Spontaneous CP breaking and the axion potential: an effective Lagrangian approach

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Foreword

- This talk is based on the work done together with
  Giancarlo Rossi, Gabriele Veneziano and Shimon Yankielowicz,
  JHEP **1712** (2017) 104, arXiv:1709.00731v3 [hep-th]

- Related work
  D. Gaiotto, Z. Komargodski and N. Seiberg
  JHEP **1801** (2018) 110, arxiv:1708.06806 [hep-th]

- as well as older papers by
  A. Smilga, Phys. Rev D59 (1999) 114021
  M. Tytgat, Phys. Rev. D **61** (2000) 114009
  M. Creutz, Phys. Rev. Lett. 92 (2004) 201601

Plan of the talk

Introduction

- Dashen in 1971 noticed that a phase in the quark mass matrix would generate *CP* violation in strong interactions.
- He speculated that this could explain the *CP* violation found in the physics of K mesons.
- It turned out that one would get a too big *CP* violation.
- Although the ABJ $U(1)_A$ anomaly was known, one thought that one could rotate the phase away by a $U(1)_A$ transformation because the topological charge was a total derivative.
- In this way one got rid of *CP* violation.
- This, however, brought in another problem: the $U(1)$ problem.
- The split in the quark masses is not sufficient to explain the mass spectrum of the pseudoscalar mesons.

- After the discovery of instanton solutions and the presence of different topological sectors it was soon realized that the $U(1)$ problem might be solved.
- Although it remained controversial for a while.
- The observation that, in the framework of large-N QCD, the quark mass matrix contained an extra parameter corresponding to the topological susceptibility of pure Yang-Mills theory, opened the way to a quantitative resolution of the $U(1)$ problem. [Witten,1978 and Veneziano,1978]
- It is based on an effective Lagrangian for the pseudoscalar mesons that is valid for small $\frac{m}{\Lambda}$ and $\frac{1}{N}$ with $\frac{mN}{\Lambda}$ fixed.
- m is the mass of the quarks, N is the number of colors.
- Unfortunately, the resolution of the $U(1)$ problem brings back the problem of CP violation of strong interactions.

- The Lagrangian contains an extra term proportional to the topological charge density of Yang-Mills theory that breaks *CP*.
- Actually, by performing a $U(1)_A$ transformation, one can eliminate the phase from the mass matrix and having it to contribute to the topological term:

$$L_{QCD} = \cdots - \theta \int d^4x \ Q(x)$$

$$\theta = \bar{\theta} + arg \ \ \det m \quad ; \quad Q(x) = \frac{g^2}{32\pi^2} F^a \tilde{F}^a$$

  Under *CP*: $Q \to -Q$.
- The *CP* violation induced by this term was used to estimate the electric dipole moment of the neutron [Baluni, 1979].
- It was later refined by identifying a leading logarithmic contribution thus establishing a limit on $\theta \sim 10^{-9} - 10^{-10}$ [Crewter et al, 1979]
- For the smallness of it QCD had, on its own, no explanation.

- A natural way to put the θ-angle to zero is by the Peccei-Quinn mechanism that implies an axion [Peccei and Quinn, 1977].
- A generic axion can be incorporated in the previous effective Lagrangian [Sannino and DV, 2014].
- Since the axion mass is much smaller than that of the pseudoscalar mesons, one can compute the axion potential by integrating out the pseudoscalar mesons.
- This is certainly justified in a certain region of the parameter space of the effective Lagrangian.
- At zero temperature we are certainly in this region where this way of computing the axion potential is reliable.
- On the other hand, there are other regions where the mass of one pseudoscalar meson goes to zero.
- The question is if at finite temperature this region can be reached.
- If this is this case the usual calculation of the axion potential is not correct.
- Lattice gauge theory calculations are not yet precise enough to see if this is the case.
- In this seminar I will describe how all this comes about.

Low-energy effective Lagrangian of QCD

- Strong interactions are described by the QCD Lagrangian:

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\Psi} \left(i\gamma^\mu D_\mu - m \right) \Psi - \theta Q(x)$$

- Mass matrix can always be put in a diagonal form

$$m_{ij} = m_i \delta_{ij} \;\; ; \;\; i, j = 1 \ldots N_f$$

- Topological charge density

$$Q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \;\; ; \;\; \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

- For massless quarks the transformations (A and B are $N_f \times N_f$ unitary matrices)

$$\Psi^i_R \to A^{ij} \Psi^j_R \;\; ; \;\; \Psi^i_L \to B^{ij} \Psi^j_L \;\; ; \;\; \Psi_{R,L} = \frac{1 \pm \gamma^5}{2} \Psi$$

  are a symmetry of the QCD Lagrangian:

  $U(N_f) \times U(N_f)$ chiral invariance

- This symmetry is spontaneously broken to the vectorial $U(N_f)$ generated by the transformations for which $A = B$.
- Pseudoscalar mesons are Goldstone bosons associated to the spontaneous breaking of chiral symmetry.
- They get a non-zero mass from the quark mass matrix ($m \neq 0$).
- But the splittling in the quark masses:

  $$\frac{m_u}{m_d} = 0.56 \;\; ; \;\; \frac{m_s}{m_d} = 20.1 \;\; ; \;\; \bar{m}_d|_{\mu=2GeV} = (3.1 \pm 1)MeV$$

  cannot explain the mass spectrum of the pseudoscalar mesons:

  $$m_\pi = 139MeV \,;\, m_\eta = 547MeV \,;\, m_{\eta'} = 957MeV \,;\, m_K = 498MeV$$

  This problem was called $U(1)$-problem.
- $U(1)$ axial anomaly:

  $$\partial_\mu \left[\bar{\Psi}_i \gamma^\mu \gamma^5 \Psi_i \right] = 2N_f q(x) + 2im_i \bar{\Psi}_i \gamma_5 \Psi_i$$

  At large N, $g^2 N$ is kept fixed and the anomaly is negligible.
- How can we incorporate the effect of the anomaly in the meson mass matrix?

- At low energy ($E << \Lambda$) and low quark masses ($m_i << \Lambda$) we can neglect all (the heavy) degrees of freedom keeping only the Goldstone bosons (pseudoscalar mesons).
- They are described by the following chiral Lagrangian:

$$L = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(\mu^2 (U + U^\dagger)\right) \quad ; \quad UU^\dagger = \frac{F_\pi^2}{2}$$

- The constraint implies that $U \in U(N_f)$:

$$U(x) = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \quad ; \quad \Phi(x) = \Pi^a \tau^a + \frac{S}{\sqrt{N_f}} \; ; \; Tr[\tau^a \tau^b] = \delta^{ab}$$

  $F_\pi \sim 95 MeV$ is the pion decay constant measured in $\pi \to \mu\nu$.
- For $N_f = 3$, Π corresponds to the octet of pseudoscalar mesons:

$$\Pi^a \tau^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi^0 + \eta_8/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta_8/\sqrt{3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\eta_8/\sqrt{3} \end{pmatrix}$$

- As the quark mass matrix, also μ^2 can be chosen to be diagonal:

$$\mu_{ij}^2 = \mu_i^2 \delta_{ij}$$

- Gell-Mann-Oakes-Renner relation

$$\mu_i^2 F_\pi^2 = -2 m_i < \bar{\Psi}_i \Psi_i >$$

In first approximation the ratio m_i / μ_i^2 is independent on i.

- If $\mu^2 = 0$ the previous Lagrangian is invariant under $U(N_f) \times U(N_f)$ transformations:

$$U \rightarrow A U B^\dagger \ \ ; \ \ U^\dagger \rightarrow B U^\dagger A^\dagger \ \ ; \ \ A^{-1} = A^\dagger \ \ ; \ \ B^{-1} = B^\dagger$$

- It has the same global symmetries as QCD with massless quarks, but it does not take care of the $U(1)$ axial anomaly !!

