\alpha(\epsilon)\, n_\alpha(\tilde{\epsilon})}{D_\alpha}$$

CHY formulas

$$\mathcal{A} = \int d\mu\, \mathcal{I} \qquad \mathcal{I}_{\text{YM}} = \text{Pf}\,' M(\epsilon) \times \mathcal{C} \qquad \mathcal{I}_{\text{grav}} = \text{Pf}\,' M(\epsilon) \times \text{Pf}\,' M(\tilde{\epsilon})$$