- We have to add a term that is invariant under $SU(N_f) \times SU(N_f) \times U(1)_V$ and trasforms under the $U(1)_A$ to reproduce the axial anomaly:

$$L \to L + 2N_f q(x)\alpha \;\; ; \;\; U \to e^{-2i\alpha} U$$

- This insures that the effective Lagrangian satisfies the same anomalous Ward identities as the fundamental QCD Lagrangian.
- This brings us to the following modified Lagrangian:

$$L = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} Tr\left(\mu^2 (U + U^\dagger)\right) +$$

$$+\frac{i}{2} Q(x) Tr\left(\log U - \log U^\dagger\right)$$

- In general we could add a generic term of the form:

$$\sum_{i=0}^{\infty} L_{2i}(U, U^\dagger)[q(x)]^{2i}$$

preserving parity and $U(N_f) \times U(N_f)$.

- It turns out that, for large N, only one term of the previous sum contributes.
- This brings us to:

$$L(U,U^{\dagger},q)=\frac{1}{2}Tr(\partial_{\mu}U\partial_{\mu}U^{\dagger})+\frac{F_{\pi}}{2\sqrt{2}}Tr\left(\mu^{2}(U+U^{\dagger})\right)$$

$$+\frac{i}{2}Q(x)Tr\left(\log U-\log U^{\dagger}\right)+\frac{1}{2}\frac{Q^{2}}{\chi_{YM}}-\theta Q(x)$$

[Veneziano and DV, 1980; Witten, 1980; Rosenzweig, Schechter and Trahern, 1980 ; Nath and Arnowitt, 1980]

- θ term is added to study the dependence of phys. quantities on θ and χ is a constant $\sim\mathcal{O}(1)$ for large N whose meaning will become clear later.
- Our aim in the following is
  1. Show how the $U(1)$ problem is solved
  2. Determine the dependence of physical quantities on the θ parameter

Finding the minimum

- Eliminate Q by using its algebraic equation of motion:

$$L(U, U^{\dagger}, q) = \frac{1}{2} Tr(\partial_{\mu} U \partial_{\mu} U^{\dagger}) + \frac{F_{\pi}}{2\sqrt{2}} Tr\left(\mu^2 (U + U^{\dagger})\right)$$

$$-\frac{\chi_{YM}}{2}\left[\theta - \frac{i}{2} Tr\left(\log U - \log U^{\dagger}\right)\right]^2$$

- Find the value of $< U >$ that minimizes the potential.
- Since $UU^{\dagger} \sim 1$ and μ^2 is diagonal we can take:

$$< U_{ij} >= e^{-i\phi_i} \delta_{ij} \frac{F_{\pi}}{\sqrt{2}} \;\; ; \quad U_{ij} \equiv e^{-i\phi_i} V_{ij} \Longrightarrow < V_{ij} >= \delta_{ij} \frac{F_{\pi}}{\sqrt{2}}$$

- One obtains $(\mu^2_{ij}(\phi_i) = \mu^2_i \cos\phi_i \delta_{ij})$:

$$L = \frac{1}{2} Tr(\partial_\mu V \partial_\mu V^\dagger) + \frac{\chi_{YM}}{8}\left[Tr\left(\log V - \log V^\dagger\right)\right]^2$$

$$+\frac{F_\pi}{2\sqrt{2}} Tr\left(\mu^2(\phi_i)\left[(V+V^\dagger) - \frac{2F_\pi}{\sqrt{2}}\right]\right)$$

$$+\frac{F_\pi^2}{2}\sum_{i=1}^{N_f}\mu_i^2\cos\phi_i - \frac{\chi_{YM}}{2}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)^2 \quad \leftarrow -\mathcal{V}$$

$$+\frac{i}{2}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)\chi Tr\left[(\log V - \log V^\dagger) - \frac{\sqrt{2}}{F_\pi}(V - V^\dagger)\right]$$

- In the last line we have used the relation:

$$\mu_i^2 \sin\phi_i = a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right)$$

- The parametrs ϕ_i are determined by minimizing the potential:

$$\mathcal{V} = \frac{F_\pi^2}{2}\left[\frac{a}{2}(\theta - \sum_{i=1}^{N_f}\phi_i)^2 - \sum_{i=1}^{N_f}\mu_i^2\cos\phi_i\right] \quad ; \quad \chi_{YM} \equiv \frac{aF_\pi^2}{2}$$

Since $\chi \sim \mathcal{O}(1)$ and $F_\pi^2 \sim \mathcal{O}(N)$ at large N, then $a \sim \mathcal{O}(\frac{1}{N})$.

- This implies the following equations:

$$\mu_i^2 \sin\phi_i = a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right) \quad ; \quad i = 1 \ldots N_f$$

- Finally in terms of Φ we get:

$$L = \frac{1}{2} Tr(\partial_\mu V \partial_\mu V^\dagger) - \frac{aN_f}{2} S^2 + \frac{F_\pi^2}{2} Tr\left[\mu^2(\theta)\left(\cos\frac{\sqrt{2}\Phi}{F_\pi} - 1\right)\right] +$$

$$+\frac{aF_\pi}{\sqrt{2}}\left(\theta - \sum_{i=1}^{N_f}\phi_i\right) Tr\left[\frac{F_\pi}{\sqrt{2}}\sin\frac{\sqrt{2}\Phi}{F_\pi} - \Phi\right] \quad \leftarrow \textit{CP violation}$$

where

$$V = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \;\; ; \;\; \Phi = \tau^a \Pi^a + \frac{S}{\sqrt{N_f}} \;\; ; \;\; \mu_{ij}^2(\theta) = \mu_i^2 \cos\phi_i \delta_{ij}$$

- We proceed as follows:
  1. First we solve the minimization equations that determine ϕ_i as functions of θ, a and μ_i^2.
  2. Then we insert them in L that will in general be a function of θ, a and μ_i^2.

- Physical quantities are invariant under $\theta \to \theta + 2\pi$!!
  If we have found a solution $\phi_i(\theta)$ of the minimization equations, then the following is also a solution:

$$\phi_i(\theta + 2\pi) = \phi_i(\theta) + 2\pi \quad ; \quad \phi_j(\theta + 2\pi) = \phi_j(\theta) \;\; ; \;\; i \neq j = 1 \dots N_f$$

  But the physical quantities depend only on $e^{i\phi_i}$ and therefore are invariant under a shift of 2π of θ.
- The quadratic part of the previous Lagrangian is:

$$L_2 = \frac{1}{2} Tr\left(\partial_\mu \Phi \partial^\mu \Phi\right) - \frac{a}{2} Tr\left(\Phi\right) Tr\left(\Phi\right) - \frac{1}{2} Tr\left(\mu^2(\theta)\Phi^2\right)$$

- It is convenient to decompose the matrix Φ as follows:

$$\Phi_{ij} = \tilde{\Pi}^{\alpha\beta}\tilde{\tau}_{ij}^{\alpha\beta} + v_i\delta_{ij}$$

$\tilde{\tau}_{ij}^{\alpha\beta}$ are the $N_f(N_f - 1)$ non-diagonal generators of $SU(N_f)$.

- One gets:

$$< \tilde{\Pi}^{\alpha\beta}(x)\tilde{\Pi}^{\gamma\delta}(y) >^{F.T.} = \frac{i\delta^{\alpha\gamma}\delta^{\beta\delta}}{p^2 - \mu^2_{\alpha\beta}} \;\; ; \;\; \mu^2_{\alpha\beta} = \frac{\mu^2_\alpha(\theta) + \mu^2_\beta(\theta)}{2}$$

$$< v_i(x)v_j(y) >^{F.T.} = iA^{-1}_{ij}(p^2)$$

$$A_{ij}(p^2) = (p^2 - \mu_i^2(\theta))\delta_{ij} - a \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ . & . & . & . & . \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

- The masses $M_i^2(\theta)$ of the physical states are determined by the following equation:

$$\det A = \prod_{i=1}^{N_f} (p^2 - M_i^2(\theta)) = \prod_{i=1}^{N_f} (p^2 - \mu_i^2(\theta))$$
$$\times \left[1 - a \sum_{j=1}^{N_f} \frac{1}{p^2 - \mu_j^2(\theta)} \right] = 0 \;\; ; \;\; \mu_i^2(\theta) = \mu_i^2 \cos \phi_i$$

- In the case with three flavours, $\theta = 0$ and in the limit $\mu_1^2 \sim \mu_2^2 << \mu_3^2$ one gets the following masses for η and η':

$$M_\pm^2 = m_K^2 + \frac{3}{2} a \pm \frac{1}{2} \sqrt{(2m_K^2 - 2m_\pi^2 - a)^2 + 8a^2}$$

$$\tan \phi = \sqrt{2} - \frac{3}{2\sqrt{2}} \cdot \frac{m_\eta^2 - m_\pi^2}{m_K^2 - m_\pi^2} \;\; ; \;\; |\eta\rangle = \cos \phi |8\rangle + \sin \phi |1\rangle$$

- We get a from the sum of the masses:

$$a = \frac{m_\eta^2 + m_{\eta'}^2 - 2m_K^2}{3} \sim 0.24(GeV)^2$$

- Using this value of a and neglecting the square term in the square root we get:

$$m_\eta^2 \sim m_K^2 + \frac{3-2\sqrt{2}}{2} a = 0.27(GeV)^2 \;\; ; \;\; [Exp.\ 0.30]$$

$$m_{\eta'}^2 \sim m_K^2 + \frac{3+2\sqrt{2}}{2} a = 0.95(GeV)^2 \;\; ; \;\; [Exp.\ 0.92]$$

and

$$\phi \sim 14 \; [Exp.\, 11]$$

[Veneziano, 1978]

- a is related to the parameter χ that enters in the effective Lagrangian (as the coefficient of the Q^2 term) by $\chi_{YM} \equiv \frac{1}{2} a F_\pi^2$.

- How can we extract it from the underlying QCD?
- In the large N limit quark loops can be neglected with respect to gluons loops.
- Therefore, for large N, a can be extracted from the topological susceptibility of pure Yang-Mills theory:

$$\lim_{q\to 0}(-i)\int d^4y \, \mathrm{e}^{iqx} < Q(x)Q(y) >_{YM} = \frac{1}{2} a F_\pi^2 \sim (180 MeV)^4$$

$$[Exp. \; Lattice] = (194(5) MeV)^4$$

- This should not be confused with the topological susceptibility computed in large N QCD or, equivalently, from the previous effective Lagrangian (using the GMOR relation):

$$\lim_{q\to 0}(-i)\int d^4x \mathrm{e}^{iqx} \langle Q(x)Q(0)\rangle_{QCD} \equiv \chi_{QCD} = \frac{\chi_{YM}}{1 + a\sum_{i=1}^{N_f} \frac{1}{\mu_i^2}}$$

$$= \chi_{YM}\left(1 - \sum_{k=1}^{N_f} \frac{\chi_{YM}}{(m_i \langle\bar{\psi}\psi\rangle)}\right)^{-1}$$

- $\langle\bar{\psi}\psi\rangle$ is the quark condensate in the large N limit i.e. in the planar theory.
- $\langle\bar{\psi}\psi\rangle$ has logarithmic terms that, however, are subleading for large N.
- In the lattice calculation of $\langle\bar{\psi}\psi\rangle$ one must then extract the leading term for large N that is free from logarithmic terms.
- In other words, one has to compute:

$$\langle\bar{\psi}\psi\rangle_{planar} = 3 \lim_{N\to\infty} \frac{\langle\bar{\psi}\psi\rangle(N)}{N}$$

- χ_{QCD} reduces to the theory with $N_f - 1$ flavors when one of the quark mass becomes large.
- If all quarks become very heavy (possible for large N) it becomes equal to χ_{YM}.
- $\chi_{QCD} = 0$ in the chiral limit, when at least one of the $\mu_i^2 = 0$, for reasons that will become clear soon.

- Using

$$\mu_1^2 = 0.7 m_\pi^2 \;\; ; \quad \mu_2^2 = 1.3 m_\pi^2 \;\; ; \;\; a = \frac{m_\eta^2 + m_{\eta'}^2 - 2m_K^2}{3} = 0.24(GeV)^2$$

we get

$$\chi_{QCD} = (78.5 MeV)^4$$

- These numerical values come from the spectrum of the pseudoscalar mesons.
- They can be computed in QCD on the lattice and the values obtained are in good agreement with the previous values.

The usual form of χ_{QCD}

- In the literature the topological susceptibility in QCD is given by the following formula:

$$\chi_{QCD} = \frac{\partial^2 F(\theta)}{\partial \theta^2}|_{\theta=0}$$

where $F(\theta)$ is the free energy that in our case is given by the potential:

$$\mathcal{V}(\theta) \equiv F(\theta) = \frac{F_\pi^2}{2}\left[-\sum_{i=1}^{N_f}\mu_i^2\cos\phi_i + \frac{a}{2}(\theta - \sum_{i=1}^{N_f}\phi_i)^2\right]$$

where the angles ϕ_i are determined by solving the conditions.

$$\mu_i^2 \sin\phi_i = a(\theta - \sum_j \phi_j) \quad ; \quad i = 1 \ldots N_f$$

- These conditions imply:

$$\mu_i^2 \cos\phi_i \frac{\partial \phi_i}{\partial \theta} = a(1 - \sum_i \frac{\partial \phi_i}{\partial \theta})$$

- At $\theta = 0$ (where $\phi_i = 0$) the previous condition implies:

$$G \equiv \mu_i^2 \frac{\partial \phi_i}{\partial \theta} = a(1 - \sum_j \frac{\partial \phi_i}{\partial \theta}) = a(1 - G \sum_j \frac{1}{\mu_j^2}) \;\; ; \;\; i = 1 \ldots N_f$$

where G is independent on i.

- It implies that

$$G = a(1 - \sum_j \frac{\partial \phi_i}{\partial \theta}) = \frac{1}{\frac{1}{a} + \sum_i \frac{1}{\mu_i^2}}$$

- Using the previous equations we can compute:

$$\frac{2}{F_\pi^2} \frac{\partial^2 F(\theta)}{\partial \theta^2}|_{\theta=0} = a(1 - \sum_i \frac{\partial \phi_i}{\partial \theta})^2 + \sum_i \mu_i^2 \cos \phi_i \left(\frac{\partial \phi_i}{\partial \theta} \right)^2$$

$$= a(1 - \sum_i \frac{\partial \phi_i}{\partial \theta}) = \frac{1}{\frac{1}{a} + \sum_i \frac{1}{\mu_i^2}}$$

$$\Longrightarrow \chi_{QCD} = \frac{\partial^2 F(\theta)}{\partial \theta^2}|_{\theta=0} = \frac{F_\pi^2/2}{\frac{1}{a} + \sum_i \frac{1}{\mu_i^2}} = \frac{\chi_{YM}}{1 + a \sum_i \frac{1}{\mu_i^2}}$$

QCD phase diagrams: $N_f = 1$

- Keep F_π and a fixed and study phase diagrams (for zero temperature and chemical potential) varying μ_i^2 and θ.
- For $N_f = 1$ we have the following potential:

$$\frac{V(\phi)}{a} = -\epsilon \cos\phi + \frac{1}{2}(\theta - \phi)^2 \;\; ; \;\; \epsilon \equiv \frac{\mu^2}{a} ,$$

and its derivatives with respect to ϕ

$$\frac{V'}{a} = \epsilon \sin\phi + \phi - \theta \;\; ; \;\; \frac{V''}{a} = \epsilon\cos\phi + 1$$
$$\frac{V'''}{a} = -\epsilon\sin\phi \;\; ; \;\; \frac{V''''}{a} = -\epsilon\cos\phi .$$

- If $\epsilon < 1$, $V'' > 0$ and there is only a single stable minimum with positive mass.
- At $\theta = 0$ the minimum is at $\phi = 0$, while at $\theta = \pi$ it is at $\phi = \pi$.
- In both cases *CP* is unbroken; *CP* is broken for other values of θ.

- If $\epsilon > 1$, V'' can be negative and some stationary points can correspond to maxima rather than minima of V.
- For a zero mass ground state we should require $V' = V'' = 0$.
- For it to be the absolute minimum we should also have $V''' = 0$ and $V'''' > 0$.
- However, we see that $V''' = 0$ is only possible if $\phi = \pi \bmod(\pi)$ and therefore only if $\theta = \pi$.
- For $\theta = \pi$ there is always a stationary point at $\phi = \pi$ which, however, for the case $\epsilon > 1$, corresponds to a maximum ($V'' < 0$).
- Since V is bounded from below there should also be minima.
- Indeed, for $\epsilon = 1 + \bar{\delta}, \bar{\delta} \ll 1$, one easily finds two (degenerate) minima.
- For $\epsilon = 1$ the three stationary points degenerate at $\phi = \pi$ and the stable minimum corresponds to a massless CP conserving ground state.

- We can determine the solution around $\phi = \pi$ by writing $\phi = \pi - \delta$ and plugging it into V' getting

$$\delta \left(\frac{\delta^2 \epsilon}{6} + 1 - \epsilon \right) = 0 .$$

- In this way we find again the solution $\delta = 0$, which corresponds to a maximum , for $\epsilon > 1$, and two stable minima related by *CP* at

$$\delta_{\pm} = \pm \sqrt{\frac{6(\epsilon - 1)}{\epsilon}} .$$

- This can be seen from

$$\left. \frac{V''}{a} \right|_{\delta=0} = 1 - \epsilon \quad ; \quad \left. \frac{V''}{a} \right|_{\delta_{\pm}} = 2(\epsilon - 1) .$$

- This implies that the solution with $\delta = 0$ is a stable one for $\epsilon \leq 1$, while the two other solutions are stable for $\epsilon > 1$.
- The twofold degeneracy at $\theta = \pi$ implies that *CP* is spontaneous broken: the two states are transformed into each other by a *CP* transformation.

- Under CP $U \to U^\dagger \implies \phi \to -\phi$ and the two solutions are:

$$\phi_+ = \pi + \sqrt{\frac{6(\epsilon - 1)}{\epsilon}} \;\; ; \;\; \phi_- - 2\pi = -\pi - \sqrt{\frac{6(\epsilon - 1)}{\epsilon}}$$

- At $\epsilon = 1$ there is a second order phase transition where the PNGB becomes massless.
- Indeed the mass square is given by the second derivative of the potential computed at the minimum, yielding

$$M^2 = \mu^2(\theta) + a = \mu^2 \cos\phi + a\,,$$

- Notice that M^2 goes to zero for $\epsilon = 1, \theta = \phi = \pi$.
- The second order phase transition is also signalled by the divergence of the topological susceptibility (defined as the $\langle Q\, Q\rangle$ correlator at zero momentum) at $\epsilon = 1, \theta = \pi$.
- One gets

$$\chi_{QCD} = \frac{\chi_{YM}}{1 + \frac{a}{\mu^2(\theta)}} = \frac{\chi_{YM}\epsilon\cos\phi}{1 + \epsilon\cos\phi}\,,$$

which diverges for $\epsilon = 1$ at $\theta = \phi = \pi$.

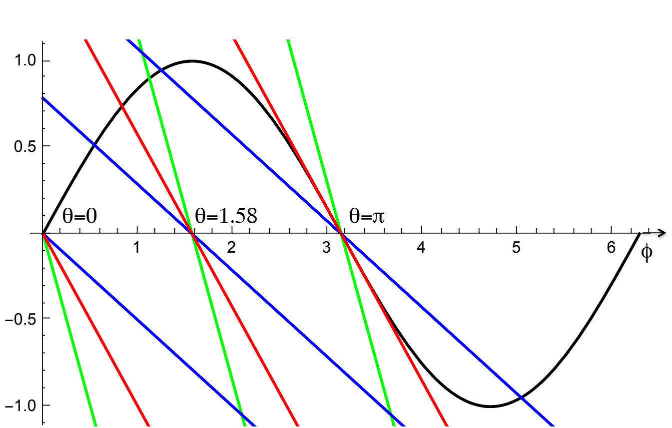

Figure: Solutions of $V'(\phi) = 0$ are given by the intersections of the curve $\sin\phi$ (black) with the straight lines $(\theta - \phi)/\epsilon$ for $\theta = 0$, $\theta = \pi$ and a generic value taken to be $\theta = 1.58$. Code color is as follows: $\epsilon < 1$ green lines, $\epsilon = 1$ red lines, $\epsilon > 1$ blue lines.

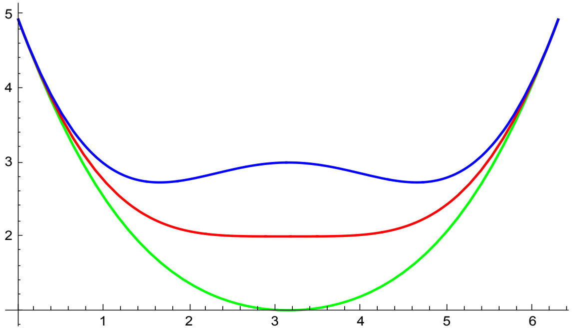

Figure: $V(\phi)$ at $\theta = \pi$, and $\epsilon = 0.5$ (green curve), $\epsilon = 1.0$ (red) and $\epsilon = 2.0$ (blue).

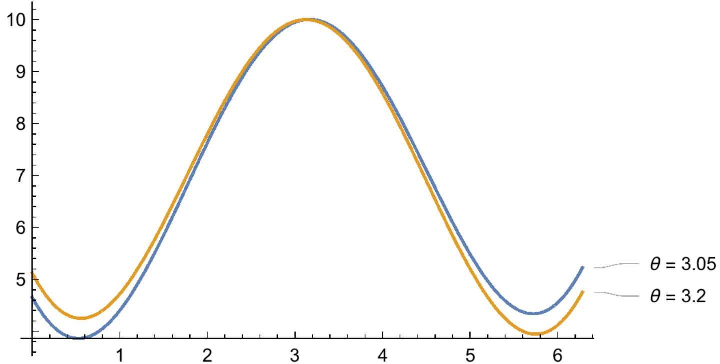

Figure: $V(\phi)$ for two values of θ on opposite sides of π and $\epsilon = 5$. The true minimum swaps abruptly as one goes through $\theta = \pi$.

- If we move away from $\theta = \pi$ while $\epsilon > 1$ we can have different situations.
- Below a critical $\epsilon(\theta)$ there is only one minimum while above it an extra couple of stationary points pops out.
- One of them is a local maximum, the other a local minimum.
- Which is the absolute minimum depends on θ.
- For $\theta < \pi$ the true minimum is at $\phi < \theta$ while for $\theta > \pi$ it is at $\phi > \theta$.
- Precisely at $\theta = \pi$ there is a two-fold degeneracy easily understood as due to the spontaneous breaking of *CP*.

- This abrupt change in the minimum of the potential around $\theta = \pi$ signals a first order phase transition all along the line $\mu^2 e^{i\theta} = [-\infty, -a^2]$ ending at the second order phase transition point $\theta = \pi, \mu^2 = a$.

QCD phase diagrams: $N_f = 2$

- In the case $N_f = 2$ with unequal masses (say, $\mu_1^2 < \mu_2^2$) the equations to be solved are

$$\epsilon_1 \sin\phi_1 = \epsilon_2 \sin\phi_2 = \theta - \phi_1 - \phi_2 \quad ; \quad \epsilon_i \equiv \frac{\mu_i^2}{a} \, .$$

- For $\theta = \pi$ the solutions are simply

$$\phi_1 = \pi \quad ; \quad \phi_2 = 0 \ \text{ or } \ \phi_1 = 0 \quad ; \quad \phi_2 = \pi \, .$$

- The masses of the two pseudoscalar mesons are given by

$$M_{1,2}^2 = a + \frac{\mu_1^2(\theta) + \mu_2^2(\theta)}{2} \pm \sqrt{a^2 + \left(\frac{\mu_1^2(\theta) - \mu_2^2(\theta)}{2}\right)^2} \, ,$$

valid for arbitrary θ.

- It is easy to check that the mass squared with the minus sign is massless if the following condition is satisfied

$$a(\mu_2^2(\theta) + \mu_1^2(\theta)) = \left(\frac{\mu_1^2(\theta) - \mu_2^2(\theta)}{2} \right)^2 - \left(\frac{\mu_1^2(\theta) + \mu_2^2(\theta)}{2} \right)^2 .$$

- Notice that, if both $\mu_{1,2}^2(\theta)$ are positive, the previous condition cannot be satisfied because the r.h.s. is always negative, while the l.h.s. is always positive.
- In particular, it cannot be satisfied at $\theta = 0$.
- But at $\theta = \phi_1 = \pi$, the previous condition becomes

$$a(\mu_2^2 - \mu_1^2) = \mu_1^2 \mu_2^2 \Longrightarrow \frac{1}{a} + \frac{1}{\mu_2^2} = \frac{1}{\mu_1^2} .$$

- This means that, if the condition

$$\frac{1}{\mu_1^2} - \frac{1}{\mu_2^2} \geq \frac{1}{a}$$

is fulfilled, *CP* is unbroken because $\theta - \phi_1 - \phi_2 = 0$.

- Although the second solution conserves *CP*, it does not correspond to the absolute minimum.
- On the other hand, if $\mu_1^{-2} < \mu_2^{-2} + a^{-1}$ not even the first solution corresponds to a minimum and other solutions take over.
- We see clearly that, as in the $N_f = 1$ case, the critical surface $\mu_1^{-2} = \mu_2^{-2} + a^{-1}$ separates the situation with a single solution from the one with several solutions.
- In the latter case *CP* is spontaneously broken and the ground state jumps as we go from $\theta < \pi$ to $\theta > \pi$.
- On the critical surface there is a massless excitation and the QCD topological susceptibility blows up.

- For the case $N_f = 2$ at $\theta = \pi$, we find a critical surface characterized by $(\mu_1^2 < \mu_2^2)$

$$\frac{1}{\mu_1^2} = \frac{1}{\mu_2^2} + \frac{1}{a}$$

where there is a massless excitation and the QCD topological susceptibility blows up.
- In the region where

$$\frac{1}{\mu_1^2} > \frac{1}{\mu_2^2} + \frac{1}{a}$$

there is only one solution and *CP* is conserved $(\theta = \phi_1 + \phi_2)$, while in the region

$$\frac{1}{\mu_1^2} < \frac{1}{\mu_2^2} + \frac{1}{a}$$

there are more solutions and *CP* is spontaneously broken $(\theta \neq \phi_1 + \phi_2)$.
- For equal masses *CP* is always spontaneously broken.

$N_f = 2$ with equal masses

- Potential for $\mu_1^2 = \mu_2^2 \equiv \mu^2$ and $\phi_1 = \phi_2 \equiv \phi$:

$$\frac{2}{aF_\pi^2} V = \frac{1}{2}(\theta - 2\phi)^2 - 2\epsilon \cos\phi \quad ; \quad \epsilon = \frac{\mu^2}{a}$$

- Minimization equation:

$$\theta - 2\phi = \epsilon \sin\phi$$

- For $\epsilon << 1$ its solution is given by

$$\phi = \frac{\theta}{2} - \frac{\epsilon}{2}\sin\frac{\theta}{2} + \frac{\epsilon^2}{4}\sin\theta + \mathcal{O}(\epsilon^3)$$

Since $\theta - 2\phi \neq 0$ CP is always broken (spontaneously at $\theta = \pi$) and the potential becomes

$$\frac{2}{F_\pi^2} V \sim a\left(-2\epsilon\cos\frac{\theta}{2} - \frac{\epsilon^2}{2}\sin^2\frac{\theta}{2} + \mathcal{O}(\epsilon^3)\right)$$

that, if $\theta = \pi$, becomes very flat (goes to zero) for $a >> \mu^2$!

- In particular, at $\theta = \pi$, it becomes of the order $a\epsilon^2 = \frac{\mu^4}{a}$ and in principle we should have added terms of order μ^4 to our effective Lagrangian.
- However, these terms of order μ^4 are negligible (at large N) with respect to those that we have found of order $\frac{\mu^4}{a}$ and cannot change this result.
- At $\theta = \pi$ the two masses are

$$M_1^2 = \frac{\mu^2 \epsilon}{2} + \mathcal{O}(\epsilon^2) \quad ; \quad M_2^2 = 2a + \frac{\mu^2 \epsilon}{2} + \mathcal{O}(\epsilon^2)$$

- When a becomes very large

$$M_1^2 \to 0 \quad ; \quad M_2^2 \to \infty$$

- Actually, by writing the potential as follows

$$\frac{2}{aF_\pi^2} V = \frac{1}{2}(\theta - \phi_1 - \phi_2)^2 - \epsilon\left(\cos\phi_1 + \cos\phi_2\right) \quad ; \quad \epsilon \equiv \frac{\mu^2}{a}$$

we get the following minimization conditions:

$$\theta - \phi_1 - \phi_2 = \epsilon \sin\phi_1 = \epsilon \sin\phi_2$$

- At $\theta = \pi$ we have two solutions that break spontaneously flavor symmetry:

$$\phi_1 = \pi, \phi_2 = 0 \quad ; \qquad \phi_1 = 0, \phi_2 = \pi$$

- They correspond to two other minima of the potential, but they are not absolute minima.
- In fact, at the two minima, the potential is exactly vanishing, while the potential corresponding to the absolute minimum, discussed above, is negative.

QCD phase diagrams: $N_f > 2$

- We have to solve the following equations:

$$\mu_i^2 \sin\phi_i = a\left(\theta - \sum_{i=1}^{N_f}\phi_i\right) \quad ; \quad i = 1\ldots N_f$$

- For $\theta = \pi$ we have the following solution that generalizes to N_f flavors what we found for two flavors:

$$\phi_1 = \pi \quad ; \quad \phi_2 = \phi_3 = \cdots = \phi_{N_f} = 0 \;\; ; \;\; \mu_1^2 \leq \mu_i^2 \;\; \text{for } i \neq 1\,.$$

- It can be immediately checked that the determinant of the mass matrix is positive if the condition

$$\Delta \equiv \frac{1}{\mu_1^2} - \frac{1}{a} - \sum_{i=2}^{N_f}\frac{1}{\mu_i^2} > 0$$

is satisfied.

- In the corresponding region of parameter space we have a *CP* conserving stable solution since $\theta - \sum_{i=1}^{N_f} \phi_i = 0$.
- On the surface where the inequality sign is replaced by an equality, the topological susceptibility diverges and there is a massless state,
- This signals a second order phase transition.
- In the region where, instead, $\Delta < 0$, the solution ceases to be a minimum and we have to look for new solutions corresponding to minima where we will find that *CP* is spontaneously broken.

- For arbitrary N_f and $\theta = \pi$, the critical surface is characterized by ($\mu_1^2 < \mu_i^2$)

$$\frac{1}{\mu_1^2} = \sum_{i=2}^{N_f} \frac{1}{\mu_i^2} + \frac{1}{a}$$

where there is a massless excitation and the QCD topological susceptibility blows up.

- In the region where

$$\frac{1}{\mu_1^2} > \sum_{i=2}^{N_f} \frac{1}{\mu_i^2} + \frac{1}{a}$$

CP is conserved, while in the region

$$\frac{1}{\mu_1^2} < \sum_{i=2}^{N_f} \frac{1}{\mu_i^2} + \frac{1}{a}$$

CP is spontaneously broken.

- For all equal masses *CP* is always spontaneously broken.

CP violation

- If $\theta \neq \sum_i \phi_i$ CP is broken and one can compute quantities that break CP.
- We get

$$\Gamma(\eta \to \pi^+\pi^-) = \theta^2 \cdot (135 \; KeV) \quad : \quad \frac{\Gamma(\eta \to \pi^+\pi^-)}{\Gamma_{tot}} = 159 \; \theta^2$$

- From experiments we get:

$$\frac{\Gamma(\eta \to \pi^+\pi^-)}{\Gamma_{tot}} < 3 \cdot 10^{-4}$$

  that gives an upper limit to the value of $\theta < 10^{-3}$
- Stronger constraint from electric dipole moment of the neutron

$$D_n = \frac{1}{4\pi^2 m_N} \cdot g_{\pi NN} \bar{g}_{\pi NN} \log \frac{m_N}{m_\pi} = 3.6 \cdot 10^{-16} \theta cm$$

  in units where the electric charge $e = 1$.
- The experimental limit is:

$$D_n < 6 \cdot 10^{-26} \Longrightarrow \theta < 10^{-9}$$

Including the axion

- θ is very small and actually consistent with zero.
- Can we make it to be zero in a natural way?
- The vanishing of m_u would be a way because it allows to rotate θ away.
- But m_u seems to be $\neq 0$.
- The Peccei-Quinn solution of the strong CP problem includes in the matter sector of QCD some new d.o.f. with an extra $U(1)_{PQ}$ symmetry that is broken by an anomaly exactly as $U(1)_A$.

- Denoting with $\alpha_{PQ}$the coefficient of the $U(1)_{PQ}$ anomaly and with F_α the scale of its spontaneous breaking, we can extend our previous Lagrangian [Sannino and DV, 2014] as follows:

$$L = \frac{1}{2} Tr(\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{2}\partial_\mu N \partial_\mu N^\dagger + \frac{F_\pi}{2\sqrt{2}} Tr\left(\mu^2 (U + U^\dagger)\right) +$$

$$-\theta Q + \frac{Q^2}{aF_\pi^2} + \frac{i}{2} Q(x) \left(Tr(\log U - \log U^\dagger) + \alpha_{PQ}(\log N - \log N^\dagger) \right)$$

where

$$U(x) = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\Phi(x)/F_\pi} \quad ; \quad N(x) = \frac{F_\alpha}{\sqrt{2}} e^{i\sqrt{2}\alpha(x)/F_\alpha}$$

$$\Phi = \Pi^a T^a + \frac{S}{\sqrt{N_f}}$$

- Under the two $U(1)$ transformations:

$$U \to e^{i\beta} U \quad ; \quad N \to e^{i\gamma} N \,,$$

the effective Lagrangian transforms as follows:

$$L \to L - (N_f \beta + a_{PQ} \gamma)\, q(x)$$

- It is invariant under the $U(1)$ determined by the condition: $N_f \beta + a_{PQ} \gamma = 0$.
- This is an anomaly-free $U(1)$ subgroup, whose spontaneous and explicit breaking (by quark masses) implies a new, pseudo-Goldstone boson, the (Peccei-Quinn-Weinberg-Wilczek) axion.

- Restricting to the fields in the Cartan sub-algebra of the QCD pseudoscalar mesons with $\frac{\sqrt{2}}{F_\pi} v_i \to -\phi_i + \frac{\sqrt{2}}{F_\pi} v_i$, the previous Lagrangian becomes

$$L = \frac{1}{2} \sum_{i=1}^{N_f} \partial_\mu v_i \partial^\mu v_i + \frac{F_\pi^2}{2} \sum_{i=1}^{N_f} \mu_i^2 \cos\left(-\phi_i + \frac{\sqrt{2}}{F_\pi} v_i\right) + \frac{Q^2}{2\chi_{YM}}$$

$$+\frac{1}{2}(\partial_\mu \sigma)^2 - Q\left(\theta - \sum_{i=1}^{N_f} \phi_i - \beta + \frac{\sqrt{2}}{F_\pi} \sum_{i=1}^{N_f} v_i + \frac{\alpha_{PQ}\sqrt{2}}{F_\alpha}\sigma\right) ,$$

where again we have allowed for a non-trivial expectation $\langle U \rangle$.
- We have also introduced an expectation value for $\alpha(x)$ and a shifted axion field σ as $\alpha(x) = -\frac{\alpha_{PQ}\sqrt{2}}{F_\alpha}\beta + \sigma(x)$.
- We determine the phases ϕ_i and β by minimizing

$$V(\phi_i, \beta) = -\frac{F_\pi^2}{2} \sum_{i=1}^{N_f} \mu_i^2 \cos\phi_i + \frac{\chi_{YM}}{2}\left(\theta - \sum_{i=1}^{N_f} \phi_i - \beta\right)^2 .$$

- The stationary points of this potential are solutions of the equations

$$-\frac{F_\pi^2}{2}\mu_i^2 \sin\phi_i + \chi_{YM}(\theta - \sum_i \phi_i - \beta) = 0 \quad ; \quad i = 1, 2, \dots, N_f$$

$$\theta - \sum_i \phi_i - \beta = 0\,,$$

- They are given by

$$\hat{\phi}_i = 0 \quad \mathrm{mod}\,(\pi) \;\; ; \quad \hat{\beta} = \theta - \sum_{i=1}^{N_f} \phi_i\,.$$

- The choice

$$\hat{\phi}_i = 0 \;\; ; \;\; i = 1, 2, \dots, N_f \quad ; \quad \hat{\beta} = \theta$$

corresponds to the minimum of the potential.

- The other choices correspond to maxima or to saddle points.

- We get

$$L = -V(\hat{\phi}_i, \hat{\beta}) + \frac{1}{2}\sum_{i=1}^{N_f} \partial_\mu v_i \partial^\mu v_i + \frac{F_\pi^2}{2}\sum_{i=1}^{N_f} \mu_i^2 \left(\cos\left(\frac{\sqrt{2}}{F_\pi} v_i \right) - 1 \right)$$

$$-\frac{\chi_{YM}}{2} \left(\frac{\sqrt{2}}{F_\pi} \sum_{i=1}^{N_f} v_i + \frac{\alpha_{PQ}\sqrt{2}}{F_\alpha}\sigma \right)^2$$

$$+\frac{1}{2\chi_{YM}} \left(Q - \chi_{YM} \left(\frac{\sqrt{2}}{F_\pi} \sum_{i=1}^{N_f} v_i + \frac{\alpha_{PQ}\sqrt{2}}{F_\alpha}\sigma \right) \right)^2 .$$

- The mass spectrum of the system can be found by diagonalizing the quadratic part

$$L_2 = \frac{1}{2}\sum_{i=1}^{N_f} \partial_\mu v_i \partial^\mu v_i - \frac{1}{2}\sum_{i=1}^{N_f} \mu_i^2 v_i^2 - \frac{\chi_{YM}}{2} \left(\frac{\sqrt{2}}{F_\pi} \sum_{i=1}^{N_f} v_i + \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma \right)^2$$

$$+\frac{1}{2}(\partial_\mu \sigma)^2 = \frac{1}{2}\sum^{N_f+1} \partial_\mu H_a \partial^\mu H_a - \frac{1}{2} H^T A H ,$$

- H is an $N_f + 1$-column vector and A is the squared-mass matrix

$$H = \begin{pmatrix} \sigma \\ v_1 \\ v_2 \\ . \\ . \\ v_{N_f} \end{pmatrix} \quad ; \quad A = \begin{pmatrix} b^2 a & ba & ba & ba & \dots & ba \\ ba & \mu_1^2 + a & a & a & \dots & a \\ ba & a & \mu_2^2 + a & a & \dots & a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ ba & a & a & a & \dots & \mu_{N_f}^2 + a \end{pmatrix}$$

- The mass spectrum is the result of the diagonalization of A and can be read off from

$$\det\left(p^2 \delta_{ij} - A_{ij}\right) = p^2 \prod_{i=1}^{N_f} (p^2 - \mu_i^2) \left[1 - a \left(\sum_{i=1}^{N_f} \frac{1}{p^2 - \mu_i^2} + \frac{b^2}{p^2} \right) \right]$$

$$= \prod_{i=1}^{N_f+1} \left(p^2 - M_i^2 \right) \quad ; \quad a = \frac{2\chi_{YM}}{F_\pi^2} \quad ; \quad b = \frac{F_\pi \alpha_{PQ}}{F_\alpha}$$

M_i are the masses of the physical states that diagonalize the mass matrix.

- By going to $p^2 = 0$

$$\det A = ab^2 \prod_{i=1}^{N_f} \mu_i^2 = \prod_{j=1}^{N_f+1} M_j^2 \,,$$

  where the product on the r.h.s. includes the axion as well as the Cartan PNGB masses.
- For small b, the mass of the axion is given by looking for a zero at small p^2 of the term in square brackets. Neglecting p^2 with respect to μ_i^2 one obtains

$$M^2_{axion} = \frac{b^2}{\frac{1}{a} + \sum_{i=1}^{N_f} \frac{1}{\mu_i^2}} \quad ; \quad b \equiv \frac{F_\pi \alpha_{PQ}}{F_\alpha}$$

- This reduces to the usual expression for axion mass in the limit $a, \mu_s^2 \gg \mu_{u,d}^2$.

- Alternatively using the definition of b and χ_{QCD}, we can write

$$M^2_{axion} = \frac{2\alpha^2_{PQ}}{F^2_\alpha}\chi_{QCD}\,,$$

another formula often used in the literature.

- Finally, we get the following two-point correlation function

$$\langle Q(x)Q(y)\rangle^{F.T.} = i\chi_{YM}\frac{p^2\prod_{i=1}^{N_f}(p^2-\mu_i^2)}{\prod_{i=1}^{N_f+1}(p^2-M_i^2)}$$

$$= \frac{i\chi_{YM}}{\left[1 - a\left(\sum_{i=1}^{N_f}\frac{1}{p^2-\mu_i^2} + \frac{b^2}{p^2}\right)\right]}\,,$$

- It vanishes at $p^2 = 0$ signalling that the topological susceptibility in a theory where QCD is "augmented" by another sector that includes the axion, is zero.
- This is consistent with the fact that the dependence on the θ parameter disappears.

- For the physically interesting case we have to take $b \ll 1$ so that the spectrum should contain a very light pseudo-scalar, the physical axion, which is the original field σ up to an O(b) admixture of PNGBs.
- This is all well known. We will now discuss how things take an interesting turn when we go from properties of the spectrum (i.e. of small fluctuations around the minimum of V) to those of the full potential at a finite distance from its minimum.

The axion potential

- The axion potential is given by

$$V(v_i, \sigma) =$$
$$= \frac{F_\pi^2}{2}\left[-\sum_{i=1}^{N_f} \mu_i^2 \cos\left(\frac{\sqrt{2}}{F_\pi} v_i\right) + \frac{a}{2}\left(\sum_{i=1}^{N_f} \frac{\sqrt{2}}{F_\pi} v_i + \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma\right)^2\right]$$

- It is the same as that obtained in QCD with the substitutions:

$$\phi_i \to -\frac{\sqrt{2}}{F_\pi} v_i \quad ; \quad \theta \to \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma$$

- The minimum of the potential must satisfy the two equations:

$$\frac{\partial V}{\partial v_i} = \frac{F_\pi}{\sqrt{2}}\left[\sum_{i=1}^{N_f} \mu_i^2 \sin\left(\frac{\sqrt{2}}{F_\pi} v_i\right) + a\left(\sum_{i=1}^{N_f} \frac{\sqrt{2}}{F_\pi} v_i + \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma\right)\right] = 0$$

- and

$$\frac{\partial V}{\partial \sigma} \sim \left(\sum_{i=1}^{N_f} \frac{\sqrt{2}}{F_\pi} v_i + \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma \right) = 0$$

- In order to see if one gets a minimum, one must compute the mass matrix, i.e. the matrix of the second derivatives.
- Usually the axion potential is computed by integrating out the other (mesonic) fields since their mass is much bigger than the axion mass because $F_\alpha >> F_\pi$.
- This is done
  - by imposing the conditions $\frac{\partial V}{\partial v_i} = 0$ at fixed σ
  - under the assumption (well satisfied at zero temperature) that $\epsilon_i \equiv \frac{\mu_i^2}{a} << 1$.

- For $N_f = 1$ one obtains the following axion potential

$$V_{axion}(\sigma) = -\frac{F_\pi^2}{2}\mu^2 \cos\left(\frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma\right) + \mathcal{O}(\frac{\mu^2}{a})$$

- For $N_f = 2$ one gets:

$$V_{axion}(\sigma) = -\frac{F_\pi^2}{2}\sqrt{(\mu_1^2+\mu_2^2)^2 - 4\mu_1^2\mu_2^2 \sin^2\left(\frac{\alpha_{PQ}\sigma}{\sqrt{2}F_\alpha}\right)} + \mathrm{O}(\mu_i^2/a)$$

- But, is this always allowed?
- In particular, is it allowed near the point where the PNGB becomes massless?

The axion potential: $N_f = 1$

- Let us consider, for simplicity, the case of one flavor, where the previous equations become:

$$\frac{F_\pi}{\sqrt{2}}\mu^2 \sin\left(\frac{\sqrt{2}}{F_\pi}v\right) + a(v + b\sigma) = 0$$
$$ba(v + b\sigma) = 0$$

- They are satisfied if $v = \sigma = 0$ or if $-\frac{\sqrt{2}}{F_\pi}v = \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma = \pi$.
- To decide which is a minimum we compute the matrix of the sec. derivatives:

$$\begin{pmatrix} b^2 a & ab \\ ab & \mu^2 \cos\left(\frac{\sqrt{2}}{F_\pi}v\right) + a \end{pmatrix} \; ; \quad b \equiv \frac{\alpha_{PQ}F_\pi}{F_\alpha}$$

whose eigenvalues give the mass of the axion and of the pseudoscalar meson.

- $v = \sigma = 0$ is a minimum with both masses being positive.
- The other is a saddle point with one *mass*^{2} being positive and the other negative (tachyon).
- If $\epsilon = \frac{\mu^2}{a} << 1$ one can neglect the term with the sin, integrating out the mesonic field, and one gets:

$$v + b\sigma = 0$$

- Inserting it in the potential we get

$$V_{axion}(\sigma) = -\frac{F_\pi^2}{2}\mu^2 \cos\left(\frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma\right) + \mathcal{O}(\frac{\mu^2}{a})$$

In this way we have integrated out the heavy meson fields.

- If instead $\frac{\mu^2}{a} >> 1$, we get a small value for v

$$v = -\frac{a\alpha_{PQ}F_\pi}{\mu^2 F_\alpha}\sigma$$

- Inserted in the potential, we get

$$V_{axion}(\sigma) = \frac{\chi_{YM}}{2}\left(\frac{\sqrt{2}\alpha_{PQ}\sigma}{F_\alpha}\right)^2 + \mathcal{O}(\frac{a}{\mu^2})$$

Also in this case we have integrated out the heavy meson fields.

- In this regime the axion mass is determined by a and not by μ^2.
- The 2π periodicity of the axion potential requires in this regime a spike at $\frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma = \pi$.

- There is, however, a third regime where $|1 - \frac{\mu^2}{a}| << 1$.
- Taking the mass matrix around the saddle point:

$$-\frac{\sqrt{2}}{F_\pi} v = \frac{\sqrt{2}\alpha_{PQ}}{F_\alpha}\sigma = \pi$$

- In this case the mass matrix becomes:

$$A = \begin{pmatrix} b^2 a & ab \\ ab & -\mu^2 + a \end{pmatrix} \;\; ; \;\; b \equiv \frac{\alpha_{PQ} F_\pi}{F_\alpha}$$

- If $|\mu^2 - a| \sim ba$, then the two eigenvectors are strongly mixed w.r.t. the original basis (axion-PNGB).
- The mixing is maximal if $\mu^2 = a(1 - b^2)$ and A becomes

$$A = \begin{pmatrix} b^2 a & ab \\ ab & b^2 a \end{pmatrix}$$

  whose eigenvectors are $(1, \pm 1)$ with eigenvalues $b^2 a \pm ba$.
- In this range of values of $\frac{\mu^2}{a}$ and σ, it is not possible to describe the system only in terms of an axion potential.
- The potential must involve both light fields.

The axion potential: $N_f = 2, 3$

- In the real world the quarks playing a role at low energy are the up, down and strange.
- At zero temperature, the quantitative solution of the $U(1)$ problem requires $\mu_u^2 < \mu_d^2 << \mu_s^2 \sim a$.
- The ratios $\mu_u^2 : \mu_d^2 : \mu_s^2 : a$ are about $1 : 2 : 40 : 18$.
- Use these numbers together with the results we obtained from the large-N effective action approach, even if in the real world $N = 3$.
- Quark mass ratios are expected to be constant below the QCD deconfining temperature (they depend on phenomena occurring at the electroweak-breaking scale).
- The temperature dependence of χ_{YM} could possibly differ from that of the quark condensate implying a possible (strong?) T-dependence of $\mu_i^2/a \sim -\frac{m_i<\bar{\psi}_i\psi_i>}{\chi_{YM}}$.
- An increase of that ratio by an order of magnitude would bring us inside the CP broken region.

- The available lattice measurements do not seem to favor this possibility; but better measurements of $\frac{<\bar{\psi}_i \psi_i>}{\chi_{YM}}$ are needed.
- Consider the case of two or three quark flavors of different masses and allow for arbitrary ratios μ_i^2/a.
- The stationary points of the potential are

$$\frac{\sqrt{2}\alpha_{PQ}\sigma}{F_\alpha} = 0, \pi \bmod (2\pi) \; ; \; \frac{\sqrt{2}v_i}{F_\pi} = 0, \pi$$

- The absolute minimum is as usual the trivial one $\sigma = v_i = 0$.
- In general, it is legitimate to integrate out the PNGB degrees of freedom by minimizing their potential at fixed σ and then insert the solution $\hat{v}_i(\sigma)$ in $V(\sigma, v_i)$.
- If $\mu_i^2 \ll a$ this can be easily done and for $N_f = 2$ one gets the well-known potential:

$$V_{axion}(\sigma) = -\frac{F_\pi^2}{2}\sqrt{(\mu_1^2+\mu_2^2)^2 - 4\mu_1^2\mu_2^2 \sin^2\left(\frac{\alpha_{PQ}\sigma}{\sqrt{2}F_\alpha}\right)} + O(\mu_i^2/a)$$

[Veneziano and DV, 1980]

[Grilli di Cortona, Hardy, Pardo Vega and Villadoro, 2016]

- For $N_f = 3$ the result still holds up to corrections $\mathrm{O}(\mu_{u,d}^2/\mu_s^2)$.
- We have seen that the condition for having a massless boson (in the absence of the axion) is

$$\frac{1}{\mu_u^2} = \frac{1}{a} + \frac{1}{\mu_d^2} + \frac{1}{\mu_s^2} \sim \frac{1}{a} + \frac{1}{\mu_d^2} \quad \Rightarrow a(\mu_d^2 - \mu_u^2) = \mu_u^2 \mu_d^2 \, .$$

- Precisely around this point we expect a large mixing to occur between the would-be massless PNGB and the axion.
- We have solved, using Mathematica, the minimization conditions at fixed σ and reconstructed in this way the axion potential.
- We then clearly see that, at small $\mu_{u,d}^2/a$ the potential has a regular maximum around $\frac{\sqrt{2}\alpha_{PQ}\sigma}{F_\alpha} = \pi$ which coincides with the one usually used for the axion and agrees well with it elsewhere.

- As we increase $\mu^2_{u,d}/a$ above the critical value $1 - \mu^2_u/\mu^2_d$, the potential is lower that the previous potential even at $\frac{\sqrt{2}\alpha_{PQ}\sigma}{F_\alpha} = \pi$ and, by periodicity must develop a spike at that point.
- As we finally go much beyond the critical point, the true potential has nothing to do with the conventional one.

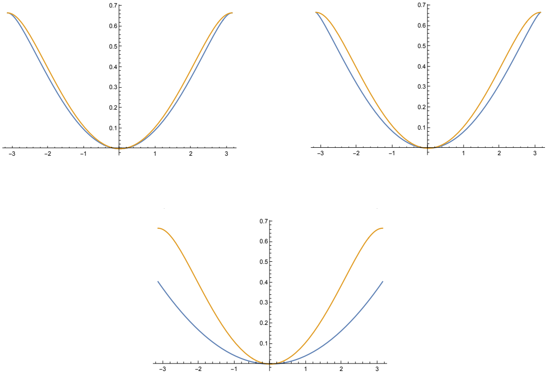

Figure: Comparing the conventional axion potential (yellow curves) with the “exact" one (blue curves) for $N_f = 2, \mu_d^2 = 2\mu_u^2$ and at three values of μ_u^2/a: $0.25, 0.5$ (critical value), 2.5. In the first two cases the two potentials (but not necessarily their derivatives) agree at $\frac{\sqrt{2}}{F_\alpha}\sigma = \pm\pi$ while in the third (overcritical) case even the values of the potentials disagree at the boundary of the periodicity interval.

Conclusions and outlook

- Using the effective Lagrangian for the pseudoscalar mesons valid at low energy and below the deconfinement temperature, we have seen that, for certain values of the parameters (a, μ_i^2), there is a second order phase transition.
- This is signalled by the fact that one of the mesons becomes massless and the topological susceptibility diverges.
- Including an axion in the previous Lagrangian one can eliminate the θ dependence of the physical quantities and compute the axion potential.
- This potential is the one that is normally used for cosmological applications for the physical value of the parameters at $T = 0$.
- It is possible, however, that their dependence on T could bring them into the region where one meson becomes massless.
- In this case one cannot compute the axion potential, as done in the literature, assuming that $\epsilon_i << \frac{\mu_i^2}{a}$.

- Lattice QCD calculations, measuring both χ_{YM} and $< \bar{\psi}_i \psi_i >_{pl}$ by the same group and with the same Monte Carlo configurations, are needed to answer this question.
- Use holography with the model of Witten-Sakai-Sugimoto to study the temperature dependence following the approach by [Bigazzi et al, 2015 +2017].